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Monotone Iterative Schemes for Positive Solutions of a Fractional Differential System with Integral Boundary Conditions on an Infinite Interval

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Abstract. In this paper, using the monotone iterative technique and the Banach contraction mapping principle, we study a class of fractional differential system with integral boundary on an infinite interval. Some explicit monotone iterative schemes for approximating the extreme positive solutions and the unique positive solution are constructed.

1. Introduction

The purpose of this paper is to study monotone iterative schemes of positive solutions for the following fractional differential system with integral boundary conditions

$$\begin{cases} D^{\alpha_1}u(t) + f_1(t, u(t), v(t), D^{\alpha_1-1}u(t), D^{\alpha_2-1}v(t)) = 0, \ n_1 - 1 < \alpha_1 \le n_1, \\ D^{\alpha_2}v(t) + f_2(t, u(t), v(t), D^{\alpha_1-1}u(t), D^{\alpha_2-1}v(t)) = 0, \ n_2 - 1 < \alpha_2 \le n_2, \\ u(0) = u'(0) = \cdots = u^{(n_1-2)}(0) = 0, D^{\alpha_1-1}u(+\infty) = \displaystyle\int_0^{+\infty} h_1(t)u(t)\mathrm{d}t, \\ v(0) = v'(0) = \cdots = v^{(n_2-2)}(0) = 0, D^{\alpha_2-1}v(+\infty) = \displaystyle\int_0^{+\infty} h_2(t)v(t)\mathrm{d}t, \end{cases} \tag{1}$$

where $t \in J = [0, +\infty)$, $f_i \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, J)$, $n_i \in N^+$, $h_i(t) \in L[0, +\infty)$, D^{α_i} are the standard Riemann-Liouville fractional derivative of order $\alpha_i, i = 1, 2$. Here we emphasize that the nonlinearity terms f_i rely on the lower-order fractional derivative of multiple unknown functions and the fractional infinite boundary value rely on the infinite integral of unknown functions.

In recent decades, there has been a rapid growth in the number of fractional calculus from both theoretical and applied perspectives, more detailed description of the subject can be found in the books [1–4]. We note that most of the current results on the existence of fractional differential equations are focused on the finite

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interval, see [5–23]. On the other hand, some authors have also focused on the solvability of fractional differential equations on the infinite intervals, some excellent results were obtained, see [24–36].

In [27] by applying standard fixed point theorems, the authors obtained the existence and uniqueness of solutions for a coupled system of fractional differential equations with m-point fractional boundary conditions

$$\begin{cases} D^p u(t) + f(t, v(t)) = 0, \ p \in (2,3), \\ D^q v(t) + g(t, u(t)) = 0, \ q \in (2,3), \\ u(0) = u'(0) = 0, \ D^{p-1}u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \\ v(0) = v'(0) = 0, \ D^{q-1}v(+\infty) = \sum_{i=1}^{m-2} \gamma_i v(\xi_i), \end{cases}$$

where $t \in J = [0, +\infty), f, g \in C(J \times \mathbb{R}, \mathbb{R}), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty, \beta_i, \gamma_i > 0$, such that $0 < \sum_{i=1}^{m-2} \beta_i u(\xi_i) < \Gamma(p)$ and $0 < \sum_{i=1}^{m-2} \gamma_i v(\xi_i) < \Gamma(q)$, D^p, D^q are the Riemann-Liouville fractional derivatives.

In [30] Zhai and Ren studied a coupled system of fractional differential equations on an unbounded domain:

$$\begin{cases} D^\alpha u(t) + \varphi(t, v(t), D^{\gamma_1}v(t)) = 0, \ \alpha \in (2,3], \gamma_1 \in (0,1), \\ D^\beta v(t) + \psi(t, u(t), D^{\gamma_2}u(t)) = 0, \ \beta \in (2,3], \gamma_2 \in (0,1), \\ I^{3-\alpha}u(0) = 0, \ D^{\alpha-2}u(0) = \displaystyle\int_0^h g_1(s)u(s)\mathrm{d}s, \ D^{\alpha-1}u(+\infty) = Mu(\xi) + a, \\ I^{3-\beta}v(0) = 0, \ D^{\beta-2}v(0) = \displaystyle\int_0^h g_2(s)v(s)\mathrm{d}s, \ D^{\beta-1}v(+\infty) = Nv(\eta) + b, \end{cases} \tag{2}$$

where $t \in J = [0, +\infty), \varphi, \psi \in C(J \times \mathbb{R} \times \mathbb{R}, J), M, N$ are real numbers satisfying $0 < M\xi^{\alpha-1} < \Gamma(\alpha), 0 < N\eta^{\beta-1} < \Gamma(\beta), \xi, \eta, h > 0$, and $a, b \in \mathbb{R}^+, g_1, g_2 \in L^1[0, h]$ are nonnegative functions. By applying fixed point theorems, sufficient conditions for the existence and uniqueness of solutions to the system (2) are provided , which is a natural expansion of the results in [28].

In [33] Zhang et al. applied a monotone iterative method to study a nonlinear fractional boundary value problem on a half line

$$\begin{cases} D^\alpha u(t) + f(t, u(t), D^{\alpha-1}u(t)) = 0, \ \alpha \in (1,2], \\ u(0) = 0, \ D^{\alpha-1}u(+\infty) = \beta u(\xi), \ \beta > 0, \end{cases}$$

where $t \in J = [0, +\infty), f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. The positive extremal solutions and iterative sequence for approximating them are derived. A similar approach is used in [37–41].

Motivated by the mentioned papers, an interesting and a nature question is if we know the existence of solution for the system (1), how can we seek it? This thought motivates the research of iterative schemes of positive solutions for the system (1).

By using the monotone iterative method, in this paper we establish two explicit monotone iterative schemes for approximating the extreme positive solutions and construct an explicit iterative schemes for approximating the unique positive solution, which are more interesting and meaningful than the traditional design route that obtains the existence of solutions. Here we obtain not only the existence of the solution for the system, but also the iterative schemes of the solution. Furthermore, we extend the iterative solution problem of a single equation to the system which is different from [11, 26, 30, 34, 37–41]. Finally, the main results extend the fractional derivative from the low-order to the high-order fractional derivatives.

2. Preliminaries

We first introduce the hypotheses that will play an important role in subsequent proof.

(H_1) $h_i(t) \in L[0, +\infty)$ and $\displaystyle\int_0^{+\infty} h_i(t)t^{\alpha_i-1}\mathrm{d}t = \Lambda_i < \Gamma(\alpha_i), f_i(t, 0, 0, 0, 0) \not\equiv 0, \forall t \in J, \ i = 1, 2.$

(H_2) The nonnegative functions $a_{i0}(t), a_{ik}(t) \in L[0, +\infty)$ and constants $\lambda_{ik} \geq 0$ satisfy

$$|f_i(t, u_1, u_2, u_3, u_4)| \leq a_{i0}(t) + \sum_{k=1}^{4} a_{ik}(t)|u_k|^{\lambda_{ik}}, \forall t \in J, \ u_k \in \mathbb{R}, i = 1, 2, \ k = 1, 2, 3, 4.$$

and

$$\int_0^{+\infty} a_{i0}(t)\mathrm{d}t = a_{i0}^* < +\infty, \int_0^{+\infty} a_{i3}(t)\mathrm{d}t = a_{i3}^* < +\infty, \int_0^{+\infty} a_{i4}(t)\mathrm{d}t = a_{i4}^* < +\infty,$$

$$\int_0^{+\infty} a_{i1}(t)(1+t^{\alpha_1-1})^{\lambda_{i1}}\mathrm{d}t = a_{i1}^* < +\infty, \int_0^{+\infty} a_{i2}(t)(1+t^{\alpha_2-1})^{\lambda_{i2}}\mathrm{d}t = a_{i2}^* < +\infty, i=1,2.$$

(H_3) The nonnegative functions $b_{ik}(t) \in L[0,+\infty)$ satisfy

$$|f_i(t,u_1,u_2,u_3,u_4) - f_i(t,\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4)| \le \sum_{k=1}^{4} b_{ik}(t)|u_k - \bar{u}_k|,$$

$$\forall t \in J,\ u_k, \bar{u}_k \in \mathbb{R},\ i=1,2,\ k=1,2,3,4.$$

and

$$\int_0^{+\infty} b_{i1}(t)(1+t^{\alpha_1-1})\mathrm{d}t = b_{i1}^* < +\infty, \int_0^{+\infty} b_{i2}(t)(1+t^{\alpha_2-1})\mathrm{d}t = b_{i2}^* < +\infty,$$

$$\int_0^{+\infty} b_{i3}(t)\mathrm{d}t = b_{i3}^* < +\infty,\ \int_0^{+\infty} b_{i4}(t)\mathrm{d}t = b_{i4}^* < +\infty, \int_0^{+\infty} |f_i(t,0,0,0,0)|\mathrm{d}t = \tau_i < +\infty,\ i=1,2.$$

(H_4) Functions $f_i(t,u_1,u_2,u_3,u_4)$ are increasing with respect to the variables $u_1,u_2,u_3,u_4, \forall t \in J, i=1,2$.

Next we list some definitions and lemmas that are helpful to the proof of principal theorems.

Definition 2.1(see [1, 3]). The Riemann-Liouville fractional integral of order $q > 0$ for an integrable function g is defined as

$$I^q g(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} g(t)\mathrm{d}t,$$

provided that the integral exists.

Definition 2.2.(see [1, 3]) The Riemann-Liouville fractional derivative of order $q > 0$ for an integrable function g is defined as

$$D^q g(x) = \frac{1}{\Gamma(n-q)}\Big(\frac{d}{dx}\Big)^n \int_0^x (x-t)^{n-q-1} g(t)\mathrm{d}t,$$

where $n = [q]+1, [\alpha]$ is the smallest integer greater than or equal to α, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 2.1.(see [1, 3]) Let $q > 0$ and $u \in C(0,1) \cap L(0,1)$. Then the general solution of fractional differential equation $D^q u(t) = 0$ is

$$u(t) = c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}, i = 1,2,\cdots,n$ and $n-1 < q < n$.

Lemma 2.2. Let $y_i \in C[0,+\infty)$ with $\int_0^{+\infty} h_i(t)t^{\alpha_i-1}\mathrm{d}t \neq \Gamma(\alpha_i), n_i - 1 < \alpha_i \le n_i, i=1,2$. Then the fractional differential system boundary value problem

$$\begin{cases} D^{\alpha_1}u(t) + y_1(t) = 0,\ n_1-1 < \alpha_1 \le n_1, \\ D^{\alpha_2}v(t) + y_2(t) = 0,\ n_2-1 < \alpha_2 \le n_2, \\ u(0) = u'(0) = \cdots = u^{(n_1-2)}(0) = 0, D^{\alpha_1-1}u(+\infty) = \displaystyle\int_0^{+\infty} h_1(t)u(t)\mathrm{d}t, \\ v(0) = v'(0) = \cdots = v^{(n_2-2)}(0) = 0, D^{\alpha_2-1}v(+\infty) = \displaystyle\int_0^{+\infty} h_2(t)v(t)\mathrm{d}t, \end{cases} \tag{3}$$

has the integral representation

$$\begin{cases} u(t) = \displaystyle\int_0^{+\infty} K_1(t,s)y_1(s)\mathrm{d}s, \\ v(t) = \displaystyle\int_0^{+\infty} K_2(t,s)y_2(s)\mathrm{d}s, \end{cases} \tag{4}$$

where

$$K_i(t,s) = K_{i1}(t,s) + K_{i2}(t,s), i = 1,2. \tag{5}$$

with

$$K_{i1}(t,s) = \frac{1}{\Gamma(\alpha_i)}\begin{cases} t^{\alpha_i-1} - (t-s)^{\alpha_i-1}, 0 \le s \le t \le +\infty, \\ t^{\alpha_i-1}, 0 \le t \le s \le +\infty, \end{cases} \tag{6}$$

$$K_{i2}(t,s) = \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i) - \Lambda_i}\int_0^{+\infty} h_i(t)K_{i1}(t,s)\mathrm{d}t. \tag{7}$$

Proof. From Lemma 2.1, we can turn differential system (3) into an equivalent integral system

$$\begin{cases} u(t) = -I^{\alpha_1}y_1(t) + c_{11}t^{\alpha_1-1} + c_{12}t^{\alpha_1-2} + \ldots + c_{1n_1}t^{\alpha_1-n_1}, \\ v(t) = -I^{\alpha_2}y_2(t) + c_{21}t^{\alpha_2-1} + c_{22}t^{\alpha_2-2} + \ldots + c_{2n_2}t^{\alpha_2-n_2}, \end{cases} \tag{8}$$

where $c_{11}, c_{12}, \cdots, c_{1n_1}, c_{21}, c_{22}, \cdots, c_{2n_2}$ are arbitrary constants. With the help of conditions $u(0) = u'(0) = \cdots = u^{(n_1-2)}(0) = 0$ and $v(0) = v'(0) = \cdots = v^{(n_2-2)}(0) = 0$, it is easy to know that $c_{12} = c_{13} = \cdots = c_{1n_1} = c_{22} = c_{23} = \cdots = c_{2n_2} = 0$. From (8) we have

$$\begin{cases} u(t) = -\dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^t (t-s)^{\alpha_1}y_1(s)\mathrm{d}s + c_{11}t^{\alpha_1-1}, \\ v(t) = -\dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^t (t-s)^{\alpha_2}y_2(s)\mathrm{d}s + c_{21}t^{\alpha_2-1}. \end{cases} \tag{9}$$

Then

$$\begin{cases} D^{\alpha_1-1}u(t) = c_{11}\Gamma(\alpha_1) - \displaystyle\int_0^t y_1(s)\mathrm{d}s, \\ D^{\alpha_2-1}v(t) = c_{21}\Gamma(\alpha_2) - \displaystyle\int_0^t y_2(s)\mathrm{d}s. \end{cases} \tag{10}$$

Hence

$$\begin{cases} D^{\alpha_1-1}u(+\infty) = c_{11}\Gamma(\alpha_1) - \displaystyle\int_0^{+\infty} y_1(s)\mathrm{d}s, \\ D^{\alpha_2-1}v(+\infty) = c_{21}\Gamma(\alpha_2) - \displaystyle\int_0^{+\infty} y_2(s)\mathrm{d}s. \end{cases} \tag{11}$$

Based on the conditions $D^{\alpha_1-1}u(+\infty) = \int_0^{+\infty} h_1(t)u(t)\mathrm{d}t$ and $D^{\alpha_2-1}v(+\infty) = \int_0^{+\infty} h_2(t)v(t)\mathrm{d}t$, we have

$$\begin{cases} c_{11} = \dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^{+\infty} h_1(t)u(t)\mathrm{d}t + \dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^{+\infty} y_1(s)\mathrm{d}s, \\ c_{21} = \dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^{+\infty} h_2(t)v(t)\mathrm{d}t + \dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^{+\infty} y_2(s)\mathrm{d}s. \end{cases} \tag{12}$$

Submitting (12) to (10), we know

$$\begin{cases} u(t) &= -\dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^t (t-s)^{\alpha_1}y_1(s)\mathrm{d}s + \dfrac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}\Big[\displaystyle\int_0^{+\infty} h_1(t)u(t)\mathrm{d}t + \int_0^{+\infty} y_1(s)\mathrm{d}s\Big] \\ &= \displaystyle\int_0^{\infty} K_{11}(t,s)y_1(s)\mathrm{d}s + \dfrac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}\displaystyle\int_0^{+\infty} h_1(t)u(t)\mathrm{d}t, \\ v(t) &= -\dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^t (t-s)^{\alpha_2}y_2(s)\mathrm{d}s + \dfrac{t^{\alpha_2-1}}{\Gamma(\alpha_2)}\Big[\displaystyle\int_0^{+\infty} h_2(t)v(t)\mathrm{d}t + \int_0^{+\infty} y_2(s)\mathrm{d}s\Big] \\ &= \displaystyle\int_0^{\infty} K_{21}(t,s)y_2(s)\mathrm{d}s + \dfrac{t^{\alpha_2-1}}{\Gamma(\alpha_2)}\displaystyle\int_0^{+\infty} h_2(t)v(t)\mathrm{d}t. \end{cases} \tag{13}$$

Multiplying both sides of the above equality by $h_1(t)$ and $h_2(t)$ and integrating from 0 to $+\infty$, we obtain

$$\begin{cases} \int_0^{+\infty} h_1(t)u(t)\mathrm{d}t = \dfrac{\Gamma(\alpha_1)}{\Gamma(\alpha_1)-\Lambda_1}\int_0^{+\infty} h_1(t)\int_0^{+\infty} K_{11}(t,s)y_1(s)\mathrm{d}s\mathrm{d}t, \\ \int_0^{+\infty} h_2(t)v(t)\mathrm{d}t = \dfrac{\Gamma(\alpha_2)}{\Gamma(\alpha_2)-\Lambda_2}\int_0^{+\infty} h_2(t)\int_0^{+\infty} K_{21}(t,s)y_2(s)\mathrm{d}s\mathrm{d}t. \end{cases}$$

Combining (13), we have

$$\begin{cases} u(t) &= \int_0^{+\infty} K_{11}(t,s)y_1(s)\mathrm{d}s + \dfrac{t^{\alpha_1-1}}{\Gamma(\alpha_1)-\Lambda_1}\int_0^{+\infty} h_1(t)\int_0^{+\infty} K_{11}(t,s)y_1(s)\mathrm{d}s\mathrm{d}t \\ &= \int_0^{\infty} K_{11}(t,s)y_1(s)\mathrm{d}s + \int_0^{\infty} K_{12}(t,s)y_1(s)\mathrm{d}s, \\ &= \int_0^{\infty} K_1(t,s)y_1(s)\mathrm{d}s, \\ v(t) &= \int_0^{+\infty} K_{21}(t,s)y_2(s)\mathrm{d}s + \dfrac{t^{\alpha_2-1}}{\Gamma(\alpha_2)-\Lambda_2}\int_0^{+\infty} h_2(t)\int_0^{+\infty} K_{21}(t,s)y_2(s)\mathrm{d}s\mathrm{d}t \\ &= \int_0^{+\infty} K_{21}(t,s)y_2(s)\mathrm{d}s + \int_0^{+\infty} K_{22}(t,s)y_2(s)\mathrm{d}s \\ &= \int_0^{+\infty} K_2(t,s)y_2(s)\mathrm{d}s. \end{cases}$$

The proof is completed.

Remark 2.1. From (4), (5), (6) and (7), by direct calculation, we have

$$\begin{cases} D^{\alpha_1-1}u(t) = \int_0^{+\infty} K_1^*(t,s)y_1(s)\mathrm{d}s, \\ D^{\alpha_2-1}v(t) = \int_0^{+\infty} K_2^*(t,s)y_2(s)\mathrm{d}s, \end{cases}$$

where

$$K_i^*(t,s) = K_{i1}^*(t,s) + K_{i2}^*(t,s),\ i = 1,2.$$

with

$$K_{i1}^*(t,s) = \begin{cases} 0, 0 \le s \le t \le +\infty, \\ 1, 0 \le t \le s \le +\infty, \end{cases}\quad K_{i2}^*(t,s) = \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i)-\Delta_i}\int_0^{+\infty} h_i(t)K_{i1}(t,s)\mathrm{d}t.$$

Lemma 2.3. For $(s,t) \in J \times J$, if hypothesis (H_1) is satisfied, then

$$0 \le K_i(t,s) \le \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i)-\Lambda_i},\ \ 0 \le \frac{K_i(t,s)}{1+t^{\alpha_i-1}} \le \frac{1}{\Gamma(\alpha_i)-\Lambda_i},\ i = 1,2.$$

Proof. From (6) and (7), it is obvious that

$$0 \le K_{i1}(t,s) \le \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i)}, \forall (t,s) \in J \times J,$$

and

$$0 \le K_{i2}(t,s) \le \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i)-\Lambda_i}\int_0^{+\infty} \frac{h_i(t)t^{\alpha_i-1}}{\Gamma(\alpha_i)}\mathrm{d}t = \frac{\Lambda_i t^{\alpha_i-1}}{\Gamma(\alpha_i)(\Gamma(\alpha_i)-\Lambda_i)}, \forall (t,s) \in J \times J.$$

So

$$0 \le K_i(t,s) = K_{i1}(t,s) + K_{i2}(t,s) \le \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i)-\Lambda_i}, \forall (t,s) \in J \times J.$$

Furthermore

$$0 \le \frac{K_i(t,s)}{1+t^{\alpha_i-1}} \le \frac{1}{\Gamma(\alpha_i)-\Lambda_i}, \forall (t,s) \in J \times J.$$

The proof is completed.

Remark 2.2. From Remark 2.1, by direct calculation, we can easily know that

$$0 \le K_i^*(t,s) = K_{i1}^*(t,s) + K_{i2}^*(t,s) \le 1 + \frac{\Lambda_i}{\Gamma(\alpha)-\Lambda_i} = \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i)-\Lambda_i}, \forall (t,s) \in J \times J,\ i = 1,2,$$

Let $E = \{u \in C(J,\mathbb{R}) | \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha_1-1}} < +\infty\}$ and $X = \{u \in E, D^{\alpha_1-1}u \in C(J,\mathbb{R}) |\ \sup_{t\in J} |D^{\alpha_1-1}u(t)| < +\infty\}$ be equipped with the norm

$$\|u\|_X = \max\{\|u\|_0, \|D^{\alpha_1-1}u\|_1\},$$

where $\|u\|_0 = \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha_1-1}}$ and $\|D^{\alpha_1-1}u\|_1 = \sup_{t\in J} |D^{\alpha_1-1}u(t)|$. Also let $F = \{v \in C(J,\mathbb{R}) | \sup_{t\in J} \frac{|v(t)|}{1+t^{\alpha_2-1}} < +\infty\}$ and $Y = \{v \in F, D^{\alpha_2-1}v \in C(J,\mathbb{R}) | \sup_{t\in J} |D^{\alpha_2-1}v(t)| < +\infty\}$ be equipped with the norm

$$\|u\|_Y = \max\{\|v\|_0, \|D^{\alpha_2-1}v\|_1\},$$

where $\|v\|_0 = \sup_{t\in J} \frac{|v(t)|}{1+t^{\alpha_2-1}}$ and $\|D^{\alpha_2-1}v\|_1 = \sup_{t\in J} |D^{\alpha_2-1}v(t)|$. Thus the space $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two Banach spaces which have been shown in [24]. Moreover, the product space $(X \times Y, \|\cdot\|_{X\times Y})$ is also a Banach space with the norm

$$\|\cdot\|_{X\times Y} = \max\{\|u\|_X, \|v\|_Y\}.$$

Lemma 2.4. If hypothesis (H_2) is satisfied, then for $\forall (u,v) \in X \times Y$, we have

$$\int_0^{+\infty} |f_i(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))| \mathrm{d}s \le a_{i0}^* + \sum_{k=1}^{4} a_{ik}^* \|(u,v)\|_{X\times Y}^{\lambda_{ik}}, i = 1,2.$$

Proof. For $\forall (u,v) \in X \times Y$, by hypothesis ($H_2$), we have

$$\begin{aligned}
&\int_0^{+\infty} |f_i(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))| \mathrm{d}s \\
\le &\int_0^{+\infty} \left(a_{i0}(s) + a_{i1}(s)|u(s)|^{\lambda_{i1}} + a_{i2}(s)|v(s)|^{\lambda_{i2}} + a_{i3}(s)|D^{\alpha_1-1}u(s))|^{\lambda_{i3}} + a_{i4}(s)|D^{\alpha_2-1}v(s))|^{\lambda_{i4}}\right) \mathrm{d}s \\
\le &a_{i0}^* + \int_0^{+\infty} a_{i1}(s))(1+s^{\alpha_1-1})^{\lambda_{i1}} \frac{|u(s)|^{\lambda_{i1}}}{(1+s^{\alpha_1-1})^{\lambda_{i1}}} ds + \int_0^{+\infty} a_{i2}(s))(1+s^{\alpha_2-1})^{\lambda_{i2}} \frac{|v(s)|^{\lambda_{i2}}}{(1+s^{\alpha_2-1})^{\lambda_{i2}}} ds \\
&+ \int_0^{+\infty} a_{i3}(s)|D^{\alpha_1-1}u(s)|^{\lambda_{i3}} ds + \int_0^{+\infty} a_{i4}(s)|D^{\alpha_2-1}v(s)|^{\lambda_{i4}} ds \\
\le &a_{i0}^* + a_{i1}^*\|u\|_X^{\lambda_{i1}} + a_{i2}^*\|v\|_Y^{\lambda_{i2}} + a_{i3}^*\|u\|_X^{\lambda_{i3}} + a_{i4}^*\|v\|_Y^{\lambda_{i4}} \\
\le &a_{i0}^* + \sum_{k=1}^{4} a_{ik}^* \|(u,v)\|_{X\times Y}^{\lambda_{ik}},\ i = 1,2.
\end{aligned}$$

Lemma 2.5. If hypothesis (H_3) is satisfied, then for $\forall (u,v) \in X \times Y$, we have

$$\int_0^{+\infty} |f_i(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))| \mathrm{d}s \le \sum_{k=1}^{4} b_{ik}^* \|(u,v)\|_{X\times Y} + \tau_i,\ i = 1,2.$$

Proof. For $\forall (u,v) \in X \times Y$, by hypothesis (H_3), we have

$$\begin{aligned}
&\int_0^{+\infty} |f_i(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))|ds \\
=&\int_0^{+\infty} |f_i(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s)) - f_i(s,0,0,0,0) + f_i(s,0,0,0,0)|ds \\
\le&\int_0^{+\infty} |f_i(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s)) - f_i(s,0,0,0,0)|ds + \int_0^{+\infty} |f_i(s,0,0,0,0)|ds \\
\le&\int_0^{+\infty} b_{i1}(s)(1+s^{\alpha_1-1})\frac{|u(s)|}{1+s^{\alpha_1-1}}ds + \int_0^{+\infty} b_{i2}(s)(1+s^{\alpha_2-1})\frac{|v(s)|}{1+s^{\alpha_2-1}}ds \\
&+\int_0^{+\infty} b_{i3}(s)|D^{\alpha_1-1}u(s)|ds + \int_0^{+\infty} b_{i4}(s)|D^{\alpha_2-1}v(s)|ds + \int_0^{+\infty} |f_i(s,0,0,0,0)|ds \\
\le& b_{i1}^*\|u\|_X + b_{i2}^*\|v\|_Y + b_{i3}^*\|u\|_X + b_{i4}^*\|v\|_Y + \tau_i \\
\le&\sum_{k=1}^{4} b_{ik}^*\|(u,v)\|_{X\times Y} + \tau_i,\ i=1,2.
\end{aligned}$$

Lemma 2.6. (see [24]) Let $U \subset X$ be a bounded set. Then U is a relatively compact in X if the following conditions hold:

(i) For any $u \in U$, $\frac{u(t)}{1+t^{\alpha-1}}$ and $D^{\alpha-1}u(t)$ are equicontinuous on any compact interval of J;

(ii) For any $\varepsilon > 0$, there is a constant $C = C(\varepsilon) > 0$ such that $|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}| < \varepsilon$ and $|D^{\alpha-1}u(t_1) - D^{\alpha-1}u(t_2)| < \varepsilon$ for any $t_1, t_2 \ge C$ and $u \in U$.

We define the cone $P \subset X \times Y$ by $P = \{(u,v) \in X \times Y | u(t) \ge 0, v(t) \ge 0, D^{\alpha_1-1}u(t) \ge 0, D^{\alpha_2-1}v(t) \ge 0, t \in J\}$. By Lemma 2.2, let $T : P \to P$ be the operator defined as

$$T(u,v)(t) = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} K_1(t,s)f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))ds \\ \int_0^{+\infty} K_2(t,s)f_2(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))ds \end{pmatrix} \tag{14}$$

By Remark 2.1, we also define

$$\begin{pmatrix} D^{\alpha_1-1}T_1(u,v)(t) \\ D^{\alpha_2-1}T_2(u,v)(t) \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} K_1^*(t,s)f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))ds \\ \int_0^{+\infty} K_2^*(t,s)f_2(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))ds \end{pmatrix} \tag{15}$$

It is easy to know that the system (1) has a solution if and only if the operator equation $(u,v) = T(u,v)$ has a fixed point, where T is given by (14). In fact, if (u,v) is a solution for the system (1), by lemma 2.2, we can obtain

$$\begin{aligned}
u &= \int_0^{+\infty} K_1(t,s)f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))ds = T_1(u,v), \\
v &= \int_0^{+\infty} K_2(t,s)f_2(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))ds = T_2(u,v).
\end{aligned}$$

That is, (u,v) is a fixed point for the operator equation $(u,v) = T(u,v)$. On the contrary, the Riemann-Liouville fractional derivation on both sides of the operator equation is

$$\begin{aligned}
D^{\alpha_1}u(t) &= D^{\alpha_1}T_1(u,v)(t) = -f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s)), \\
D^{\alpha_2}v(t) &= D^{\alpha_2}T_2(u,v)(t) = -f_2(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s)).
\end{aligned}$$

Combining (11) and (12), we can obtain

$$D^{\alpha_1-1}u(+\infty)=\int_0^{+\infty}h_1(t)u(t)\mathrm{d}t,\ \ D^{\alpha_2-1}v(+\infty)=\int_0^{+\infty}h_2(t)v(t)\mathrm{d}t,$$

That is, (u,v) is a solution for the system (1).

Lemma 2.7. If the hypotheses (H_1) and (H_2) are satisfied, then the operator $T:P\to P$ is completely continuous.

Proof. First it is easy to know $T:P\to P$. Since $K_i(t,s)\geq 0$ and $f_i\geq 0$, we have $T_i(u,v)(t)\geq 0,\forall(u,v)\in P,\ t\in J,\ i=1,2$.

Next we prove in three steps that the operator $T:P\to P$ is relatively compact.

Step 1 Let $U=\{(u,v)|(u,v)\in P,\|(u,v)\|_{X\times Y}\leq M\}$. For $\forall(u,v)\in U$, by Lemma 2.3, Remark 2.2 and Lemma 2.4, we obtain

$$\begin{aligned}\|T_1(u,v)\|_0 &= \sup_{t\in J}\Big|\int_0^{+\infty}\frac{K_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\mathrm{d}s\Big|\\ &\leq \frac{1}{\Gamma(\alpha_1)-\Lambda_1}\int_0^{+\infty}|f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))|\mathrm{d}s\\ &\leq \frac{1}{\Gamma(\alpha_1)-\Lambda_1}\Big[a_{10}^*+\sum_{k=1}^{4}a_{1k}^*\|(u,v)\|_{X\times Y}^{\lambda_{1k}}\Big]\end{aligned}\tag{16}$$

and

$$\begin{aligned}\|T_1(u,v)\|_1 &= \sup_{t\in J}\Big|\int_0^{\infty}K_1^*(t,s)f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\mathrm{d}s\Big|\\ &\leq \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1)-\Lambda_1}\int_0^{+\infty}|f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))|\mathrm{d}s\\ &\leq \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1)-\Lambda_1}\Big[a_{10}^*+\sum_{k=1}^{4}a_{1k}^*\|(u,v)\|_{X\times Y}^{\lambda_{1k}}\Big].\end{aligned}\tag{17}$$

Thus

$$\|T_1(u,v)\|_X=\max\Big\{\|T_1(u,v)\|_0,\|T_1(u,v)\|_1\Big\}\leq\frac{\max\{1,\Gamma(\alpha_1)\}}{\Gamma(\alpha_1)-\Lambda_1}\Big[a_{10}^*+\sum_{k=1}^{4}a_{1k}^*M^{\lambda_{1k}}\Big].$$

Similarly

$$\|T_2(u,v)\|_Y=\max\Big\{\|T_2(u,v)\|_0,\|T_2(u,v)\|_1\Big\}\leq\frac{\max\{1,\Gamma(\alpha_2)\}}{\Gamma(\alpha_2)-\Lambda_2}\Big[a_{20}^*+\sum_{k=1}^{4}a_{2k}^*M^{\lambda_{2k}}\Big].$$

Then

$$\begin{aligned}\|T(u,v)\|_{X\times Y} &= \max\Big\{\|T_1(u,v)\|_X,\|T_2(u,v)\|_Y\Big\}\\ &\leq \max\Big\{\frac{\max\{1,\Gamma(\alpha_1)\}}{\Gamma(\alpha_1)-\Lambda_1}\Big(a_{10}^*+\sum_{k=1}^{4}a_{1k}^*M^{\lambda_{1k}}\Big),\\ &\qquad\frac{\max\{1,\Gamma(\alpha_2)\}}{\Gamma(\alpha_2)-\Lambda_2}\Big(a_{20}^*+\sum_{k=1}^{4}a_{2k}^*M^{\lambda_{2k}}\Big)\Big\},\end{aligned}$$

which means that TU is uniformly bounded.

Step 2 Let $I \subset J$ be any compact interval. Then, for all $t_1, t_2 \in I, t_2 > t_1$ and $(u, v) \in U$, we have

$$\begin{aligned} &\Big|\frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}} - \frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}}\Big| \\ \leq\ &\Big|\int_0^{+\infty} \Big(\frac{K_1(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{K_1(t_1,s)}{1+t_1^{\alpha_1-1}}\Big) f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\mathrm{d}s\Big| \\ \leq\ &\int_0^{+\infty} \Big|\frac{K_1(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{K_1(t_1,s)}{1+t_1^{\alpha_1-1}}\Big|\Big|f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\Big|\mathrm{d}s \end{aligned} \tag{18}$$

Noticing that $K_1(t,s)/(1+t^{\alpha_1-1})$ is uniformly continuous for any $(t,s) \in I \times I$. In the meantime, the function $K_1(t,s)/(1+t^{\alpha_1-1})$ only relys on t for $s \geq t$, which infers that $K_1(t,s)/(1+t^{\alpha_1-1})$ is uniformly continuous on $I \times (J \setminus I)$. Therefore, for all $s \in J$ and $t_1, t_2 \in I$, we have

$$\forall \epsilon > 0, \exists \delta(\epsilon) \text{ such that if } |t_1 - t_2| < \delta, \text{ then } \Big|\frac{K_1(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{K_1(t_1,s)}{1+t_1^{\alpha_1-1}}\Big| < \epsilon. \tag{19}$$

By Lemma 2.4, for all $(u, v) \in U$, we can obtain

$$\int_0^{+\infty} |f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))|ds \leq \Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^* M^{\lambda_{1k}}\Big] < \infty,$$

together (18) and (19), which means that $T_1(u,v)(t)/(1+t^{\alpha_1-1})$ is equicontinuous on I.

Note that

$$D^{\alpha_1-1}T_1(u,v)(t) = \int_0^{+\infty} K_1^*(t,s) f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\mathrm{d}s$$

and the function $K_1^*(t,s) \in C(J \times J)$ doesn't rely on t, which means that $D^{\alpha_1-1}T_1(u,v)(t)$ is equicontinuous on I. In the same way, we can show that $T_2(u,v)(t)/(1+t^{\alpha_2-1})$ and $D^{\alpha_2-1}T_2(u,v)(t)$ are equicontinuous. Thus T_1 and T_2 is equicontinuous on I.

As a natural result, the operator T is equicontinuous for all $(u, v) \in U$ on any compact interval I of J.

Step 3 We show the operator T is equiconvergent at $+\infty$. Since

$$\lim_{t\to+\infty} \frac{K_i(t,s)}{1+t^{\alpha_i-1}} = \frac{1}{\Gamma(\alpha_i)} + \frac{1}{\Gamma(\alpha_i)-\Lambda_i}\int_0^{+\infty} h(t)K_{i1}(t,s)\mathrm{d}t \leq \frac{1}{\Gamma(\alpha_i)-\Lambda_i} < +\infty,\ i = 1, 2,$$

by knowledge of limit theory, we can deduce that for any $\epsilon > 0$, there exists a constant $C = C(\epsilon) > 0$, for any $t_1, t_2 \geq C$ and $s \in J$, such that

$$\Big|\frac{K_i(t_2,s)}{1+t_2^{\alpha_i-1}} - \frac{K_i(t_1,s)}{1+t_1^{\alpha_i-1}}\Big| < \epsilon,\ i = 1, 2,$$

Therefore, by Lemma 2.4 and (18), we conclude that $T_i(u,v)(t)/1+t^{\alpha_i-1} (i = 1, 2)$ are equiconvergent at $+\infty$. As the function $K_i^*(t,s) (i = 1, 2)$ don't rely on t, we can easily infer that $D^{\alpha_i-1}T_i(u,v)(t) (i = 1, 2)$ is equiconvergent at $+\infty$.

From the above three steps, Lemma 2.6 is satisfied. So the operator $T : P \to P$ is relatively compact.

Finally we show that the operator $T : P \to P$ is continuous. Let $(u_n, v_n), (u, v) \in P$, such that $(u_n, v_n) \to (u, v) (n \to \infty)$. Then $\|(u_n, v_n)\|_{X\times Y} < +\infty, \|(u, v)\|_{X\times Y} < +\infty$. Similar to (16) and (17), we have

$$\begin{aligned} \|T_1(u_n,v_n)\|_0 =& \sup_{t\in J}\Big|\int_0^{+\infty} \frac{K_1(t,s)}{1+t^{\alpha_1-1}} f_1(s,u_n(s),v_n(s),D^{\alpha_1-1}u_n(s),D^{\alpha_2-1}v_n(s))\mathrm{d}s\Big| \\ \leq& \frac{1}{\Gamma(\alpha_1)-\Lambda_1}\Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^* \|(u_n,v_n)\|_{X\times Y}^{\lambda_{1k}}\Big], \end{aligned}$$

and

$$\|T_1(u_n,v_n)\|_1 = \sup_{t\in J} |\int_0^{+\infty} K_1^*(t,s)f_1(s,u_n(s),v_n(s),D^{\alpha_1-1}u_n(s),D^{\alpha_2-1}v_n(s))ds| \\ \leq \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1)-\Lambda_1}\Big[a_{10}^* + \sum_{k=1}^{4} a_{1k}^*\|(u_n,v_n)\|_{X\times Y}^{\lambda_{1k}}\Big].$$

By continuity of function f_1 and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^{+\infty}\frac{K_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u_n(s),v_n(s),D^{\alpha_1-1}u_n(s),D^{\alpha_2-1}v_n(s))\mathrm{d}s \\ = \int_0^{+\infty}\frac{K_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\mathrm{d}s,$$

and

$$\lim_{n\to\infty}\int_0^{+\infty}K_1^*(t,s)f_1(s,u_n(s),v_n(s),D^{\alpha_1-1}u_n(s),D^{\alpha_2-1}v_n(s))\mathrm{d}s \\ = \int_0^{\infty}K_1^*(t,s)f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\mathrm{d}s.$$

Then

$$\|T_1(u_n,v_n)-T_1(u,v)\|_0 \leq \sup_{t\in J}\int_0^{+\infty}\frac{K_1(t,s)}{1+t^{\alpha-1}}\Big|f_1(s,u_n(s),v_n(s),D^{\alpha_1-1}u_n(s),D^{\alpha_2-1}v_n(s)) \\ - f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\Big|\mathrm{d}s \to 0,\ n\to\infty,$$

and

$$\|T_1(u_n,v_n)-T_1(u,v)\|_1 \leq \sup_{t\in J}\int_0^{+\infty}K_1^*(t,s)\Big|f_1(s,u_n(s),v_n(s),D^{\alpha_1-1}u_n(s),D^{\alpha_2-1}v_n(s)) \\ - f_1(s,u(s),v(s),D^{\alpha_1-1}u(s),D^{\alpha_2-1}v(s))\Big|\mathrm{d}s \to 0,\ n\to\infty.$$

So, as $n\to\infty$,

$$\|T_1(u_n,v_n)-T_1(u,v)\|_X = \max\{\|T_1(u_n,v_n)-T_1(u,v)\|_0, \|T_1(u_n,v_n)-T_1(u,v)\|_1\} \to 0.$$

This means that the operator T_1 is continuous. At the same way, we can show than the operator T_2 is continuous. That is, the operator T is continuous.

In view of the above all arguments, we deduce that the operator $T: P\to P$ is completely continuous. Therefore proof is completed.

3. Main results

For convenience, we set

$$L_i = \frac{1}{\Gamma(\alpha_i)-\Lambda_i}, i=1,2,\ \ L=\max\{L_1,L_2,\Gamma(\alpha_1)L_1,\Gamma(\alpha_2)L_2\}.$$

Define a partial order over the product space:

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \geq \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

if $u_1(t) \geq u_2(t), v_1(t) \geq v_2(t), D^{\alpha_1-1}u_1(t) \geq D^{\alpha_1-1}u_2(t), D^{\alpha_2-1}v_1(t) \geq D^{\alpha_2-1}v_2(t), t \in J$.

Theorem 3.1. Assume that (H_1),(H_2) and (H_4) hold. There exists a positive constant R such that the system (1) have two positive solutions (u^*, v^*) and (w^*, z^*) satisfying $0 \leq \|(u^*, v^*)\|_{X\times Y} \leq R$ and $0 \leq \|(w^*, z^*)\|_{X\times Y} \leq R$. Moreover, $\lim_{n\to\infty}(u_n, v_n) = (u^*, v^*)$ and $\lim_{n\to\infty}(w_n, z_n) = (w^*, z^*)$, (u_n, v_n) and (w_n, z_n) can be given by the following monotone iterative schemes

$$(u_n, v_n) = T(u_{n-1}, v_{n-1}) = \begin{pmatrix} T_1(u_{n-1}, v_{n-1})(t) \\ T_2(u_{n-1}, v_{n-1})(t) \end{pmatrix}, n = 1, 2, \ldots, \ with\ (u_0(t), v_0(t)) = \begin{pmatrix} Rt^{\alpha_1} \\ Rt^{\alpha_2} \end{pmatrix} \tag{20}$$

and

$$(w_n, z_n) = T(w_{n-1}, z_{n-1}) = \begin{pmatrix} T_1(w_{n-1}, z_{n-1})(t) \\ T_2(w_{n-1}, z_{n-1})(t) \end{pmatrix}, n = 1, 2, \ldots, \ with\ (w_0(t), z_0(t)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{21}$$

In addition

$$\begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix} \leq \begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} w^* \\ z^* \end{pmatrix} \leq \cdots \leq \begin{pmatrix} u^* \\ v^* \end{pmatrix} \leq \cdots \leq \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}$$

$$\leq \cdots \leq \begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix} \leq \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix} \leq \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} \tag{22}$$

and

$$\begin{pmatrix} D^{\alpha_1-1}w_0(t) \\ D^{\alpha_2-1}z_0(t) \end{pmatrix} \leq \begin{pmatrix} D^{\alpha_1-1}w_1(t) \\ D^{\alpha_2-1}z_1(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} D^{\alpha_1-1}w_n(t) \\ D^{\alpha_2-1}z_n(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} D^{\alpha_1-1}w^* \\ D^{\alpha_2-1}z^* \end{pmatrix} \leq \cdots \leq \begin{pmatrix} D^{\alpha_1-1}u^* \\ D^{\alpha_2-1}v^* \end{pmatrix}$$

$$\leq \cdots \leq \begin{pmatrix} D^{\alpha_1-1}u_n(t) \\ D^{\alpha_2-1}v_n(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} D^{\alpha_1-1}u_2(t) \\ D^{\alpha_2-1}v_2(t) \end{pmatrix} \leq \begin{pmatrix} D^{\alpha_1-1}u_1(t) \\ D^{\alpha_2-1}v_1(t) \end{pmatrix} \leq \begin{pmatrix} D^{\alpha_1-1}u_0(t) \\ D^{\alpha_2-1}v_0(t) \end{pmatrix}. \tag{23}$$

Proof. First, Lemma 2.7 leads to the fact that $T(P) \subset P$ for any $(u, v) \in P, t \in J$.

Next, for $0 \leq \lambda_{1k}, \lambda_{2k} < 1 (k = 1, 2, 3, 4)$, choose

$$R \geq \max\left\{5a^*_{10}, 5a^*_{20}, (5La^*_{1k})^{1/(1-\lambda_{1k})}, (5La^*_{2k})^{1/(1-\lambda_{2k})}\right\}, k = 1, 2, 3, 4,$$

and define $U_R = \{(u, v) \in P : \|(u, v)\|_{X\times Y} \leq R\}$. For any $(u, v) \in U_R$, similar to (16) and (17), we obtain

$$\|T_1(u, v)\|_0 \leq L_1\Big[a^*_{10} + \sum_{k=1}^{4} a^*_{1k}\|(u, v)\|^{\lambda_{1k}}_{X\times Y}\Big] \leq L\Big[a^*_{10} + \sum_{k=1}^{4} a^*_{1k}R^{\lambda_{1k}}\Big] \leq R$$

and

$$\|T_1(u, v)\|_1 \leq L_1\Big[a^*_{10} + \sum_{k=1}^{4} a^*_{1k}\|(u, v)\|^{\lambda_{1k}}_{X\times Y}\Big] \leq L\Big[a^*_{10} + \sum_{k=1}^{4} a^*_{1k}R^{\lambda_{1k}}\Big] \leq R.$$

This implies that $\|T_1(u, v)\|_X \leq R$ for all $(u, v) \in U_R$. In the same way, $\|T_2(u, v)\|_Y \leq R$. Consequently we have

$$\|T(u, v)\|_{X\times Y} = \Big\{\|T_1(u, v)\|_X, \|T_2(u, v)\|_Y\Big\} \leq R.$$

That is, $T(U_R) \subset U_R$.

According to (20) and (21), it is obvious that $(u_0(t), v_0(t)), (w_0(t), z_0(t)) \in U_R$. By the complete continuity of the operator T, we define the schemes (u_n, v_n) and (w_n, z_n) by $(u_n, v_n) = T(u_{n-1}, v_{n-1}), (w_n, z_n)= T(w_{n-1}, z_{n-1})$ for $n = 1, 2, \ldots$. Since $T(B) \subset B$, we can know that $(u_n, v_n), (w_n, z_n) \in T(B)$ for $n = 1, 2, \ldots$. Hence we need show that there exist (u^*, v^*) and (w^*, z^*) satisfying $\lim_{n\to\infty}(u_n, v_n) = (u^*, v^*)$ and $\lim_{n\to\infty}(w_n, z_n) = (w^*, z^*)$, which are two monotone schemes for positive solutions of the system (1).

For $t \in J$, by Lemma 2.3 and (20), we know

$$\begin{aligned} u_1(t) = T_1(u_0, v_0)(t) = &\int_0^{+\infty} K_1(t,s) f_1(s, u_0(s), v_0(s), D^{\alpha_1-1}u_0(s), D^{\alpha_2-1}v_0(s))\mathrm{d}s \\ \le & t^{\alpha_1-1}L_1\Big[a^*_{10} + \sum_{k=1}^{4} a^*_{1k}R^{\lambda_{1k}}\Big] \\ \le & Rt^{\alpha_1-1} = u_0(t) \end{aligned}$$

and

$$\begin{aligned} v_1(t) = T_2(u_0, v_0)(t) = &\int_0^{+\infty} K_2(t,s) f_2(s, u_0(s), v_0(s), D^{\alpha_1-1}u_0(s), D^{\alpha_2-1}v_0(s))\mathrm{d}s \\ \le & t^{\alpha_2-1}L_2\Big[a^*_{20} + \sum_{k=1}^{4} a^*_{2k}R^{\lambda_{2k}}\Big] \\ \le & Rt^{\alpha_2-1} = v_0(t), \end{aligned}$$

that is

$$T(u,v)(t) = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} T_1(u_0, v_0)(t) \\ T_2(u_0, v_0)(t) \end{pmatrix} \le \begin{pmatrix} Rt^{\alpha_1-1} \\ Rt^{\alpha_2-1} \end{pmatrix} = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix}. \tag{24}$$

And then we study the monotonicity of the fractional derivative of (u, v). By (24) we know

$$\begin{aligned} D^{\alpha_1-1}u_1(t) = & D^{\alpha_1-1}T_1(u_0, v_0)(t) = \int_0^{+\infty} K^*_1(t,s) f_1(s, u_0(s), v_0(s)\ D^{\alpha_1-1}u_0(s), D^{\alpha_2-1}v_0(s))\mathrm{d}s \\ & \le \Gamma(\alpha_1)L_1\Big[a^*_{10} + \sum_{k=1}^{4} a^*_{1k}R^{\lambda_{1k}}\Big] \le \Gamma(\alpha_1)R = D^{\alpha_1-1}u_0(t), \\ D^{\alpha_2-1}v_1(t) = & D^{\alpha_2-1}T_2(u_0, v_0)(t) = \int_0^{+\infty} K^*_2(t,s) f_2(s, u_0(s), v_0(s)\ D^{\alpha_1-1}u_0(s), D^{\alpha_2-1}v_0(s))\mathrm{d}s \\ & \le \Gamma(\alpha_2)L_2\Big[a^*_{20} + \sum_{k=1}^{4} a^*_{2k}R^{\lambda_{1k}}\Big] \le \Gamma(\alpha_2)R = D^{\alpha_2-1}v_0(t), \end{aligned}$$

that is

$$T(u,v)(t) = \begin{pmatrix} D^{\alpha_1-1}u_1(t) \\ D^{\alpha_2-1}v_1(t) \end{pmatrix} = \begin{pmatrix} D^{\alpha_1-1}T_1(u_0, v_0)(t) \\ D^{\alpha_2-1}T_2(u_0, v_0)(t) \end{pmatrix} \le \begin{pmatrix} \Gamma(\alpha_1)R \\ \Gamma(\alpha_2)R \end{pmatrix} = \begin{pmatrix} D^{\alpha_1-1}u_0(t) \\ D^{\alpha_2-1}v_0(t) \end{pmatrix}. \tag{25}$$

Thus, from (24) and (25), for $\forall t \in J$, by the monotonicity hypothesis (H_4) of the functions f_i, we do the second iteration

$$\begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} T_1(u_1, v_1)(t) \\ T_2(u_1, v_1)(t) \end{pmatrix} \le \begin{pmatrix} T_1(u_0, v_0)(t) \\ T_2(u_0, v_0)(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix},$$

$$\begin{pmatrix} D^{\alpha_1-1}u_2(t) \\ D^{\alpha_2-1}v_2(t) \end{pmatrix} = \begin{pmatrix} D^{\alpha_1-1}T_1(u_1, v_1)(t) \\ D^{\alpha_2-1}T_2(u_1, v_1)(t) \end{pmatrix} \le \begin{pmatrix} D^{\alpha_1-1}T_1(u_0, v_0)(t) \\ D^{\alpha_2-1}T_2(u_0, v_0)(t) \end{pmatrix} = \begin{pmatrix} D^{\alpha_1-1}u_1(t) \\ D^{\alpha_2-1}v_1(t) \end{pmatrix}.$$

By recursion, for $t \in J$, the scheme $\{(u_n, v_n)\}_{n=0}^{\infty}$ satisfies

$$\begin{pmatrix} u_{n+1}(t) \\ v_{n+1}(t) \end{pmatrix} \leq \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \quad \begin{pmatrix} D^{\alpha_1-1}u_{n+1}(t) \\ D^{\alpha_2-1}v_{n+1}(t) \end{pmatrix} \leq \begin{pmatrix} D^{\alpha_1-1}u_n(t) \\ D^{\alpha_2-1}v_n(t) \end{pmatrix}.$$

By the aid of the iterative scheme $(u_{n+1}, v_{n+1}) = T(u_n, v_n)$ and the complete continuity of the operator T, it is easy to infer that $(u_n, v_n) \to (u^*, v^*)$ and $T(u^*, v^*) = (u^*, v^*)$.

For the scheme $\{(w_n, z_n)\}_{n=0}^{\infty}$, we use a similar discussion. For $t \in J$, we have

$$\begin{aligned} \begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} = \begin{pmatrix} T_1(w_0, z_0)(t) \\ T_2(w_0, z_0)(t) \end{pmatrix} &= \begin{pmatrix} \int_0^{+\infty} K_1(t,s) f_1(t, w_0(t), z_0(t), D^{\alpha_1-1}w_0(t), D^{\alpha_2-1}z_0(t))ds \\ \int_0^{+\infty} K_2(t,s) f_2(t, w_0(t), z_0(t), D^{\alpha_1-1}w_0(t), D^{\alpha_2-1}z_0(t))ds \end{pmatrix} \\ &\geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} D^{\alpha_1-1}w_1(t) \\ D^{\alpha_2-1}z_1(t) \end{pmatrix} &= \begin{pmatrix} \int_0^{+\infty} K_1^*(t,s) f_1(t, w_0(t), z_0(t), D^{\alpha_1-1}w_0(t), D^{\alpha_2-1}z_0(t))ds \\ \int_0^{+\infty} K_2^*(t,s) f_2(t, w_0(t), z_0(t), D^{\alpha_1-1}w_0(t), D^{\alpha_2-1}z_0(t))ds \end{pmatrix} \\ &\geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} D^{\alpha_1-1}w_0(t) \\ D^{\alpha_2-1}z_0(t) \end{pmatrix}. \end{aligned}$$

Using the the monotonicity hypothesis (H_4) of the functions f_i, we have

$$\begin{pmatrix} w_2(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} T_1(w_1, z_1)(t) \\ T_2(w_1, z_1)(t) \end{pmatrix} \geq \begin{pmatrix} T_1(w_0, z_0)(t) \\ T_2(w_0, z_0)(t) \end{pmatrix} = \begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix},$$

$$\begin{pmatrix} D^{\alpha_1-1}w_2(t) \\ D^{\alpha_2-1}z_2(t) \end{pmatrix} = \begin{pmatrix} D^{\alpha_1-1}T_1(w_1, z_1)(t) \\ D^{\alpha_2-1}T_2(w_1, z_1)(t) \end{pmatrix} \geq \begin{pmatrix} D^{\alpha_1-1}T_1(w_0, z_0)(t) \\ D^{\alpha_2-1}T_2(w_0, z_0)(t) \end{pmatrix} = \begin{pmatrix} D^{\alpha_1-1}w_1(t) \\ D^{\alpha_2-1}z_1(t) \end{pmatrix}.$$

Analogously, for $n = 0, 1, 2, \ldots$ and $t \in J$, we have

$$\begin{pmatrix} w_{n+1}(t) \\ z_{n+1}(t) \end{pmatrix} \geq \begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix}, \quad \begin{pmatrix} D^{\alpha_1-1}w_{n+1}(t) \\ D^{\alpha_2-1}z_{n+1}(t) \end{pmatrix} \geq \begin{pmatrix} D^{\alpha_1-1}w_n(t) \\ D^{\alpha_2-1}z_n(t) \end{pmatrix}.$$

Combining the iterative scheme $(w_{n+1}, z_{n+1}) = T(w_n, z_n)$ and the complete continuity of the operator T, it is easy to infer that $(w_n, z_n) \to (w^*, z^*)$ and $T(w^*, z^*) = (w^*, z^*)$. Finally we show that (u^*, v^*) and (w^*, z^*) are the minimal and maximal positive solutions of the system (1). Suppose that $(\xi(t), \eta(t))$ is any positive solution of the system (1), then $T(\xi(t), \eta(t)) = (\xi(t), \eta(t))$ and

$$\begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} Rt^{\alpha_1-1} \\ Rt^{\alpha_2-1} \end{pmatrix} = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix},$$

$$\begin{pmatrix} D^{\alpha_1-1}w_0(t) \\ D^{\alpha_2-1}z_0(t) \end{pmatrix} \leq \begin{pmatrix} D^{\alpha_1-1}\xi(t) \\ D^{\alpha_2-1}\eta(t) \end{pmatrix} \leq \begin{pmatrix} D^{\alpha_1-1}u_0(t) \\ D^{\alpha_2-1}v_0(t) \end{pmatrix}.$$

Applying the monotone property of the operator T, we know that

$$\begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} = \begin{pmatrix} T_1(w_0, z_0)(t) \\ T_2(w_0, z_0)(t) \end{pmatrix} \leq \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} T_1(u_0, v_0)(t) \\ T_2(u_0, v_0)(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix},$$

$$\begin{pmatrix} D^{\alpha_1-1}w_1(t) \\ D^{\alpha_2-1}z_1(t) \end{pmatrix} \leq \begin{pmatrix} D^{\alpha_1-1}\xi(t) \\ D^{\alpha_2-1}\eta(t) \end{pmatrix} \leq \begin{pmatrix} D^{\alpha_1-1}u_1(t) \\ D^{\alpha_2-1}v_1(t) \end{pmatrix}.$$

Repeating the above steps, we have

$$\begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix} \le \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \le \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}$$

$$\begin{pmatrix} D^{\alpha_1-1}w_n(t) \\ D^{\alpha_2-1}z_n(t) \end{pmatrix} \le \begin{pmatrix} D^{\alpha_1-1}\xi(t) \\ D^{\alpha_2-1}\eta(t) \end{pmatrix} \le \begin{pmatrix} D^{\alpha_1-1}u_n(t) \\ D^{\alpha_2-1}v_n(t) \end{pmatrix},$$

From the above results, combine $\lim_{n\to\infty}(w_n, z_n) = (w^*, z^*)$ and $\lim_{n\to\infty}(u_n, u_n) = (u^*, v^*)$, we get the results (22) and (23).

Again $f(t, 0, 0, 0, 0) \neq 0$ for all $t \in J$, we know that $(0, 0)$ isn't a solution of the system (1). By (22) and (23), it is obvious that (w^*, z^*) and (u^*, v^*) are the extreme positive solutions of system (1), which can be constructed by means of two monotone iterative schemes in (20) and (21).

With regard to the difference scope of parameters $\lambda_{ik}(i = 1, 2, k = 1, 2, 3, 4)$, the method is similar, so we omit the details, thus the proof is completed.

Theorem 3.2. Suppose the hypotheses (H_1) and (H_3) are satisfied. If

$$m = L\max\Big\{\sum_{k=1}^{4} b_{1k}, \sum_{k=1}^{4} b_{2k}\Big\} < 1, \tag{26}$$

then the system (1) has a unique positive solution $(\overline{u}(t), \overline{v}(t))$ in P. Moreover, there is a iterative scheme (u_n, v_n), such that $(u_n, v_n) \to (\overline{u}, \overline{u})$ as $n \to \infty$ uniformly on any finite interval of J, where

$$(u_n, v_n) = T(u_{n-1}, v_{n-1}) = \begin{pmatrix} T_1(u_{n-1}, v_{n-1})(t) \\ T_2(u_{n-1}, v_{n-1})(t) \end{pmatrix}, n = 1, 2, \dots. \tag{27}$$

In addition, there is an error estimate for the approximation scheme.

$$\|(u_n, v_n) - (\overline{u}, \overline{v})\|_{X\times Y} = \frac{m^n}{1-m}\|(u_1, v_1) - (u_0, v_0)\|_{X\times Y}, n = 1, 2, \dots. \tag{28}$$

Proof Choose

$$r \ge L\tau/(1-m),$$

where m is defined by (26) and $\tau = \max\{\tau_1, \tau_2\}, \tau_i$ is defined by the hypothesis (H_3).

First we prove that $TU_r \subset U_r$, where $U_r = \{(u, v) \in P, \|(u, v)\|_{X\times Y} \le r\}$. For any $(u, v) \in U_r$, by Lemma 2.3, Remark 2.2 and Lemma 2.5, we have

$$\|T_1(u, v)\|_0 \le L\Big(\sum_{k=1}^{4} b^*_{1k} r + \tau_1\Big)$$

and

$$\|T_1(u, v)\|_1 \le L\Big(\sum_{k=1}^{4} b^*_{1k} r + \tau_1\Big),$$

which implies

$$\|T_1(u, v)\|_X \le L\Big(\sum_{k=1}^{4} b^*_{1k} r + \tau_i\Big) \le mr + L\tau_1, \ \forall (u, v) \in U_r.$$

Similar

$$\|T_2(u, v)\|_Y \le L\Big(\sum_{k=1}^{4} b^*_{2k} r + \tau_2\Big) \le mr + L\tau_2, \ \forall (u, v) \in U_r.$$

So we have

$$\|T(u, v)\|_{X\times Y} \le mr + L\tau \le r. \ \forall (u, v) \in U_r.$$

Now we show that T is a contraction. For any $(u_1, v_1), (u_2, v_2) \in U_r$, by hypothesis (H$_3$), we obtain

$$\begin{aligned}
&\|T_1(u_1,v_1) - T_1(u_2,v_2)\|_0\\
\le& \sup_{t\in J}\int_0^{+\infty}\frac{K_1(t,s)}{1+t^{\alpha_1-1}}\Big|f_1(s,u_1(s),v_1(s),D^{\alpha_1-1}u_1(s),D^{\alpha_2-1}v_1(s))\\
&- f_1(s,u_2(s),v_2(s),D^{\alpha_1-1}u_2(s),D^{\alpha_2-1}v_2(s))\Big|ds\\
\le& L\int_0^{+\infty}\Big[b_{11}(s)(1+s^{\alpha_1-1})\frac{|u_1(s)-u_2(s)|}{1+s^{\alpha_1-1}} + b_{12}(s)(1+s^{\alpha_2-1})\frac{|v_1(s)-v_2(s)|}{1+s^{\alpha_2-1}}\\
&+ b_{13}(s)|D^{\alpha_1-1}u_1(s) - D^{\alpha_1-1}u_2(s)|\Big]ds + b_{14}(s)|D^{\alpha_2-1}v_1(s) - D^{\alpha_2-1}v_2(s)|\Big]ds\\
\le& L\sum_{k=1}^{4} b_{1k}^{*}\|(u_1,v_1)-(u_2,v_2)\|_{X\times Y}
\end{aligned}$$

and

$$\begin{aligned}
\|T_1(u_1,v_1) - T_1(u_2,v_2)\|_1 \le& \sup_{t\in J}\int_0^{+\infty}K_1^{*}(t,s)\Big|f_1(s,u_1(s),v_1(s),D^{\alpha_1-1}u_1(s),D^{\alpha_2-1}v_1(s))\\
&- f_1(s,u_2(s),v_2(s),D^{\alpha_1-1}u_2(s),D^{\alpha_2-1}v_2(s))\Big|ds\\
\le& L\sum_{k=1}^{4} b_{1k}^{*}\|(u_1,v_1)-(u_2,v_2)\|_{X\times Y},
\end{aligned}$$

which implies

$$\|T_1(u_1,v_1) - T_1(u_2,v_2)\|_X \le L\sum_{k=1}^{4} b_{1k}^{*}\|(u_1,v_1)-(u_2,v_2)\|_{X\times Y}. \tag{29}$$

In the same way, we have

$$\|T_2(u_1,v_1) - T_2(u_2,v_2)\|_Y \le L\sum_{k=2}^{4} b_{2k}^{*}\|(u_1,v_1)-(u_2,v_2)\|_{X\times Y}. \tag{30}$$

From (29) and (30), we have

$$\|T(u_1,v_1) - T(u_2,v_2)\|_{X\times Y} \le m\|(u_1,v_1)-(u_2,v_2)\|_{X\times Y}, , \forall (u_1,v_1),(u_2,v_2)\in U_r. \tag{31}$$

Since $m < 1$, then T is a contraction. Hence the Banach fixed-point theorem ensures that T has a unique fixed point $(\overline{u}, \overline{v})$ in P. That is, the system (1) has a unique positive solution $(\overline{u}, \overline{v})$.

Furthermore, for any $(u_0, v_0) \in P, \|(u_n, v_n) - (\overline{u}, \overline{v})\|_{X\times Y} \to 0$ as $n \to \infty$, where $u_n = T_1(u_{n-1}, v_{n-1}), v_n = T_2(u_{n-1}, V_{n-1}), n = 1, 2, \ldots$. By (31), we obtain

$$\|(u_n,v_n) - (u_{n-1},v_{n-1})\|_{X\times Y} \le m^{n-1}\|(u_1,v_1)-(u_0,v_0)\|_{X\times Y},$$

and

$$\begin{aligned}
\|(u_n,v_n) - (u_j,v_j)\|_{X\times Y} \le& \|(u_n,v_n) - (u_{n-1},v_{n-1})\|_{X\times Y} + \|(u_{n-1},v_{n-1}) - (u_{n-2},v_{n-2})\|_{X\times Y}\\
&+ \cdots + \|(u_{j+1},v_{j+1}) - (u_j,v_j)\|_{X\times Y}\\
\le& \frac{m^n(1-m^{j-n})}{1-m}\|(u_1,v_1)-(u_0,v_0)\|_{X\times Y}.
\end{aligned} \tag{32}$$

Letting $j \to +\infty$ on both sides of (32), we have

$$\|(u_n, v_n) - (\overline{u}, \overline{v})\|_{X\times Y} \leq \frac{m^n}{1-m}\|u_1 - u_0\|_{X\times Y}.$$

Hence the proof of theorem 3.2 is completed.

Now we give two examples to illustrate the application of the main results.

Example 3.1. Consider the following fractional differential system on an infinite interval

$$\begin{cases} -D^{2.5}u(t) = \dfrac{2}{(10+t)^2} + \dfrac{e^{-t}|u(t)|^{0.1}}{(1+\sqrt{t^3})^{0.1}} + \dfrac{e^{-2t}|v(t)|^{0.3}}{(1+\sqrt{t})^{0.3}} + \dfrac{2t|D^{1.5}u(t)|^{0.2}}{(3+t^2)^2} + \dfrac{|D^{0.5}v(t)|^{0.4}}{1+t^2}, \\ -D^{1.5}v(t) = \dfrac{1}{(20+t)^3} + \dfrac{e^{-3t}|u(t)|^{0.2}}{(1+\sqrt{t^3})^{0.2}} + \dfrac{e^{-4t}|v(t)|^{0.4}}{(1+\sqrt{t})^{0.4}} + \dfrac{3t^2|D^{1.5}u(t)|^{0.2}}{(3+t^3)^2} + \dfrac{2|D^{0.5}v(t)|^{0.6}}{1+t^2}, \\ u(0) = u'(0) = 0,\ D^{1.5}u(+\infty) = \displaystyle\int_0^{+\infty} t^{-1.5}e^{-t}u(t)dt, \\ u(0) = 0,\ D^{0.5}v(+\infty) = \displaystyle\int_0^{+\infty} t^{-0.5}e^{-2t}v(t)dt, \end{cases} \tag{33}$$

where $\alpha_1 = 2.5, \alpha_1 = 1.5, h_1(t) = t^{-1.5}e^{-t}, h_2(t) = t^{-0.5}e^{-2t}, \lambda_{11} = 0.1, \lambda_{12} = 0.3, \lambda_{13} = 0.2, \lambda_{14} = 0.4, \lambda_{21} = 0.2, \lambda_{22} = 0.4, \lambda_{23} = 0.2, \lambda_{24} = 0.6$ and

$$f_1(t, u_1, u_2, u_3, u_4) = \frac{2}{(10+t)^2} + \frac{e^{-t}|u_1|^{0.1}}{(1+\sqrt{t^3})^{0.1}} + \frac{e^{-2t}|u_2|^{0.3}}{(1+\sqrt{t})^{0.3}} + \frac{2t|u_3|^{0.2}}{(3+t^2)^2} + \frac{|u_4|^{0.4}}{1+t^2},$$

$$f_2(t, u_1, u_2, u_3, u_4) = \frac{1}{(20+t)^3} + \frac{e^{-3t}|u_1|^{0.2}}{(1+\sqrt{t^3})^{0.2}} + \frac{e^{-4t}|u_2|^{0.4}}{(1+\sqrt{t})^{0.4}} + \frac{3t^2|u_3|^{0.2}}{(3+t^3)^2} + \frac{2|u_4)|^{0.6}}{1+t^2},$$

It is easy to know that $\Gamma(2.5) = 1.32934 > \Lambda_1 = \int_0^{+\infty} h_1(t)t^{1.5}dt = 1$, $\Gamma(1.5) = 0.88623 > \Lambda_2 = \int_0^{+\infty} h_2(t)t^{0.5}dt = 0.5, f_i(t, 0, 0, 0, 0) \not\equiv 0, i = 1, 2$. So the hypothesis ($H_1$) is satisfied.

Noting that

$$\begin{aligned} |f_1(t, u_1, u_2, u_3, u_4)| &\leq \frac{2}{(10+t)^2} + \frac{e^{-t}|u_1|^{0.1}}{(1+\sqrt{t^3})^{0.1}} + \frac{e^{-2t}|u_2|^{0.3}}{(1+\sqrt{t})^{0.3}} + \frac{2t|u_3|^{0.2}}{(3+t^2)^2} + \frac{|u_4|^{0.4}}{1+t^2} \\ &= a_{10}(t) + a_{11}(t)|u_1|^{0.1} + a_{12}(t)|u_2|^{0.3} + a_{13}(t)|u_3|^{0.2} + a_{14}(t)|u_4|^{0.4}, \end{aligned}$$

$$\begin{aligned} |f_2(t, u_1, u_2, u_3, u_4)| &\leq \frac{1}{(20+t)^3} + \frac{e^{-3t}|u_1|^{0.2}}{(1+\sqrt{t^3})^{0.2}} + \frac{e^{-4t}|u_2|^{0.4}}{(1+\sqrt{t})^{0.4}} + \frac{3t^2|u_3|^{0.2}}{(3+t^3)^2} + \frac{2|u_4)|^{0.6}}{1+t^2} \\ &= a_{20}(t) + a_{21}(t)|u_1|^{0.2} + a_{22}(t)|u_2|^{0.2} + a_{23}(t)|u_3|^{0.2} + a_{24}(t)|u_4|^{0.6} \end{aligned}$$

and

$$a_{10}^* = \int_0^{+\infty} a_{10}(t)dt = \frac{1}{5},\ a_{11}^* = \int_0^{+\infty} a_{11}(t)(1+t^{1.5})^{0.1}dt = 1,\ a_{12}^* = \int_0^{+\infty} a_{12}(t)(1+t^{0.5})^{0.3}dt = \frac{1}{2},$$

$$a_{13}^* = \int_0^{+\infty} a_{13}(t)dt = \frac{1}{3},\ a_{14}^* = \int_0^{+\infty} a_{14}(t)dt = \frac{\pi}{2},$$

$$a_{20}^* = \int_0^{+\infty} a_{10}(t)dt = \frac{1}{800},\ a_{21}^* = \int_0^{+\infty} a_{21}(t)(1+t^{1.5})^{0.2}dt = \frac{1}{3},\ a_{22}^* = \int_0^{+\infty} a_{22}(t)(1+t^{0.5})^{0.4}dt = \frac{1}{4},$$

$$a_{23}^* = \int_0^{+\infty} a_{23}(t)dt = \frac{1}{3},\ a_{24}^* = \int_0^{+\infty} a_{24}(t)dt = \pi.$$

which means that the hypothesis (H_2) is satisfied.

From the expression of the function f_i, it is obvious that f_i is increasing respect to the variables $u_1, u_2, u_3, u_4, \forall t \in J, i = 1, 2$. Thus the hypothesis ($H_4$) is satisfied. By Theorem 3.1, it follows that the system (33) have two positive solution, which can be given by the limits means of two explicit monotone iterative scheme in (20) and (21).

Example 3.2. Consider the following fractional differential system an infinite interval

$$\begin{cases} -D^{2.5}u(t) = \dfrac{2}{(10+t)^2} + \dfrac{e^{-20t}|u(t)|}{1+\sqrt{t^3}} + \dfrac{e^{-15t}|v(t)|}{1+\sqrt{t}} + \dfrac{t|D^{1.5}u(t)|}{5(3+t^2)^2} + \dfrac{t|D^{0.5}v(t)|}{10(1+t^2)^2}, \\ -D^{1.5}v(t) = \dfrac{1}{(20+t)^3} + \dfrac{e^{-18t}|u(t)|}{1+\sqrt{t^3}} + \dfrac{e^{-16t}|v(t)|}{1+\sqrt{t}} + \dfrac{3t^2|D^{1.5}u(t)|}{7(3+t^3)^2} + \dfrac{|D^{0.5}v(t)|}{20(1+t^2)^2}, \\ u(0) = u'(0) = 0,\ D^{1.5}u(+\infty) = \displaystyle\int_0^{+\infty} t^{-1.5}e^{-t}u(t)dt, \\ u(0) = 0,\ D^{0.5}v(+\infty) = \displaystyle\int_0^{+\infty} t^{-0.5}e^{-2t}v(t)dt, \end{cases} \tag{34}$$

where $\alpha_1 = 2.5, \alpha_1 = 1.5, h_1(t) = t^{-1.5}e^{-t}, h_2(t) = t^{-0.5}e^{-2t}$ and

$$f_1(t, u_1, u_2, u_3, u_4) = \frac{2}{(10+t)^2} + \frac{e^{-20t}|u_1|}{1+\sqrt{t^3}} + \frac{e^{-15t}|u_2|}{1+\sqrt{t}} + \frac{t|u_3|}{5(3+t^2)^2} + \frac{t|u_4|}{10(1+t^2)^2},$$

$$f_2(t, u_1, u_2, u_3, u_4) = \frac{1}{(20+t)^3} + \frac{e^{-18t}|u_1|}{1+\sqrt{t^3}} + \frac{e^{-16t}|u_2|}{1+\sqrt{t}} + \frac{3t^2|u_3|}{7(3+t^3)^2} + \frac{|u_4)|}{20(1+t^2)},$$

Similar to the example 3.1, it is easy to verify that the hypothesis (H_1) is satisfied.

Observing that

$$\begin{aligned} & |f_1(t, u_1, u_2, u_3, u_4) - f_1(t, \overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4)| \\ \le\ & \frac{e^{-20t}}{1+\sqrt{t^3}}|u_1 - \overline{u}_1| + \frac{e^{-15t}|}{1+\sqrt{t}}|u_2 - \overline{u}_2| + \frac{t}{5(3+t^2)^2}|u_3 - \overline{u}_3| + \frac{t}{10(1+t^2)^2}|u_4 - \overline{u}_4| \\ =\ & b_{11}(t)|u_1 - \overline{u}_1| + b_{12}(t)|u_2 - \overline{u}_2| + b_{13}(t)|u_3 - \overline{u}_3| + b_{14}(t)|u_4 - \overline{u}_4|, \end{aligned}$$

$$\begin{aligned} & |f_2(t, u_1, u_2, u_3, u_4) - f_2(t, \overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4)| \\ \le\ & \frac{e^{-18t}}{1+\sqrt{t^3}}|u_1 - \overline{u}_1| + \frac{e^{-16t}|}{1+\sqrt{t}}|u_2 - \overline{u}_2| + \frac{3t^2}{7(3+t^3)^2}|u_3 - \overline{u}_3| + \frac{1}{20(1+t^2)}|u_4 - \overline{u}_4| \\ =\ & b_{21}(t)|u_1 - \overline{u}_1| + b_{22}(t)|u_2 - \overline{u}_2| + b_{23}(t)|u_3 - \overline{u}_3| + b_{24}(t)|u_4 - \overline{u}_4|, \end{aligned}$$

and

$$b^*_{11} = \int_0^{+\infty} b_{11}(t)(1+t^{1.5})dt = \frac{1}{20},\ b^*_{12} = \int_0^{+\infty} b_{12}(t)(1+t^{0.5})dt = \frac{1}{15},$$

$$b^*_{13} = \int_0^{+\infty} b_{13}(t)dt = \frac{1}{30},\ b^*_{14} = \int_0^{+\infty} b_{14}(t)dt = \frac{1}{20},$$

$$b^*_{21} = \int_0^{+\infty} b_{21}(t)(1+t^{1.5})dt = \frac{1}{18},\ b^*_{22} = \int_0^{+\infty} b_{22}(t)(1+t^{0.5})dt = \frac{1}{16},$$

$$b^*_{23} = \int_0^{+\infty} b_{23}(t)dt = \frac{1}{21},\ b^*_{24} = \int_0^{+\infty} a_{24}(t)dt = \frac{\pi}{40}.$$

$$\lambda_1 = \int_0^{+\infty} f_1(t, 0, 0, 0, 0)dt = \int_0^{+\infty} \frac{2}{(10+t)^2}dt = \frac{1}{5},$$

$$\lambda_2 = \int_0^{+\infty} f_2(t, 0, 0, 0, 0)dt = \int_0^{+\infty} \frac{1}{(20+t)^3}dt = \frac{\pi}{8000},$$

which means that the hypothesis (H_3) is satisfied. By direct computation, we have

$$m = L \max\Big\{\sum_{k=1}^{4} b_{1k}, \sum_{k=1}^{4} b_{2k}\Big\} = 4.03638 \times \max\Big\{0.2, 0.24422\Big\} = 0.98576 < 1.$$

So all conditions of Theorem 3.2 are satisfied. Then the system (34) has a unique positive solution, which can be obtained by the limits from the iterative sequences in (27).

4. Conclusions

In this paper, we apply the monotone iterative technique and the Banach contraction mapping principle to study a class of fractional differential system with integral boundary in an infinite interval. We first transform the system (1) into an equivalent operator equation (14), and then construct some norm inequalities related to nonlinear terms $f_i (i = 1, 2)$ by means of hypothesis conditions. Finally some explicit monotone iterative schemes for approximating the extreme positive solutions and the unique positive solution are established.

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