



On Asymptotically λ – Statistical Equivalent Sequences of Order α in Probability

Kübra Elif Akbaş^a, Mahmut Işık^b

^aFaculty of Medicine; Firat University; 23119 Elazığ; TURKEY

^bFaculty of Education; Harran University; Osmanbey Campus; 63190 Şanlıurfa; TURKEY

Abstract. In this study, we introduce and examine the concepts of asymptotically λ –statistical equivalent sequences of order α in probability and strong asymptotically λ –equivalent sequences of order α in probability. We give some relations connected to these concepts.

1. Introduction and Background

The idea of statistical convergence was introduced by Steinhaus [27] and Fast [11] and later reintroduced by Schoenberg [24] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Cinar *et al.* [6], Connor [3], Colak [4], Colak and Bektas [5], Et *et al.* ([8],[9],[10]), Fridy [12], Gadjiev and Orhan [13], Ghosal *et al.* ([7],[14],[15]), Isik *et al.* ([16],[17],[18]), Mursaleen [20], Sengul and Et [25], Sengul [26] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone–Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

The idea of statistical convergence depends upon the density of subsets of the set \mathbb{N} . The density of a subset E of \mathbb{N} is defined by

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_E(k) \text{ provided the limit exists,}$$

where χ_E is the characteristic function of E . It is clear that any finite subset of \mathbb{N} has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

A sequence $x = (x_n)$ of real numbers is said to be statistically convergent to a real number L if for each $\varepsilon > 0$, the set $K = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ has natural density of zero and in this case we write $x_n \xrightarrow{s} L$.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by $t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$, where $I_n = [n - \lambda_n + 1, n]$

2010 Mathematics Subject Classification. 40C05, 47N30, 60B10, 46A45

Keywords. Statistical convergence of order α in probability, Cesaro summability of order α in probability, asymptotic statistical equivalence

Received: 09 January 2020 Revised: 14 February 2020; Accepted: 18 March 2020

Communicated by Eberhard Malkowsky

Email addresses: elifet41@gmail.com (Kübra Elif Akbaş), misik63@yahoo.com (Mahmut Işık)

for $n = 1, 2, \dots$. A sequence $x = (x_k)$ is said to be (V, λ) -summable to a number L if $t_n(x) \rightarrow L$ as $n \rightarrow \infty$. If $\lambda_n = n$, then (V, λ) -summability is reduced to Cesàro summability. By Λ we denote the class of all non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$.

Marouf [19] introduced definitions for asymptotically equivalent sequences and asymptotic regular matrices. Patterson [21] extended these concepts by presenting an asymptotically statistically equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Later on asymptotically equivalent sequences have been studied in ([1], [2],[22],[23]).

In this paper we introduce and examine the concepts of asymptotically λ -statistical equivalent sequences of order α in probability and strong asymptotically λ -equivalent sequences of order α in probability.

Let X_n ($n \in \mathbb{N}$) be a random variable which is defined on a given event space S with respect to a given class of events Δ and a probability function $P : \Delta \rightarrow \mathbb{R}$, then we say that $X_1, X_2, X_3, \dots, X_n \dots$ is a sequence of random variables. A sequence of random variables is denoted by $\{X_n\}_{n \in \mathbb{N}}$.

A sequence of random variables $\{X_n\}$ is said to be bounded in probability, if for every $\delta > 0$ there exists $M > 0$ such that

$$P(|X_n| > M) < \delta, \text{ for all } n \in \mathbb{N}$$

that is ;

$$\lim_{n \rightarrow \infty} P(|X_n| > M) = 0.$$

Definition 1.1. Let (S, Δ, P) be a probability space, $\lambda = (\lambda_n)$ be a sequence as above and α be a positive real number such that $0 < \alpha \leq 1$. Two nonnegative sequences of random variables $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are said to be asymptotically λ -statistical equivalent of order α in probability provided that for every $\varepsilon, \delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \delta \right\} \right| = 0.$$

In this case we write $X \overset{PS^\alpha}{\sim}_\lambda Y$. In case of $\alpha = 1$ we write $X \overset{PS^1}{\sim}_\lambda Y$ instead of $X \overset{PS^\alpha}{\sim}_\lambda Y$ and in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ we write $X \overset{PS^\alpha}{\sim} Y$ instead of $X \overset{PS^\alpha}{\sim}_\lambda Y$ and also we write $X \overset{PS}{\sim} Y$ instead of $X \overset{PS^\alpha}{\sim}_\lambda Y$, if $\alpha = 1$ and $\lambda_n = n$ for all $n \in \mathbb{N}$.

Definition 1.2. Let (S, Δ, P) be a probability space, $\lambda = (\lambda_n)$ be a sequence as above and α be a positive real number such that $0 < \alpha \leq 1$. Two nonnegative sequences of random variables $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are said to be strong asymptotically λ -equivalent of order α in probability provided that for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) = 0.$$

In this case we write $X \overset{PN^\alpha}{\sim}_\lambda Y$. In case of $\alpha = 1$ we write $X \overset{PN^1}{\sim}_\lambda Y$ instead of $X \overset{PN^\alpha}{\sim}_\lambda Y$ and in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ we write $X \overset{PN^\alpha}{\sim} Y$ instead of $X \overset{PN^\alpha}{\sim}_\lambda Y$ and also we write $X \overset{PN}{\sim} Y$ instead of $X \overset{PN^\alpha}{\sim}_\lambda Y$, if $\alpha = 1$ and $\lambda_n = n$ for all $n \in \mathbb{N}$.

2. Main Results

In this section we give the main results of this study.

The proof of the following theorem is obtained by using the standard techniques, therefore we give it unproven.

Theorem 2.1. Let the sequence $\lambda = (\lambda_n)$ be as above and α, β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then

- i) $X \overset{PS^\alpha}{\sim}_\lambda Y$ implies $X \overset{PS^\beta}{\sim}_\lambda Y$,
- ii) $X \overset{PN^\alpha}{\sim}_\lambda Y$ implies $X \overset{PN^\beta}{\sim}_\lambda Y$,
- iii) $X \overset{PN^\alpha}{\sim}_\lambda Y$ implies $X \overset{PS^\beta}{\sim}_\lambda Y$.

Theorem 2.2. Let the sequence $\lambda = (\lambda_n)$ be as above and α be a fixed real number such that $0 < \alpha \leq 1$, if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n^\alpha}{n^\alpha} > 0 \tag{1}$$

then $X \overset{PS^\alpha}{\sim} Y$ implies $X \overset{PS^\alpha_\lambda}{\sim} Y$.

Proof. For given $\varepsilon, \delta > 0$ we have

$$\left\{ k \leq n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \supset \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\}.$$

Therefore

$$\begin{aligned} \frac{1}{n^\alpha} \left| \left\{ k \leq n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right| &\geq \frac{1}{n^\alpha} \left| \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right| \\ &= \frac{\lambda_n^\alpha}{n^\alpha} \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right|. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and using (1), we get $X \overset{PS^\alpha}{\sim} Y$. \square

Corollary 2.3. Let the sequence $\lambda = (\lambda_n)$ be as above, if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} > 0$$

then $X \overset{PS}{\sim} Y$ implies $X \overset{PS_\lambda}{\sim} Y$.

Theorem 2.4. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. Then each of the following assertions holds true;

(i) If

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} > 0 \tag{2}$$

then $X \overset{PS^\beta_\mu}{\sim} Y$ implies $X \overset{PS^\alpha_\lambda}{\sim} Y$.

(ii) If

$$\lim_{n \rightarrow \infty} \frac{\mu_n}{\lambda_n^\beta} = 1 \tag{3}$$

then $X \overset{PS^\alpha_\lambda}{\sim} Y$ implies $X \overset{PS^\beta_\mu}{\sim} Y$.

Proof. (i) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let (2) be satisfied. For given $\varepsilon, \delta > 0$ we have

$$\left\{ k \in J_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \supseteq \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\}$$

where $I_n = [n - \lambda_n + 1, n]$ and $J_n = [n - \mu_n + 1, n]$. Therefore we can write

$$\frac{1}{\mu_n^\beta} \left| \left\{ k \in J_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right| \geq \frac{\lambda_n^\alpha}{\mu_n^\beta} \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right|$$

and this gives the proof.

(ii) Let $X \overset{PS^\alpha}{\sim} Y$ and (3) be satisfied. Since $I_n \subset J_n$ for $\varepsilon, \delta > 0$ we may write

$$\begin{aligned} & \frac{1}{\mu_n^\beta} \left| \left\{ k \in J_n : P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \delta \right\} \right| \\ &= \frac{1}{\mu_n^\beta} \left| \left\{ n - \mu_n + 1 < k \leq n - \lambda_n : P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \delta \right\} \right| \\ & \quad + \frac{1}{\mu_n^\beta} \left| \left\{ k \in I_n : P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \delta \right\} \right| \\ &\leq \frac{\mu_n - \lambda_n}{\mu_n^\beta} + \frac{1}{\lambda_n^\beta} \left| \left\{ k \in I_n : P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \delta \right\} \right| \\ &\leq \left(\frac{\mu_n - \lambda_n}{\lambda_n^\beta} \right) + \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \delta \right\} \right| \\ &\leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1 \right) + \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \delta \right\} \right| \end{aligned}$$

for all $n \in \mathbb{N}$. This implies that the condition $X \overset{PS^\alpha}{\sim} Y$ implies $X \overset{PS^\beta}{\sim} Y$. \square

From Theorem 2.4 we obtain the following results.

Corollary 2.5. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$.

If (2) holds, then

- (i) $X \overset{PS^\mu}{\sim} Y$ implies $X \overset{PS^\alpha}{\sim} Y$ for each $\alpha \in (0, 1]$,
- (ii) $X \overset{PS^\mu}{\sim} Y$ implies $X \overset{PS^\alpha}{\sim} Y$ for each $\alpha \in (0, 1]$,
- (iii) $X \overset{PS^\mu}{\sim} Y$ implies $X \overset{PS^\lambda}{\sim} Y$.

If (3) holds, then

- (i) $X \overset{PS^\alpha}{\sim} Y$ implies $X \overset{PS^\mu}{\sim} Y$ for each $\alpha \in (0, 1]$,
- (ii) $X \overset{PS^\alpha}{\sim} Y$ implies $X \overset{PS^\mu}{\sim} Y$ for each $\alpha \in (0, 1]$,
- (iii) $X \overset{PS^\lambda}{\sim} Y$ implies $X \overset{PS^\mu}{\sim} Y$.

Theorem 2.6. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then we have

- (i) Let the condition in (2) hold, then $X \overset{PN^\beta}{\sim} Y$ implies $X \overset{PN^\alpha}{\sim} Y$,
- (ii) Let the condition in (3) hold and $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ be two bounded sequences of random variables, then $X \overset{PN^\alpha}{\sim} Y$ implies $X \overset{PN^\beta}{\sim} Y$.

Proof. (i) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let (2) be satisfied. The proof follows from the following inequality:

$$\frac{1}{\mu_n^\beta} \sum_{k \in J_n} P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right) \geq \frac{\lambda_n^\alpha}{\mu_n^\beta \lambda_n^\alpha} \sum_{k \in I_n} P \left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon \right).$$

(ii) Let $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ be two bounded sequences of random variables, $X \overset{PN^\alpha}{\sim} Y$ and suppose that (3) holds. Since $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are bounded, we can find $M_1, M_2 > 0$ such that $\lim_{n \rightarrow \infty} P(|X_n| > M_1) = 0$,

$\lim_{n \rightarrow \infty} P(|Y_n| > M_2) = 0$ and $\left| \frac{X_n}{Y_n} \right| \leq \frac{M_1}{M_2} < M$ for all $n \in \mathbb{N}$. Since $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$, we may write

$$\begin{aligned} \frac{1}{\mu_n^\beta} \sum_{k \in J_n} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) &= \frac{1}{\mu_n^\beta} \sum_{k \in J_n - I_n} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) \\ &\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^\beta}\right) M + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) \\ &\leq \left(\frac{\mu_n - \lambda_n^\beta}{\lambda_n^\beta}\right) M + \frac{1}{\lambda_n^\beta} \sum_{k \in I_n} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) \\ &\leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1\right) M + \frac{1}{\lambda_n^\beta} \sum_{k \in I_n} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore $X \overset{PN_\mu^\beta}{\sim} Y$. \square

Theorem 2.6 yields the following results.

Corollary 2.7. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$.

If (2) holds, then

- (i) $X \overset{PN_\mu^\alpha}{\sim} Y$ implies $X \overset{PN_\lambda^\alpha}{\sim} Y$ for each $\alpha \in (0, 1]$,
- (ii) $X \overset{PN_\mu}{\sim} Y$ implies $X \overset{PN_\lambda^\alpha}{\sim} Y$ for each $\alpha \in (0, 1]$,
- (iii) $X \overset{PN_\mu}{\sim} Y$ implies $X \overset{PN_\lambda}{\sim} Y$.

Let the condition in (3) hold and $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ be two bounded sequences of random variables, then

- (i) $X \overset{PN_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu^\alpha}{\sim} Y$ for each $\alpha \in (0, 1]$,
- (ii) $X \overset{PN_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu}{\sim} Y$ for each $\alpha \in (0, 1]$,
- (iii) $X \overset{PN_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu}{\sim} Y$.

Theorem 2.8. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$.

(i) Let the condition in (2) hold, then $X \overset{PN_\mu^\beta}{\sim} Y$ implies $X \overset{PS_\lambda^\alpha}{\sim} Y$.

(ii) Let the condition in (3) hold and $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ be two bounded sequences of random variables then $X \overset{PS_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu^\beta}{\sim} Y$.

Proof. (i) For any sequences $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ of random variables and $\varepsilon, \delta > 0$, we have

$$\begin{aligned} &\sum_{k \in J_n} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) \\ &= \sum_{\substack{k \in J_n \\ P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) \geq \delta}} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) + \sum_{\substack{k \in J_n \\ P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) < \delta}} P\left(\left| \frac{X_k}{Y_k} - L \right| \geq \varepsilon\right) \end{aligned}$$

$$\begin{aligned}
 &\geq \sum_{\substack{k \in I_n \\ P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta}} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) + \sum_{\substack{k \in I_n \\ P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) < \delta}} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \\
 &\geq \sum_{\substack{k \in I_n \\ P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta}} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \\
 &\geq \left| \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right| \delta
 \end{aligned}$$

and so that

$$\begin{aligned}
 \frac{1}{\mu_n^\beta} \sum_{k \in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) &\geq \frac{1}{\mu_n^\beta} \left| \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right| \delta \\
 &\geq \frac{\lambda_n^\alpha}{\mu_n^\beta} \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right| \delta.
 \end{aligned}$$

Therefore $X \stackrel{PS^\alpha}{\sim} Y$.

(ii) Let the condition in (3) hold, $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ be two bounded sequences of random variables and suppose that $X \stackrel{PS^\alpha}{\sim} Y$. We can select a number $M > 0$ such as in the proof of Theorem 2.6 (ii). Then for every $\varepsilon, \delta > 0$ we may write

$$\begin{aligned}
 \frac{1}{\mu_n^\beta} \sum_{k \in J_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) &= \frac{1}{\mu_n^\beta} \sum_{k \in J_n - I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \\
 &\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^\beta}\right) M + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \\
 &\leq \left(\frac{\mu_n - \lambda_n^\beta}{\mu_n^\beta}\right) M + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \\
 &= \left(\frac{\mu_n}{\lambda_n^\beta} - 1\right) M + \frac{1}{\mu_n^\beta} \sum_{\substack{k \in I_n \\ P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta}} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \\
 &\quad + \frac{1}{\mu_n^\beta} \sum_{\substack{k \in I_n \\ P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) < \delta}} P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \\
 &\leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1\right) M + \frac{M}{\mu_n^\beta} \left| \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right| + \frac{\mu_n}{\lambda_n^\beta} \delta \\
 &\leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1\right) M + \frac{M}{\lambda_n^\alpha} \left| \left\{ k \in I_n : P\left(\left|\frac{X_k}{Y_k} - L\right| \geq \varepsilon\right) \geq \delta \right\} \right| + \frac{\mu_n}{\lambda_n^\beta} \delta
 \end{aligned}$$

for all $n \in \mathbb{N}$. Using (3) we obtain that $X \stackrel{PN^\beta}{\sim} Y$. \square

The following results easily derive from Theorem 2.8.

Corollary 2.9. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$.

Let the condition in (2) hold, then

(i) $X \overset{PN_\mu^\alpha}{\sim} Y$ implies $X \overset{PS_\lambda^\alpha}{\sim} Y$ for each $\alpha \in (0, 1]$,

(ii) $X \overset{PN_\mu}{\sim} Y$ implies $X \overset{PS_\lambda^\alpha}{\sim} Y$ for each $\alpha \in (0, 1]$,

(iii) $X \overset{PN_\mu}{\sim} Y$ implies $X \overset{PS_\lambda}{\sim} Y$.

Let the condition in (3) hold and $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ be two bounded sequences of random variables, then

(i) $X \overset{PS_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu^\alpha}{\sim} Y$ for each $\alpha \in (0, 1]$,

(ii) $X \overset{PS_\lambda^\alpha}{\sim} Y$ implies $X \overset{PN_\mu}{\sim} Y$ for each $\alpha \in (0, 1]$,

(iii) $X \overset{PS_\lambda}{\sim} Y$ implies $X \overset{PN_\mu}{\sim} Y$.

References

- [1] M. Basarir, S. Altundag, On Δ -lacunary statistical asymptotically equivalent sequences, *Filomat* 22(1) (2008) 161–172.
- [2] N. L. Braha, On asymptotically Δ_λ^m -lacunary statistical equivalent sequences, *Appl. Math. Comput.* 219(1) (2012) 280–288.
- [3] J. S. Connor, The statistical and strong p -Cesaro convergence of sequences, *Analysis* 8 (1988) 47–63.
- [4] R. Colak, Statistical convergence of order α , *Modern Methods in Analysis and Its Applications*, New Delhi, India: Anamaya Pub. 2010 (2010) 121–129.
- [5] R. Colak, C. A. Bektas, λ -Statistical convergence of order α , *Acta Math. Sin. Engl. Ser.* 31(3) (2011) 953–959.
- [6] M. Cinar, M. Karakas, M. Et, On pointwise and uniform statistical convergence of order α for sequences of functions, *Fixed Point Theory Appl.* 2013(33) (2013) 11 pp.
- [7] P. Das, S. Ghosal, S. Som, Statistical convergence of order α in probability, *Arab J. Math. Sci.* 21(2) (2015) 253–265.
- [8] M. Et, B. C. Tripathy, A. J. Dutta, On pointwise statistical convergence of order α of sequences of fuzzy mappings, *Kuwait J. Sci.* 41(3) (2014) 17–30.
- [9] M. Et, S. A. Mohiuddine, A. Alotaibi, On λ -statistical convergence and strongly λ -summable functions of order α , *J. Inequal. Appl.* 2013(469) (2013) 8 pp.
- [10] M. Et, R. Colak, Y. Altin, Strongly almost summable sequences of order α , *Kuwait J. Sci.* 41(2) (2014) 35–47.
- [11] H. Fast, Sur la convergence statistique, *Colloq. Math.* 2 (1951) 241–244.
- [12] J. Fridy, On statistical convergence, *Analysis* 5 (1985) 301–313.
- [13] A. D. Gadjev, C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* 32(1) (2002) 129–138.
- [14] S. Ghosal, Weighted statistical convergence of order α and its applications, *J. Egyptian Math. Soc.* 24(1) (2016) 60–67.
- [15] S. Ghosal, λ -convergence of a sequence of random variables, *J. Egyptian Math. Soc.* 23(1) (2015) 85–89.
- [16] M. Isik, K. E. Akbas, On λ -statistical convergence of order α in probability, *J. Inequal. Spec. Funct.* 8(4) (2017) 57–64.
- [17] M. Isik, K. E. Et, On lacunary statistical convergence of order α in probability, *AIP Conference Proceedings* 1676, 020045 (2015), doi: <http://dx.doi.org/10.1063/1.4930471>.
- [18] M. Isik, K. E. Akbas, On asymptotically lacunary statistical equivalent sequences of order α in probability, *ITM Web of Conferences* 13, 01024 (2017), doi: 10.1051/itmconf/20171301024.
- [19] M. S. Marouf, Asymptotic equivalence and summability, *Int. J. Math. Math. Sci.* 16(4) (1993) 755–762.
- [20] M. Mursaleen, λ -statistical convergence, *Math. Slovaca* 50(1) (2000) 111–115.
- [21] R. F. Patterson, On asymptotically statistical equivalent sequences, *Demonstr. Math.* 36(1) (2003) 149–153.
- [22] R. Savas, M. Basarir, (σ, λ) -asymptotically statistical equivalent sequences, *Filomat* 20(1) (2006) 35–42.
- [23] E. Savas, Generalized asymptotically I -lacunary statistical equivalent of order α for sequences of sets, *Filomat* 31(6) (2017) 1507–1514.
- [24] I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* 66 (1959) 361–375.
- [25] H. Sengul, M. Et, On I -lacunary statistical convergence of order α of sequences of sets, *Filomat* 31(8) (2017) 2403–2412.
- [26] H. Sengul, On Wijsman I -lacunary statistical equivalence of order (η, μ) , *J. Inequal. Spec. Funct.* 9(2) (2018) 92–101.
- [27] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* 2 (1951) 73–74.