Filomat 34:13 (2020), 4297–4309
https://doi.org/10.2298/FIL2013297Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ordering of Bicyclic Signed Digraphs by Energy

Xiuwen Yang[a,b], Ligong Wang[a,b]

[a] *School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, P.R. China*
[b] *Xi'an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, P.R. China*

Abstract. Let $\mathcal{S}_n$ be the class of bicyclic signed digraphs with n vertices whose two signed directed even cycles are vertex-disjoint. In this paper, we characterize the ordering of bicyclic signed digraphs in $\mathcal{S}_n$ by energy with two positive or negative directed even cycles (resp., one positive directed even cycle and one negative directed even cycle). Furthermore, we determine extremal energy in $\mathcal{S}_n$ by the two orderings.

1. Introduction

A signed digraph (or briefly sidigraph) consists of a pair $S = (D, \sigma)$, where $D = (\mathcal{V}, \mathcal{A})$ is the underlying digraph and $\sigma : \mathcal{A} \to \{1, -1\}$ is the signed function. An arc with a +1 (−1) sign is called a positive (negative) arc of S. In general, the arcs of S are called signed arcs. The sign of a sidigraph is defined as the product of signs of its arcs. A sidigraph is said to be positive (negative) if its sign is positive (negative). A sidigraph is said to be all-positive (all-negative) if all its arcs are positive (negative). A sidigraph is said to be cycle-balanced if each of its cycles are positive, otherwise non cycle-balanced. Throughout this paper, we assume that cycle-balanced (non cycle-balanced) cycle is a positive (negative) cycle and denote them by C_n^+ (C_n^-), where n is the number of vertices.

For a sidigraph, we use uv to denote an arc from a vertex u to a vertex v. A vertex set $\{v_i | i = 1, 2, \ldots, n\}$ and a signed arc set $\{v_i v_{i+1} | i = 1, 2, \ldots, n-1\}$ make up a directed path P_n. A vertex set $\{v_i | i = 1, 2, \ldots, n\}$ and a signed arc set $\{v_i v_{i+1} | i = 1, 2, \ldots, n-1\} \cup \{v_n v_1\}$ make up a directed cycle C_n. If a underlying graph of sidigraph is connected, then the sidigraph is connected. If a connected sidigraph contains a unique singed directed cycle, then it's a unicyclic sidigraph. If a connected sidigraph contains exactly two singed directed cycles, then it's a bicyclic sidigraph. We consider the class $\mathcal{S}_n$ of bicyclic signed digraphs with n ($n \geq 4$) vertices whose two signed directed even cycles are vertex-disjoint. For a sidigraph $S = (D, \sigma)$, if it has a directed path from u to v and a directed path from v to u, where $\forall u, v \in \mathcal{V}$, then it's strongly connected. The maximal strongly connected subsidigraph of S is called a strong component of a sidigraph S.

2010 *Mathematics Subject Classification*. Primary 05C35; Secondary 05C50
Keywords. Ordering, Energy of signed digraphs, Bicyclic signed digraphs.
Received: 06 January 2020; Accepted: 24 January 2020
Communicated by Dragan S. Djordjević
Corresponding author: Ligong Wang
Supported by the National Natural Science Foundation of China (No. 11871398), the Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2018JM1032), the Fundamental Research Funds for the Central Universities (No. 3102019ghjd003), and the Seed Foundation of Innovation and Creation for Graduate Students in Northwestern Polytechnical University (No. ZZ2019031).
Email addresses: yangxiuwen1995@163.com (Xiuwen Yang), lgwangmath@163.com (Ligong Wang)

For a sidigraph S with $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, its adjacency matrix $A(S) = (a_{ij})_{n\times n}$ is the matrix whose entry a_{ij} is given by

$$a_{ij} = \begin{cases} \sigma(v_i, v_j), & \text{if } v_iv_j \in \mathcal{A}, \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial $\phi_S(x)$ of S with n vertices is $\phi_S(x) = det(xI_n - A(S))$. The eigenvalues $z_1, z_2, \ldots, z_n$ of S are the eigenvalues of $A(S)$. Since the matrix $A(S)$ is an asymmetric matrix, the eigenvalues of S are complex. So we should consider the real part of the eigenvalue of S. The energy of S is defined by

$$E(S) = \sum_{k=1}^{n} |\mathrm{Re}(z_k)|,$$

where $\mathrm{Re}(z_k)$ is the real part of eigenvalue z_k.

The concept of energy was first developed for the study of hydrocarbons. This chemical concept was proposed in the study of chemistry based on quantum theory. Furthermore, graph energy is one of the few research directions in mathematics which is driven and dominated by chemistry. Some knowledge about the energy of a graph can be found in the literature [1, 5, 10, 11, 13, 18, 19]. The digraph adds direction to the graph, making the concept of energy more widely applied to chemistry. Some knowledge about the energy of a digraph can be found in the literature [12, 16, 21, 23, 24, 26, 27].

For some results of sigraphs, Germina et al. [9] defined a sigraph energy to be the sum of absolute values of sigraph eigenvalues. Belardo et al. [3] considered the star complement technique for signed line graphs. And now, Belardo et al. [2] detected a class $\mathcal{G}$ of signed graphs whose nullity as an $A(\Gamma)$-eigenvalue did not depend on the chosen signature. And the class $\mathcal{G}$ contained, among others, all signed trees and all signed lollipop graphs. Hou et al. [15] considered the signed graphs with two distinct eigenvalues. They also characterized all connected signed graphs with maximum degree at most 4 and with just two distinct eigenvalues. Wang and Hou [25] found the maximal energy among all n-vertices connected bicyclic signed graphs with at most one odd cycle is $\mathcal{P}_n^{4,4}$. Li and Wang [20] obtained an exact estimate of energy for almost all signed graphs and established lower and upper bounds to the energy of random multipartite signed graphs.

For some results of sidigraphs, Yusuke and Iwao [28] discussed the structure of balanced coverings of signed digraphs under consideration of coverings of strongly connected digraphs. Bhat and Pirzada [4] obtained a new family of pairs of non-cospectral, equienergetic and strongly connected signed digraphs. Khan and Farooq [17] introduced a class of n vertices bicyclic sidigraphs and found sidigraphs with minimal and maximal energy in this class. Hafeez et al. [14] found the maximal energy of all bicyclic sidigraphs with fixed order and constructed few classes of non-cospectral equienergetic bicyclic sidigraphs. More results about sidigraphs see [6, 8, 22].

In 2014, Pirzada and Bhat [22] extended the notion of energy of sigraphs to sidigraphs in a similar way as graph energy has been extended to digraph energy. Then the energy of a positive (negative) directed cycle C_n^+ (C_n^-) are

$$E(C_n^+) = \sum_{k=0}^{n-1} |\cos\frac{2k\pi}{n}|,\ E(C_n^-) = \sum_{k=0}^{n-1} |\cos\frac{(2k+1)\pi}{n}|.$$

Furthermore, if C_r^+ (C_r^-) and C_s^+ (C_s^-) are two positive (negative) directed cycles with $2 \le r \le s$, then in [23], we have

$$E(C_r^+) \le E(C_s^+),\ E(C_r^-) \le E(C_s^-).$$

If $S_1, S_2, \ldots, S_k$ are strong components of a sidigraph S with n vertices, then the energy of a sidigraph is shown in [23] that

$$E(S) = \sum_{i=1}^{k} E(S_i).$$

The following formulae to calculate the energy of a positive (negative) directed cycle C_n^+ (C_n^-), where $n \geq 2$, are shown in [22].

$$E(C_n^+) = \begin{cases} 2\cot\frac{\pi}{n}, & \text{if } n \equiv 0 \pmod 4, \\ 2\csc\frac{\pi}{n}, & \text{if } n \equiv 2 \pmod 4, \\ \csc\frac{\pi}{2n}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{1}$$

$$E(C_n^-) = \begin{cases} 2\csc\frac{\pi}{n}, & \text{if } n \equiv 0 \pmod 4, \\ 2\cot\frac{\pi}{n}, & \text{if } n \equiv 2 \pmod 4, \\ \csc\frac{\pi}{2n}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{2}$$

This paper is organized as follows. In Section 2, we give some known knowledge about trigonometric functions and some energy orderings of bicyclic sidigraphs in $\mathcal{S}_n$ about $n \equiv i \pmod 4$. In Section 3, we give the complete ordering of bicyclic sidigraphs by energy with two positive or negative directed even cycles. We also give the complete ordering of bicyclic sidigraphs by energy with one positive directed even cycle and one negative directed even cycle. Finally, we determine extremal energy of bicyclic sidigraphs in $\mathcal{S}_n$.

2. Preliminaries

In this section, we present some energy orderings of bicyclic sidigraphs in $\mathcal{S}_n$ about $n \equiv i \pmod 4$ to prove our main results, where $i = 0, 1, 2, 3$. First of all, we give the monotonicity of some linear combinations of trigonometric functions.

Lemma 2.1. *[7] The function* $f(x) = 2\cot\frac{\pi}{x} + 2\cot\frac{\pi}{n-x}$ *is increasing on* $[2, \frac{n}{2}]$ *and decreasing on* $[\frac{n}{2}, n-2]$*, where* $n > 4$.

Lemma 2.2. *[26] The function* $f(x) = 2\csc\frac{\pi}{x} + 2\csc\frac{\pi}{n-x}$ *is decreasing on* $[2, \frac{n}{2}]$*, where* $n > 4$.

Lemma 2.3. *[26] The function* $f(x) = 2\csc\frac{\pi}{x} + 2\cot\frac{\pi}{n-x}$ *is decreasing on* $[2, n-2]$*, where* $n > 4$.

Firstly, we consider the energy of bicyclic sidigraphs with two positive or negative directed even cycles.

Lemma 2.4. *Let* C_m^+ *and* C_{n-m}^+ *(*C_m^- *and* C_{n-m}^-*) be to positive (negative) directed even cycles which are vertex-disjoint, where* $n \equiv 0 \pmod 4$, $m \equiv 0 \pmod 2$, $n > 4$ *and* $m \in [2, n-2]$*. Then we have*

(i) If $\frac{n}{2} \equiv 0 \pmod 4$*, then*

$$E(C_2^+) + E(C_{n-2}^+) > E(C_4^-) + E(C_{n-4}^-) > E(C_6^+) + E(C_{n-6}^+) > \cdots > E(C_{\frac{n}{2}}^-) + E(C_{\frac{n}{2}}^-)$$
$$> E(C_{\frac{n}{2}}^+) + E(C_{\frac{n}{2}}^+) > \cdots > E(C_{n-4}^+) + E(C_4^+) > E(C_{n-2}^-) + E(C_2^-).$$

(ii) If $\frac{n}{2} \equiv 2 \pmod 4$*, then*

$$E(C_2^+) + E(C_{n-2}^+) > E(C_4^-) + E(C_{n-4}^-) > E(C_6^+) + E(C_{n-6}^+) > \cdots > E(C_{\frac{n}{2}}^+) + E(C_{\frac{n}{2}}^+)$$
$$> E(C_{\frac{n}{2}}^-) + E(C_{\frac{n}{2}}^-) > \cdots > E(C_{n-4}^+) + E(C_4^+) > E(C_{n-2}^-) + E(C_2^-).$$

Proof. If $m \in [2, \frac{n}{2}]$, then $n - m \in [\frac{n}{2}, n-2]$ or $m \in [\frac{n}{2}, n-2]$, then $n - m \in [2, \frac{n}{2}]$, so we only consider $m \in [2, \frac{n}{2}]$.

Firstly, by Eq.(1) and Eq.(2), we have

$$E(C_m^+) + E(C_{n-m}^+) = \begin{cases} 2\cot\frac{\pi}{m} + 2\cot\frac{\pi}{n-m}, & \text{if } m \equiv 0 \pmod 4, \\ 2\csc\frac{\pi}{m} + 2\csc\frac{\pi}{n-m}, & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

$$E(C_m^-)+E(C_{n-m}^-)=\begin{cases}2\csc\frac{\pi}{m}+2\csc\frac{\pi}{n-m}, & \text{if } m\equiv 0 \pmod 4,\\ 2\cot\frac{\pi}{m}+2\cot\frac{\pi}{n-m}, & \text{if } m\equiv 2 \pmod 4.\end{cases}$$

Next, we find when $m\equiv 2 \pmod 4$, $E(C_m^+)+E(C_{n-m}^+)=2\csc\frac{\pi}{m}+2\csc\frac{\pi}{n-m}$ and when $m\equiv 0 \pmod 4$, $E(C_m^-)+E(C_{n-m}^-)=2\csc\frac{\pi}{m}+2\csc\frac{\pi}{n-m}$. And by Lemma 2.2, we see that $2\csc\frac{\pi}{m}+2\csc\frac{\pi}{n-m}$ is decreasing in the interval $[2,\frac{n}{2}]$. So we have

(a) If $\frac{n}{2}\equiv 0 \pmod 4$, then

$$E(C_2^+)+E(C_{n-2}^+)>E(C_4^-)+E(C_{n-4}^-)>E(C_6^+)+E(C_{n-6}^+)>\cdots>E(C_{\frac{n}{2}}^-)+E(C_{\frac{n}{2}}^-);$$

(b) If $\frac{n}{2}\equiv 2 \pmod 4$, then

$$E(C_2^+)+E(C_{n-2}^+)>E(C_4^-)+E(C_{n-4}^-)>E(C_6^+)+E(C_{n-6}^+)>\cdots>E(C_{\frac{n}{2}}^+)+E(C_{\frac{n}{2}}^+).$$

Similarly, when $m\equiv 0 \pmod 4$, $E(C_m^+)+E(C_{n-m}^+)=2\cot\frac{\pi}{m}+2\cot\frac{\pi}{n-m}$ and when $m\equiv 2 \pmod 4$, $E(C_m^-)+E(C_{n-m}^-)=2\cot\frac{\pi}{m}+2\cot\frac{\pi}{n-m}$. And by Lemma 2.1, we see that $2\cot\frac{\pi}{m}+2\cot\frac{\pi}{n-m}$ is increasing in the interval $[2,\frac{n}{2}]$. So we have

(c) If $\frac{n}{2}\equiv 0 \pmod 4$, then

$$E(C_{\frac{n}{2}}^+)+E(C_{\frac{n}{2}}^+)>\cdots>E(C_6^-)+E(C_{n-6}^-)>E(C_{n-4}^+)+E(C_4^+)>E(C_{n-2}^-)+E(C_2^-);$$

(d) If $\frac{n}{2}\equiv 2 \pmod 4$, then

$$E(C_{\frac{n}{2}}^-)+E(C_{\frac{n}{2}}^-)>\cdots>E(C_6^-)+E(C_{n-6}^-)>E(C_{n-4}^+)+E(C_4^+)>E(C_{n-2}^-)+E(C_2^-).$$

Finally, since

$$2\csc\frac{2\pi}{n}+2\csc\frac{2\pi}{n}>2\cot\frac{2\pi}{n}+2\cot\frac{2\pi}{n},$$

and by (a)-(d), the results hold. □

Lemma 2.5. *Let C_m^+ and C_{n-m}^+ (C_m^- and C_{n-m}^-) be to positive (negative) directed even cycles which are vertex-disjoint, where $n\equiv 2 \pmod 4$, $m\equiv 0 \pmod 2$, $n>4$ and $m\in[2,n-2]$. Then we have*

(i) If $\frac{n}{2}-1\equiv 0 \pmod 4$, then

$$\begin{aligned}&E(C_2^+)+E(C_{n-2}^+)>E(C_4^-)+E(C_{n-4}^-)>E(C_6^+)+E(C_{n-6}^+)>\cdots>E(C_{\frac{n}{2}-1}^-)+E(C_{\frac{n}{2}+1}^-)\\&>E(C_{\frac{n}{2}+1}^+)+E(C_{\frac{n}{2}-1}^+)>\cdots>E(C_{n-4}^+)+E(C_4^+)>E(C_{n-2}^-)+E(C_2^-).\end{aligned}$$

(ii) If $\frac{n}{2}-1\equiv 2 \pmod 4$, then

$$\begin{aligned}&E(C_2^+)+E(C_{n-2}^+)>E(C_4^-)+E(C_{n-4}^-)>E(C_6^+)+E(C_{n-6}^+)>\cdots>E(C_{\frac{n}{2}-1}^+)+E(C_{\frac{n}{2}+1}^+)\\&>E(C_{\frac{n}{2}+1}^-)+E(C_{\frac{n}{2}-1}^-)>\cdots>E(C_{n-4}^+)+E(C_4^+)>E(C_{n-2}^-)+E(C_2^-).\end{aligned}$$

Proof. If $m\in[2,\frac{n}{2}]$, then $n-m\in[\frac{n}{2},n-2]$ or $m\in[\frac{n}{2},n-2]$, then $n-m\in[2,\frac{n}{2}]$, so we only consider $m\in[2,\frac{n}{2}]$.

Firstly, by Eq.(1) and Eq.(2), we have

$$E(C_m^+)+E(C_{n-m}^+)=\begin{cases}2\cot\frac{\pi}{m}+2\csc\frac{\pi}{n-m}, & \text{if } m\equiv 0 \pmod 4,\\ 2\csc\frac{\pi}{m}+2\cot\frac{\pi}{n-m}, & \text{if } m\equiv 2 \pmod 4.\end{cases}$$

$$E(C_m^-)+E(C_{n-m}^-)=\begin{cases}2\csc\frac{\pi}{m}+2\cot\frac{\pi}{n-m}, & \text{if } m\equiv 0 \pmod 4,\\ 2\cot\frac{\pi}{m}+2\csc\frac{\pi}{n-m}, & \text{if } m\equiv 2 \pmod 4.\end{cases}$$

Next, similarly to the proof of Lemma 2.4, we find when $m \equiv 2 \pmod 4$, $E(C_m^+) + E(C_{n-m}^+) = 2\csc\frac{\pi}{m} + 2\cot\frac{\pi}{n-m}$ and when $m \equiv 0 \pmod 4$, $E(C_m^-) + E(C_{n-m}^-) = 2\csc\frac{\pi}{m} + 2\cot\frac{\pi}{n-m}$. And by Lemma 2.3, we see that $2\csc\frac{\pi}{m} + 2\cot\frac{\pi}{n-m}$ is decreasing in the interval $[2, n-2]$. So we have

(a) If $\frac{n}{2} - 1 \equiv 0 \pmod 4$, then

$$E(C_2^+) + E(C_{n-2}^+) > E(C_4^-) + E(C_{n-4}^-) > E(C_6^+) + E(C_{n-6}^+) > \cdots > E(C_{\frac{n}{2}-1}^-) + E(C_{\frac{n}{2}+1}^-);$$

(b) If $\frac{n}{2} - 1 \equiv 2 \pmod 4$, then

$$E(C_2^+) + E(C_{n-2}^+) > E(C_4^-) + E(C_{n-4}^-) > E(C_6^+) + E(C_{n-6}^+) > \cdots > E(C_{\frac{n}{2}-1}^+) + E(C_{\frac{n}{2}+1}^+).$$

Similarly, when $m \equiv 0 \pmod 4$, $E(C_m^+) + E(C_{n-m}^+) = 2\cot\frac{\pi}{m} + 2\csc\frac{\pi}{n-m}$ and when $m \equiv 2 \pmod 4$, $E(C_m^-) + E(C_{n-m}^-) = 2\cot\frac{\pi}{m} + 2\csc\frac{\pi}{n-m}$. And by Lemma 2.3, we see that $2\cot\frac{\pi}{m} + 2\csc\frac{\pi}{n-m}$ is increasing in the interval $[2, n-2]$. So we have

(c) If $\frac{n}{2} - 1 \equiv 0 \pmod 4$, then

$$E(C_{\frac{n}{2}+1}^+) + E(C_{\frac{n}{2}-1}^+) > \cdots > E(C_6^-) + E(C_{n-6}^-) > E(C_{n-4}^+) + E(C_4^+) > E(C_{n-2}^-) + E(C_2^-);$$

(d) If $\frac{n}{2} - 1 \equiv 2 \pmod 4$, then

$$E(C_{\frac{n}{2}+1}^-) + E(C_{\frac{n}{2}-1}^-) > \cdots > E(C_6^-) + E(C_{n-6}^-) > E(C_{n-4}^+) + E(C_4^+) > E(C_{n-2}^-) + E(C_2^-).$$

Finally, since

$$2\csc\frac{\pi}{\frac{n}{2}-1} + 2\cot\frac{\pi}{\frac{n}{2}+1} > 2\csc\frac{\pi}{\frac{n}{2}} + 2\cot\frac{\pi}{\frac{n}{2}} > 2\csc\frac{\pi}{\frac{n}{2}+1} + 2\cot\frac{\pi}{\frac{n}{2}-1},$$

and by (a)-(d), the results hold. □

Lemma 2.6. *Let $\mathcal{S}_n$ be the class of bicyclic signed digraphs with n ($n \geq 4$) vertices whose two signed directed even cycles are vertex-disjoint. Then we have*

(i) If $n \equiv 0 \pmod 4$, $\frac{n}{2} \equiv 0 \pmod 4$, then

$$E(C_{\frac{n-2}{2}-1}^+) + E(C_{\frac{n-2}{2}+1}^+) > E(C_{n-2}^-) + E(C_2^-) > E(C_{\frac{n-2}{2}+1}^-) + E(C_{\frac{n-2}{2}-1}^-).$$

(ii) If $n \equiv 0 \pmod 4$, $\frac{n}{2} \equiv 2 \pmod 4$, then

$$E(C_{\frac{n-2}{2}-1}^-) + E(C_{\frac{n-2}{2}+1}^-) > E(C_{n-2}^-) + E(C_2^-) > E(C_{\frac{n-2}{2}+1}^+) + E(C_{\frac{n-2}{2}-1}^+).$$

(iii) If $n \equiv 2 \pmod 4$, $\frac{n}{2} - 1 \equiv 0 \pmod 4$, then

$$E(C_{\frac{n-2}{2}}^-) + E(C_{\frac{n-2}{2}}^-) > E(C_{n-2}^-) + E(C_2^-) > E(C_{\frac{n-2}{2}}^+) + E(C_{\frac{n-2}{2}}^+).$$

(iv) If $n \equiv 2 \pmod 4$, $\frac{n}{2} - 1 \equiv 2 \pmod 4$, then

$$E(C_{\frac{n-2}{2}}^+) + E(C_{\frac{n-2}{2}}^+) > E(C_{n-2}^-) + E(C_2^-) > E(C_{\frac{n-2}{2}}^-) + E(C_{\frac{n-2}{2}}^-).$$

Proof. (i) By (1) and (2), in order to prove

$$E(C_{\frac{n-2}{2}-1}^+) + E(C_{\frac{n-2}{2}+1}^+) > E(C_{n-2}^-) + E(C_2^-) > E(C_{\frac{n-2}{2}+1}^-) + E(C_{\frac{n-2}{2}-1}^-),$$

we should prove

$$2\csc\frac{2\pi}{n-4} + 2\cot\frac{2\pi}{n} > 2\cot\frac{\pi}{n-2} > 2\csc\frac{2\pi}{n} + 2\cot\frac{2\pi}{n-4}.$$

By Lemma 2.3, $f(x) = 2\csc\frac{\pi}{x} + 2\cot\frac{\pi}{n-x}$ is decreasing on $[2, n-2]$, where $n > 4$. Then

$$2\csc\frac{2\pi}{n-4} + 2\cot\frac{2\pi}{n} > 2\csc\frac{2\pi}{n-2} + 2\cot\frac{2\pi}{n-2} > 2\csc\frac{2\pi}{n} + 2\cot\frac{2\pi}{n-4}.$$

Since

$$2\csc\frac{2\pi}{n-2} + 2\cot\frac{2\pi}{n-2} = \frac{2}{\sin\frac{2\pi}{n-2}} + \frac{2\cos\frac{2\pi}{n-2}}{\sin\frac{2\pi}{n-2}} = \frac{2 + 4\cos^2\frac{\pi}{n-2} - 2}{2\sin\frac{\pi}{n-2}\cos\frac{\pi}{n-2}} = 2\cot\frac{\pi}{n-2}. \tag{3}$$

So (i) is proved.

The proof of (ii) is similar to (i).

(iii) By (1) and (2), in order to prove

$$E(C^-_{\frac{n-2}{2}}) + E(C^-_{\frac{n-2}{2}}) > E(C^-_{n-2}) + E(C^-_2) > E(C^+_{\frac{n-2}{2}}) + E(C^+_{\frac{n-2}{2}}).$$

we should prove

$$2\csc\frac{2\pi}{n-2} + 2\csc\frac{2\pi}{n-2} > 2\csc\frac{\pi}{n-2} > 2\cot\frac{2\pi}{n-2} + 2\cot\frac{2\pi}{n-2}.$$

Since

$$2\csc\frac{2\pi}{n-2} = \frac{2}{2\sin\frac{\pi}{n-2}\cos\frac{\pi}{n-2}} > \frac{1}{\sin\frac{\pi}{n-2}} = \csc\frac{\pi}{n-2}, \tag{4}$$

and

$$2\cot\frac{2\pi}{n-2} = \frac{2(2\cos^2\frac{\pi}{n-2} - 1)}{2\sin\frac{\pi}{n-2}\cos\frac{\pi}{n-2}} < \frac{\cos\frac{\pi}{n-2}}{\sin\frac{\pi}{n-2}\cos\frac{\pi}{n-2}} = \csc\frac{\pi}{n-2}. \tag{5}$$

So (iii) is proved.

The proof of (iv) is similar to (iii). □

Next, we consider the energy of bicyclic sidigraphs with one positive directed even cycle and one negative directed even cycle.

Lemma 2.7. *Let C^+_m and C^+_{n-m} (C^-_m and C^-_{n-m}) be to positive (negative) directed even cycles which are vertex-disjoint, where $n \equiv 0 \pmod 4$, $m \equiv 0 \pmod 2$, $n > 4$ and $m \in [2, n-2]$. Then we have*

(i) If $\frac{n}{2} \equiv 0 \pmod 4$, then

$$\begin{aligned}&E(C^+_2) + E(C^-_{n-2}) > E(C^-_4) + E(C^+_{n-4}) > E(C^+_6) + E(C^-_{n-6}) > \cdots > E(C^+_{\frac{n}{2}-2}) + E(C^-_{\frac{n}{2}+2})\\ &> E(C^-_{\frac{n}{2}}) + E(C^+_{\frac{n}{2}}) > E(C^+_{\frac{n}{2}+2}) + E(C^-_{\frac{n}{2}-2}) > \cdots > E(C^-_{n-4}) + E(C^+_4) > E(C^+_{n-2}) + E(C^-_2).\end{aligned}$$

(ii) If $\frac{n}{2} \equiv 2 \pmod 4$, then

$$\begin{aligned}&E(C^+_2) + E(C^-_{n-2}) > E(C^-_4) + E(C^+_{n-4}) > E(C^+_6) + E(C^-_{n-6}) > \cdots > E(C^-_{\frac{n}{2}-2}) + E(C^+_{\frac{n}{2}+2})\\ &> E(C^+_{\frac{n}{2}}) + E(C^-_{\frac{n}{2}}) > E(C^-_{\frac{n}{2}+2}) + E(C^+_{\frac{n}{2}-2}) > \cdots > E(C^-_{n-4}) + E(C^+_4) > E(C^+_{n-2}) + E(C^-_2).\end{aligned}$$

Proof. The proof is similar to Lemma 2.5. □

Lemma 2.8. *Let C^+_m and C^+_{n-m} (C^-_m and C^-_{n-m}) be to positive (negative) directed even cycles which are vertex-disjoint, where $n \equiv 2 \pmod 4$, $m \equiv 0 \pmod 2$, $n > 4$ and $m \in [2, n-2]$. Then we have*

(i) If $\frac{n}{2} - 1 \equiv 0 \pmod 4$, then

$$\begin{aligned}&E(C^+_2) + E(C^-_{n-2}) > E(C^-_4) + E(C^+_{n-4}) > E(C^+_6) + E(C^-_{n-6}) > \cdots > E(C^-_{\frac{n}{2}-1}) + E(C^+_{\frac{n}{2}+1})\\ &> E(C^-_{\frac{n}{2}+1}) + E(C^+_{\frac{n}{2}-1}) > \cdots > E(C^-_{n-4}) + E(C^+_4) > E(C^+_{n-2}) + E(C^-_2).\end{aligned}$$

(ii) If $\frac{n}{2}-1\equiv 2\pmod 4$, *then*

$$E(C_2^+)+E(C_{n-2}^-)>E(C_4^-)+E(C_{n-4}^+)>E(C_6^+)+E(C_{n-6}^-)>\cdots>E(C_{\frac{n}{2}-1}^+)+E(C_{\frac{n}{2}+1}^-)$$
$$>E(C_{\frac{n}{2}+1}^+)+E(C_{\frac{n}{2}-1}^-)>\cdots>E(C_{n-4}^-)+E(C_4^+)>E(C_{n-2}^+)+E(C_2^-).$$

Proof. The proof is similar to Lemma 2.4. □

Lemma 2.9. *Let $\mathcal{S}_n$ be the class of bicyclic signed digraphs with n ($n\geq 4$) vertices whose two signed directed even cycles are vertex-disjoint. Then we have*

(i) If $n\equiv 0\pmod 4$, $\frac{n}{2}\equiv 0\pmod 4$, *then*

$$E(C_{\frac{n-2}{2}-1}^+)+E(C_{\frac{n-2}{2}+1}^-)>E(C_{n-2}^+)+E(C_2^-)>E(C_{\frac{n-2}{2}+1}^+)+E(C_{\frac{n-2}{2}-1}^-).$$

(ii) If $n\equiv 0\pmod 4$, $\frac{n}{2}\equiv 2\pmod 4$, *then*

$$E(C_{\frac{n-2}{2}-1}^-)+E(C_{\frac{n-2}{2}+1}^+)>E(C_{n-2}^+)+E(C_2^-)>E(C_{\frac{n-2}{2}+1}^-)+E(C_{\frac{n-2}{2}-1}^+).$$

(iii) If $n\equiv 2\pmod 4$, *then*

$$E(C_{\frac{n-2}{2}}^-)+E(C_{\frac{n-2}{2}}^+)=E(C_{n-2}^+)+E(C_2^-).$$

Proof. (i) By (1) and (2), in order to prove

$$E(C_{\frac{n-2}{2}-1}^+)+E(C_{\frac{n-2}{2}+1}^-)>E(C_{n-2}^+)+E(C_2^-)>E(C_{\frac{n-2}{2}+1}^+)+E(C_{\frac{n-2}{2}-1}^-).$$

we should prove

$$2\csc\frac{2\pi}{n-4}+2\csc\frac{2\pi}{n}>2\csc\frac{\pi}{n-2}>2\cot\frac{2\pi}{n-4}+2\cot\frac{2\pi}{n}.$$

By Lemma 2.2 and Eq.(4), we have

$$2\csc\frac{2\pi}{n-4}+2\csc\frac{2\pi}{n}>2\csc\frac{2\pi}{n-2}+2\csc\frac{2\pi}{n-2}>2\csc\frac{\pi}{n-2}.$$

By Lemma 2.1 and Eq.(5), we have

$$2\cot\frac{2\pi}{n-4}+2\cot\frac{2\pi}{n}<2\cot\frac{2\pi}{n-2}+2\cot\frac{2\pi}{n-2}<2\csc\frac{\pi}{n-2}.$$

So (i) is proved.

The proof of (ii) is similar to (i).

(iii) By (1) and (2), in order to prove

$$E(C_{\frac{n-2}{2}}^-)+E(C_{\frac{n-2}{2}}^+)=E(C_{n-2}^+)+E(C_2^-).$$

we should prove

$$2\csc\frac{2\pi}{n-2}+2\cot\frac{2\pi}{n-2}=2\cot\frac{\pi}{n-2}.$$

By (3), it is obvious, so (iii) is proved. □

3. Main results

In this section, we give our main results in the following theorems. We investigate the class $\mathcal{S}_n$ of bicyclic signed digraphs with n ($n \geq 4$) vertices whose two signed directed even cycles are vertex-disjoint. If the signed directed even cycles C_{r_1} and C_{r_2} ($2 \leq r_i \leq n-2$, $r_i \equiv 0 \pmod 2$ and $i = 1, 2$) are the strong components of $S \in \mathcal{S}_n$, then we have $E(S) = E(C_{r_1}) + E(C_{r_2})$.

Firstly, we give the complete ordering of bicyclic sidigraphs by energy with two positive or negative directed even cycles.

Theorem 3.1. *Let $\mathcal{S}_n$ be the class of bicyclic signed digraphs with n vertices whose two signed directed even cycles are vertex-disjoint, where $n \geq 4$ and $n \equiv 0 \pmod 2$. If the two directed even cycles are both positive or negative, then the complete ordering of bicyclic sidigraphs in $\mathcal{S}_n$ by energy is as follows:*

(i) Let $n \equiv 0 \pmod 4$. Then we have

(a) If $\frac{n}{2} \equiv 0 \pmod 4$, then

$$
\begin{aligned}
&E(C_2^+) + E(C_{n-2}^+) > E(C_4^-) + E(C_{n-4}^-) > E(C_6^+) + E(C_{n-6}^+) > \cdots > E(C_{\frac{n}{2}}^-) + E(C_{\frac{n}{2}}^-) \\
&> E(C_{\frac{n}{2}}^+) + E(C_{\frac{n}{2}}^+) > \cdots > E(C_{n-4}^+) + E(C_4^+) = E(C_2^+) + E(C_{n-4}^+) > E(C_4^-) + E(C_{n-6}^-) \\
&> E(C_6^+) + E(C_{n-8}^+) > \cdots > E(C_{\frac{n-2}{2}-1}^+) + E(C_{\frac{n-2}{2}+1}^+) > E(C_{n-2}^-) + E(C_2^-) > E(C_{\frac{n-2}{2}+1}^-) \\
&+ E(C_{\frac{n-2}{2}-1}^-) > \cdots > E(C_{n-6}^+) + E(C_4^+) = E(C_2^+) + E(C_{n-6}^+) > E(C_4^-) + E(C_{n-8}^-) > E(C_6^+) \\
&+ E(C_{n-10}^+) > \cdots > E(C_{\frac{n-4}{2}}^+) + E(C_{\frac{n-4}{2}}^+) > E(C_{n-4}^-) + E(C_2^-) > E(C_{\frac{n-4}{2}}^-) + E(C_{\frac{n-4}{2}}^-) > \cdots \\
&> E(C_{n-8}^+) + E(C_4^+) = E(C_2^+) + E(C_{n-8}^+) > E(C_4^-) + E(C_{n-10}^-) > E(C_6^+) + E(C_{n-12}^+) > \cdots \\
&> E(C_{\frac{n-6}{2}-1}^-) + E(C_{\frac{n-6}{2}+1}^-) > E(C_{n-6}^-) + E(C_2^-) > E(C_{\frac{n-6}{2}+1}^+) + E(C_{\frac{n-6}{2}-1}^+) > \cdots > E(C_{n-10}^+) \\
&+ E(C_4^+) = E(C_2^+) + E(C_{n-10}^+) > E(C_4^-) + E(C_{n-12}^-) > E(C_6^+) + E(C_{n-14}^+) > \cdots > E(C_{\frac{n-8}{2}}^-) \\
&+ E(C_{\frac{n-8}{2}}^-) > E(C_{n-8}^-) + E(C_2^-) > E(C_{\frac{n-8}{2}}^+) + E(C_{\frac{n-8}{2}}^+) > \cdots > E(C_{n-12}^+) + E(C_4^+) = E(C_2^+) \\
&+ E(C_{n-12}^+) > \cdots > E(C_4^+) + E(C_4^+) = E(C_2^+) + E(C_4^+) = E(C_2^+) + E(C_2^+) > E(C_6^-) + E(C_2^-) \\
&= E(C_4^-) + E(C_2^-) > E(C_2^-) + E(C_2^-).
\end{aligned}
$$

(b) If $\frac{n}{2} \equiv 2 \pmod 4$, then

$$
\begin{aligned}
&E(C_2^+) + E(C_{n-2}^+) > E(C_4^-) + E(C_{n-4}^-) > E(C_6^+) + E(C_{n-6}^+) > \cdots > E(C_{\frac{n}{2}}^+) + E(C_{\frac{n}{2}}^+) \\
&> E(C_{\frac{n}{2}}^-) + E(C_{\frac{n}{2}}^-) > \cdots > E(C_{n-4}^+) + E(C_4^+) = E(C_2^+) + E(C_{n-4}^+) > E(C_4^-) + E(C_{n-6}^-) \\
&> E(C_6^+) + E(C_{n-8}^+) > \cdots > E(C_{\frac{n-2}{2}-1}^-) + E(C_{\frac{n-2}{2}+1}^-) > E(C_{n-2}^-) + E(C_2^-) > E(C_{\frac{n-2}{2}+1}^+) \\
&+ E(C_{\frac{n-2}{2}-1}^+) > \cdots > E(C_{n-6}^+) + E(C_4^+) = E(C_2^+) + E(C_{n-6}^+) > E(C_4^-) + E(C_{n-8}^-) > E(C_6^+) \\
&+ E(C_{n-10}^+) > \cdots > E(C_{\frac{n-4}{2}}^-) + E(C_{\frac{n-4}{2}}^-) > E(C_{n-4}^-) + E(C_2^-) > E(C_{\frac{n-4}{2}}^+) + E(C_{\frac{n-4}{2}}^+) > \cdots \\
&> E(C_{n-8}^+) + E(C_4^+) = E(C_2^+) + E(C_{n-8}^+) > E(C_4^-) + E(C_{n-10}^-) > E(C_6^+) + E(C_{n-12}^+) > \cdots \\
&> E(C_{\frac{n-6}{2}-1}^+) + E(C_{\frac{n-6}{2}+1}^+) > E(C_{n-6}^-) + E(C_2^-) > E(C_{\frac{n-6}{2}+1}^-) + E(C_{\frac{n-6}{2}-1}^-) > \cdots > E(C_{n-10}^+) \\
&+ E(C_4^+) = E(C_2^+) + E(C_{n-10}^+) > E(C_4^-) + E(C_{n-12}^-) > E(C_6^+) + E(C_{n-14}^+) > \cdots > E(C_{\frac{n-8}{2}}^+) \\
&+ E(C_{\frac{n-8}{2}}^+) > E(C_{n-8}^-) + E(C_2^-) > E(C_{\frac{n-8}{2}}^-) + E(C_{\frac{n-8}{2}}^-) > \cdots > E(C_{n-12}^+) + E(C_4^+) = E(C_2^+) \\
&+ E(C_{n-12}^+) > \cdots > E(C_4^+) + E(C_4^+) = E(C_2^+) + E(C_4^+) = E(C_2^+) + E(C_2^+) > E(C_6^-) + E(C_2^-) \\
&= E(C_4^-) + E(C_2^-) > E(C_2^-) + E(C_2^-).
\end{aligned}
$$

(ii) Let $n \equiv 2 \pmod 4$. Then we have

(a) If $\frac{n}{2}-1\equiv 0 \pmod 4$*, then*

$$\begin{aligned}
&E(C_2^+)+E(C_{n-2}^+) > E(C_4^-)+E(C_{n-4}^-) > E(C_6^+)+E(C_{n-6}^+) > \cdots > E(C^-_{\frac{n}{2}-1})+E(C^-_{\frac{n}{2}+1})\\
&> E(C^+_{\frac{n}{2}+1})+E(C^+_{\frac{n}{2}-1}) > \cdots > E(C^+_{n-4})+E(C_4^+) = E(C_2^+)+E(C^+_{n-4}) > E(C_4^-)+E(C^-_{n-6})\\
&> E(C_6^+)+E(C^+_{n-8}) > \cdots > E(C^-_{\frac{n-2}{2}})+E(C^-_{\frac{n-2}{2}}) > E(C^-_{n-2})+E(C_2^-) > E(C^+_{\frac{n-2}{2}})+E(C^+_{\frac{n-2}{2}})\\
&> \cdots > E(C^+_{n-6})+E(C_4^+) = E(C_2^+)+E(C^+_{n-6}) > E(C_4^-)+E(C^-_{n-8}) > E(C_6^+)+E(C^+_{n-10})\\
&> \cdots > E(C^+_{\frac{n-4}{2}-1})+E(C^+_{\frac{n-4}{2}+1}) > E(C^-_{n-4})+E(C_2^-) > E(C^-_{\frac{n-4}{2}+1})+E(C^-_{\frac{n-4}{2}-1}) > \cdots\\
&> E(C^+_{n-8})+E(C_4^+) = E(C_2^+)+E(C^+_{n-8}) > E(C_4^-)+E(C^-_{n-10}) > E(C_6^+)+E(C^+_{n-12}) > \cdots\\
&> E(C^+_{\frac{n-6}{2}})+E(C^+_{\frac{n-6}{2}}) > E(C^-_{n-6})+E(C_2^-) > E(C^-_{\frac{n-6}{2}})+E(C^-_{\frac{n-6}{2}}) > \cdots > E(C^+_{n-10})+E(C_4^+)\\
&= E(C_2^+)+E(C^+_{n-10}) > E(C_4^-)+E(C^-_{n-12}) > E(C_6^+)+E(C^+_{n-14}) > \cdots > E(C^-_{\frac{n-8}{2}-1})+E(C^-_{\frac{n-8}{2}+1})\\
&> E(C^-_{n-8})+E(C_2^-) > E(C^+_{\frac{n-8}{2}+1})+E(C^+_{\frac{n-8}{2}-1}) > \cdots > E(C^+_{n-12})+E(C_4^+) = E(C_2^+)+E(C^+_{n-12})\\
&> \cdots > E(C_4^+)+E(C_4^+) = E(C_2^+)+E(C_4^+) = E(C_2^+)+E(C_2^+) > E(C_6^-)+E(C_2^-) = E(C_4^-)\\
&+ E(C_2^-) > E(C_2^-)+E(C_2^-).
\end{aligned}$$

(b) If $\frac{n}{2}-1\equiv 2 \pmod 4$*, then*

$$\begin{aligned}
&E(C_2^+)+E(C_{n-2}^+) > E(C_4^-)+E(C_{n-4}^-) > E(C_6^+)+E(C_{n-6}^+) > \cdots > E(C^+_{\frac{n}{2}-1})+E(C^+_{\frac{n}{2}+1})\\
&> E(C^-_{\frac{n}{2}+1})+E(C^-_{\frac{n}{2}-1}) > \cdots > E(C^+_{n-4})+E(C_4^+) = E(C_2^+)+E(C^+_{n-4}) > E(C_4^-)+E(C^-_{n-6})\\
&> E(C_6^+)+E(C^+_{n-8}) > \cdots > E(C^+_{\frac{n-2}{2}})+E(C^+_{\frac{n-2}{2}}) > E(C^-_{n-2})+E(C_2^-) > E(C^-_{\frac{n-2}{2}})+E(C^-_{\frac{n-2}{2}})\\
&> \cdots > E(C^+_{n-6})+E(C_4^+) = E(C_2^+)+E(C^+_{n-6}) > E(C_4^-)+E(C^-_{n-8}) > E(C_6^+)+E(C^+_{n-10})\\
&> \cdots > E(C^-_{\frac{n-4}{2}-1})+E(C^-_{\frac{n-4}{2}+1}) > E(C^-_{n-4})+E(C_2^-) > E(C^+_{\frac{n-4}{2}+1})+E(C^+_{\frac{n-4}{2}-1}) > \cdots\\
&> E(C^+_{n-8})+E(C_4^+) = E(C_2^+)+E(C^+_{n-8}) > E(C_4^-)+E(C^-_{n-10}) > E(C_6^+)+E(C^+_{n-12}) > \cdots\\
&> E(C^-_{\frac{n-6}{2}})+E(C^-_{\frac{n-6}{2}}) > E(C^-_{n-6})+E(C_2^-) > E(C^+_{\frac{n-6}{2}})+E(C^+_{\frac{n-6}{2}}) > \cdots > E(C^+_{n-10})+E(C_4^+)\\
&= E(C_2^+)+E(C^+_{n-10}) > E(C_4^-)+E(C^-_{n-12}) > E(C_6^+)+E(C^+_{n-14}) > \cdots > E(C^+_{\frac{n-8}{2}-1})+E(C^+_{\frac{n-8}{2}+1})\\
&> E(C^-_{n-8})+E(C_2^-) > E(C^-_{\frac{n-8}{2}+1})+E(C^-_{\frac{n-8}{2}-1}) > \cdots > E(C^+_{n-12})+E(C_4^+) = E(C_2^+)+E(C^+_{n-12})\\
&> \cdots > E(C_4^+)+E(C_4^+) = E(C_2^+)+E(C_4^+) = E(C_2^+)+E(C_2^+) > E(C_6^-)+E(C_2^-) = E(C_4^-)\\
&+ E(C_2^-) > E(C_2^-)+E(C_2^-).
\end{aligned}$$

Proof. Obviously, we have $2\cot\frac{\pi}{4}=2\csc\frac{\pi}{2}$, and together with Lemmas 2.4-2.6, we get the ordering (i) and (ii) of bicyclic sidigraphs in $\mathcal{S}_n$ by energy. □

Remark 3.2. *Actually, let* $\mathcal{S}_n$ *be the class of bicyclic signed digraphs with* n *vertices whose two signed directed even cycles are vertex-disjoint, where* $n\geq 4$ *and* $n\equiv 1 \pmod 2$*. If the two directed even cycles are both positive or negative, then the complete energy ordering of bicyclic sidigraphs in* $\mathcal{S}_n$ *is similar to Theorem 3.1. We only need change* n *in Theorem 3.1 into* $n-1$*.*

Example 3.3. *Let* $\mathcal{S}_{25}$ *be the class of bicyclic signed digraphs with* 25 *vertices, where the two signed directed even cycles with* r_1 *and* r_2 *vertices are vertex-disjoint. If the two directed even cycles are both positive or negative, then the complete energy ordering of bicyclic sidigraphs in* $\mathcal{S}_{25}$ *see the Figure 1.*

Next, we give the complete ordering of bicyclic sidigraphs by energy with one positive directed even cycle and one negative directed even cycle.

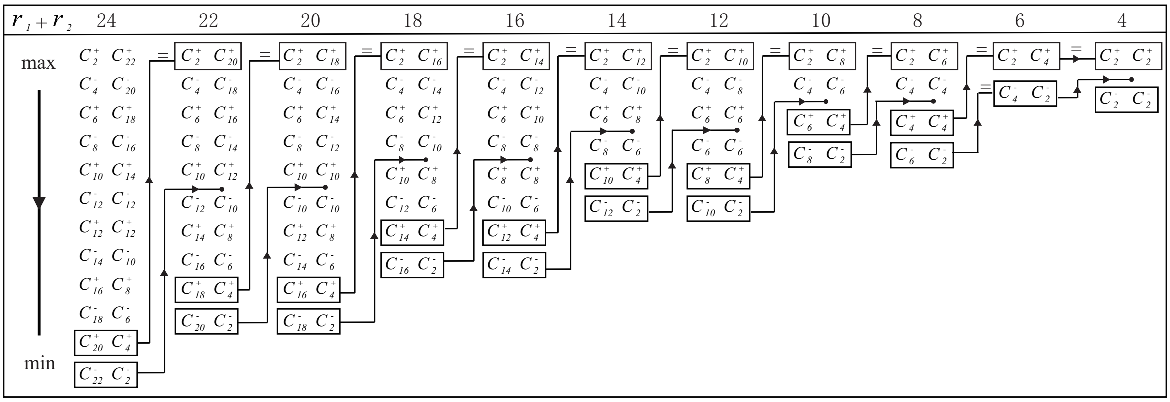

Figure 1: The complete energy ordering of bicyclic sidigraphs in $\mathcal{S}_{25}$

Theorem 3.4. *Let $\mathcal{S}_n$ be the class of bicyclic signed digraphs with n vertices whose two signed directed even cycles are vertex-disjoint, where $n \geq 4$ and $n \equiv 0 \pmod 2$. If the two directed even cycles one is positive and one is negative, then the complete ordering of bicyclic sidigraphs in $\mathcal{S}_n$ by energy is as follows:*

(i) Let $n \equiv 0 \pmod 4$. Then we have

(a) If $\frac{n}{2} \equiv 0 \pmod 4$, then

$$E(C_2^+) + E(C_{n-2}^-) > E(C_4^-) + E(C_{n-4}^+) > E(C_6^+) + E(C_{n-6}^-) > \cdots > E(C^+_{\frac{n}{2}-2}) + E(C^-_{\frac{n}{2}+2})$$
$$> E(C^-_{\frac{n}{2}}) + E(C^+_{\frac{n}{2}}) > E(C^+_{\frac{n}{2}+2}) + E(C^-_{\frac{n}{2}-2}) > \cdots > E(C^-_{n-4}) + E(C_4^+) = E(C_2^+) + E(C^-_{n-4})$$
$$> E(C_4^-) + E(C^+_{n-6}) > E(C_6^+) + E(C^-_{n-8}) > \cdots > E(C^+_{\frac{n-2}{2}-1}) + E(C^-_{\frac{n-2}{2}+1}) > E(C^+_{n-2})$$
$$+ E(C_2^-) > E(C^+_{\frac{n-2}{2}+1}) + E(C^-_{\frac{n-2}{2}-1}) > \cdots > E(C^-_{n-6}) + E(C_4^+) = E(C_2^+) + E(C^-_{n-6})$$
$$> E(C_4^-) + E(C^+_{n-8}) > E(C_6^+) + E(C^-_{n-10}) > \cdots > E(C^-_{\frac{n-4}{2}-2}) + E(C^+_{\frac{n-4}{2}+2}) > E(C^+_{\frac{n-4}{2}})$$
$$+ E(C^-_{\frac{n-4}{2}}) = E(C^+_{n-4}) + E(C_2^-) > E(C^-_{\frac{n-4}{2}+2}) + E(C^+_{\frac{n-4}{2}-2}) > \cdots > E(C^-_{n-8}) + E(C_4^+)$$
$$= E(C_2^+) + E(C^-_{n-8}) > E(C_4^-) + E(C^+_{n-10}) > E(C_6^+) + E(C^-_{n-12}) > \cdots > E(C^-_{\frac{n-6}{2}-1})$$
$$+ E(C^+_{\frac{n-6}{2}+1}) > E(C^+_{n-6}) + E(C_2^-) > E(C^-_{\frac{n-6}{2}+1}) + E(C^+_{\frac{n-6}{2}-1}) > \cdots > E(C^-_{n-10}) + E(C_4^+)$$
$$= E(C_2^+) + E(C^-_{n-10}) > E(C_4^-) + E(C^+_{n-12}) > E(C_6^+) + E(C^-_{n-14}) > \cdots > E(C^+_{\frac{n-8}{2}-2})$$
$$+ E(C^-_{\frac{n-8}{2}+2}) > E(C^-_{\frac{n-8}{2}}) + E(C^+_{\frac{n-8}{2}}) = E(C^+_{n-8}) + E(C_2^-) > E(C^+_{\frac{n-8}{2}+2}) + E(C^-_{\frac{n-8}{2}-2})$$
$$> \cdots > E(C^-_{n-12}) + E(C_4^+) = E(C_2^+) + E(C^-_{n-12}) > \cdots > E(C_4^-) + E(C_4^+) = E(C_8^+)$$
$$+ E(C_2^-) = E(C_2^+) + E(C_4^-) > E(C_6^+) + E(C_2^-) > E(C_4^+) + E(C_2^-) = E(C_2^+) + E(C_2^-).$$

(b) If $\frac{n}{2} \equiv 2 \pmod 4$, then

$$E(C_2^+) + E(C_{n-2}^-) > E(C_4^-) + E(C_{n-4}^+) > E(C_6^+) + E(C_{n-6}^-) > \cdots > E(C^-_{\frac{n}{2}-2}) + E(C^+_{\frac{n}{2}+2})$$
$$> E(C^+_{\frac{n}{2}}) + E(C^-_{\frac{n}{2}}) > E(C^-_{\frac{n}{2}+2}) + E(C^+_{\frac{n}{2}-2}) > \cdots > E(C^-_{n-4}) + E(C_4^+) = E(C_2^+) + E(C^-_{n-4})$$
$$> E(C_4^-) + E(C^+_{n-6}) > E(C_6^+) + E(C^-_{n-8}) > \cdots > E(C^-_{\frac{n-2}{2}-1}) + E(C^+_{\frac{n-2}{2}+1}) > E(C^+_{n-2})$$
$$+ E(C_2^-) > E(C^-_{\frac{n-2}{2}+1}) + E(C^+_{\frac{n-2}{2}-1}) > \cdots > E(C^-_{n-6}) + E(C_4^+) = E(C_2^+) + E(C^-_{n-6})$$
$$> E(C_4^-) + E(C^+_{n-8}) > E(C_6^+) + E(C^-_{n-10}) > \cdots > E(C^+_{\frac{n-4}{2}-2}) + E(C^-_{\frac{n-4}{2}+2}) > E(C^-_{\frac{n-4}{2}})$$
$$+ E(C^+_{\frac{n-4}{2}}) = E(C^+_{n-4}) + E(C_2^-) > E(C^+_{\frac{n-4}{2}+2}) + E(C^-_{\frac{n-4}{2}-2}) > \cdots > E(C^-_{n-8}) + E(C_4^+)$$

$$
\begin{aligned}
&= E(C_2^+) + E(C_{n-8}^-) > E(C_4^-) + E(C_{n-10}^+) > E(C_6^+) + E(C_{n-12}^-) > \cdots > E(C^+_{\frac{n-6}{2}-1}) \\
&+ E(C^-_{\frac{n-6}{2}+1}) > E(C^+_{n-6}) + E(C^-_2) > E(C^+_{\frac{n-6}{2}+1}) + E(C^-_{\frac{n-6}{2}-1}) > \cdots > E(C^-_{n-10}) + E(C^+_4) \\
&= E(C_2^+) + E(C^-_{n-10}) > E(C_4^-) + E(C^+_{n-12}) > E(C_6^+) + E(C^-_{n-14}) > \cdots > E(C^-_{\frac{n-8}{2}-2}) \\
&+ E(C^+_{\frac{n-8}{2}+2}) > E(C^+_{\frac{n-8}{2}}) + E(C^-_{\frac{n-8}{2}}) = E(C^+_{n-8}) + E(C^-_2) > E(C^-_{\frac{n-8}{2}+2}) + E(C^+_{\frac{n-8}{2}-2}) \\
&> \cdots > E(C^-_{n-12}) + E(C^+_4) = E(C_2^+) + E(C^-_{n-12}) > \cdots > E(C_4^-) + E(C_4^+) = E(C_8^+) \\
&+ E(C_2^-) = E(C_2^+) + E(C_4^-) > E(C_6^+) + E(C_2^-) > E(C_4^+) + E(C_2^-) = E(C_2^+) + E(C_2^-).
\end{aligned}
$$

(ii) Let $n \equiv 2 \pmod 4$*. Then we have*
(a) If $\frac{n}{2} - 1 \equiv 0 \pmod 4$*, then*

$$
\begin{aligned}
&E(C_2^+) + E(C_{n-2}^-) > E(C_4^-) + E(C_{n-4}^+) > E(C_6^+) + E(C_{n-6}^-) > \cdots > E(C^-_{\frac{n}{2}-1}) + E(C^+_{\frac{n}{2}+1}) \\
&> E(C^-_{\frac{n}{2}+1}) + E(C^+_{\frac{n}{2}-1}) > \cdots > E(C^-_{n-4}) + E(C^+_4) = E(C_2^+) + E(C^-_{n-4}) > E(C_4^-) + E(C^+_{n-6}) \\
&> E(C_6^+) + E(C^-_{n-8}) > \cdots > E(C^+_{\frac{n-2}{2}-2}) + E(C^-_{\frac{n-2}{2}+2}) > E(C^-_{\frac{n-2}{2}}) + E(C^+_{\frac{n-2}{2}}) = E(C^+_{n-2}) \\
&+ E(C_2^-) > E(C^+_{\frac{n-2}{2}+2}) + E(C^-_{\frac{n-2}{2}-2}) > \cdots > E(C^-_{n-6}) + E(C^+_4) = E(C_2^+) + E(C^-_{n-6}) \\
&> E(C_4^-) + E(C^+_{n-8}) > E(C_6^+) + E(C^-_{n-10}) > \cdots > E(C^+_{\frac{n-4}{2}-1}) + E(C^-_{\frac{n-4}{2}+1}) > E(C^+_{n-4}) \\
&+ E(C_2^-) > E(C^+_{\frac{n-4}{2}+1}) + E(C^-_{\frac{n-4}{2}-1}) > \cdots > E(C^-_{n-8}) + E(C^+_4) = E(C_2^+) + E(C^-_{n-8}) \\
&> E(C_4^-) + E(C^+_{n-10}) > E(C_6^+) + E(C^-_{n-12}) > \cdots > E(C^-_{\frac{n-6}{2}-2}) + E(C^+_{\frac{n-6}{2}+2}) > E(C^+_{\frac{n-6}{2}}) \\
&+ E(C^-_{\frac{n-6}{2}}) = E(C^+_{n-6}) + E(C_2^-) > E(C^-_{\frac{n-6}{2}+2}) + E(C^+_{\frac{n-6}{2}-2}) > \cdots > E(C^-_{n-10}) + E(C^+_4) \\
&= E(C_2^+) + E(C^-_{n-10}) > E(C_4^-) + E(C^+_{n-12}) > E(C_6^+) + E(C^-_{n-14}) > \cdots > E(C^-_{\frac{n-8}{2}-1}) \\
&+ E(C^+_{\frac{n-8}{2}+1}) > E(C^+_{n-8}) + E(C_2^-) > E(C^-_{\frac{n-8}{2}+1}) + E(C^+_{\frac{n-8}{2}-1}) > \cdots > E(C^-_{n-12}) + E(C^+_4) \\
&= E(C_2^+) + E(C^-_{n-12}) > \cdots > E(C_4^-) + E(C_4^+) = E(C_8^+) + E(C_2^-) = E(C_2^+) + E(C_4^-) \\
&> E(C_6^+) + E(C_2^-) > E(C_4^+) + E(C_2^-) = E(C_2^+) + E(C_2^-).
\end{aligned}
$$

(b) If $\frac{n}{2} - 1 \equiv 2 \pmod 4$*, then*

$$
\begin{aligned}
&E(C_2^+) + E(C_{n-2}^-) > E(C_4^-) + E(C_{n-4}^+) > E(C_6^+) + E(C_{n-6}^-) > \cdots > E(C^+_{\frac{n}{2}-1}) + E(C^-_{\frac{n}{2}+1}) \\
&> E(C^+_{\frac{n}{2}+1}) + E(C^-_{\frac{n}{2}-1}) > \cdots > E(C^-_{n-4}) + E(C^+_4) = E(C_2^+) + E(C^-_{n-4}) > E(C_4^-) + E(C^+_{n-6}) \\
&> E(C_6^+) + E(C^-_{n-8}) > \cdots > E(C^-_{\frac{n-2}{2}-2}) + E(C^+_{\frac{n-2}{2}+2}) > E(C^+_{\frac{n-2}{2}}) + E(C^-_{\frac{n-2}{2}}) = E(C^+_{n-2}) \\
&+ E(C_2^-) > E(C^-_{\frac{n-2}{2}+2}) + E(C^+_{\frac{n-2}{2}-2}) > \cdots > E(C^-_{n-6}) + E(C^+_4) = E(C_2^+) + E(C^-_{n-6}) \\
&> E(C_4^-) + E(C^+_{n-8}) > E(C_6^+) + E(C^-_{n-10}) > \cdots > E(C^-_{\frac{n-4}{2}-1}) + E(C^+_{\frac{n-4}{2}+1}) > E(C^+_{n-4}) \\
&+ E(C_2^-) > E(C^-_{\frac{n-4}{2}+1}) + E(C^+_{\frac{n-4}{2}-1}) > \cdots > E(C^-_{n-8}) + E(C^+_4) = E(C_2^+) + E(C^-_{n-8}) \\
&> E(C_4^-) + E(C^+_{n-10}) > E(C_6^+) + E(C^-_{n-12}) > \cdots > E(C^+_{\frac{n-6}{2}-2}) + E(C^-_{\frac{n-6}{2}+2}) > E(C^-_{\frac{n-6}{2}}) \\
&+ E(C^+_{\frac{n-6}{2}}) = E(C^+_{n-6}) + E(C_2^-) > E(C^+_{\frac{n-6}{2}+2}) + E(C^-_{\frac{n-6}{2}-2}) > \cdots > E(C^-_{n-10}) + E(C^+_4) \\
&= E(C_2^+) + E(C^-_{n-10}) > E(C_4^-) + E(C^+_{n-12}) > E(C_6^+) + E(C^-_{n-14}) > \cdots > E(C^+_{\frac{n-8}{2}-1}) \\
&+ E(C^-_{\frac{n-8}{2}+1}) > E(C^+_{n-8}) + E(C_2^-) > E(C^+_{\frac{n-8}{2}+1}) + E(C^-_{\frac{n-8}{2}-1}) > \cdots > E(C^-_{n-12}) + E(C^+_4) \\
&= E(C_2^+) + E(C^-_{n-12}) > \cdots > E(C_4^-) + E(C_4^+) = E(C_8^+) + E(C_2^-) = E(C_2^+) + E(C_4^-) \\
&> E(C_6^+) + E(C_2^-) > E(C_4^+) + E(C_2^-) = E(C_2^+) + E(C_2^-).
\end{aligned}
$$

Proof. Obviously, we have $2\cot\frac{\pi}{4} = 2\csc\frac{\pi}{2}$, and together with Lemmas 2.7-2.9, we get the ordering (i) and (ii) of bicyclic sidigraphs in $\mathcal{S}_n$ by energy. □

Remark 3.5. *Actually, let $\mathcal{S}_n$ be the class of bicyclic signed digraphs with n vertices whose two signed directed even cycles are vertex-disjoint, where $n \geq 4$ and $n \equiv 1 \pmod 2$. If the two directed even cycles one is positive and one is negative, then the complete energy ordering of bicyclic sidigraphs in $\mathcal{S}_n$ is similar to Theorem 3.4. We only need change n in Theorem 3.4 into $n-1$.*

Example 3.6. *Let $\mathcal{S}_{23}$ be the class of bicyclic signed digraphs with 23 vertices, where the two signed directed even cycles with r_1 and r_2 vertices are vertex-disjoint. If the two directed even cycles one is positive and one is negative, then the complete energy ordering of bicyclic sidigraphs in $\mathcal{S}_{23}$ see the Figure 2.*

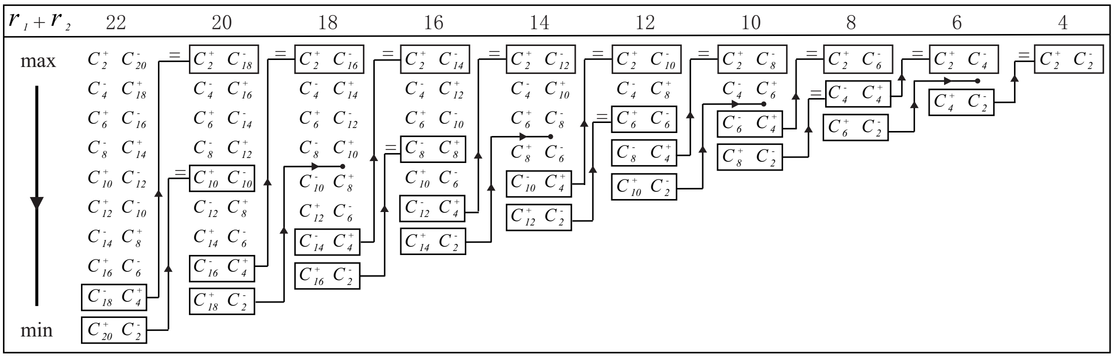

Figure 2: The complete energy ordering of bicyclic sidigraphs in $\mathcal{S}_{23}$

We can get the maximal and minimal energy of bicyclic sidigraphs in $\mathcal{S}_n$ by Theorem 3.1 and Theorem 3.4.

Theorem 3.7. *Let $\mathcal{S}_n$ be the class of bicyclic signed digraphs with n ($n \geq 4$) vertices whose two signed directed even cycles C_{r_1} and C_{r_2} are vertex-disjoint, where $2 \leq r_i \leq n-2$, $r_i \equiv 0 \pmod 2$, $i = 1, 2$. Then we have*

(i) If $n \equiv 0 \pmod 4$, then S has maximal energy if the two signed directed even cycles C_{r_1} and C_{r_2} are C_2^+ and C_{n-2}^+.

(ii) If $n \equiv 1 \pmod 4$, then S has maximal energy if the two signed directed even cycles C_{r_1} and C_{r_2} are C_2^+ and C_{n-3}^+.

(iii) If $n \equiv 2 \pmod 4$, then S has maximal energy if the two signed directed even cycles C_{r_1} and C_{r_2} are C_2^+ and C_{n-2}^-.

(iv) If $n \equiv 3 \pmod 4$, then S has maximal energy if the two signed directed even cycles C_{r_1} and C_{r_2} are C_2^+ and C_{n-3}^-.

(v) S has minimal energy if the two signed directed even cycles C_{r_1} and C_{r_2} are both C_2^-.

Proof. (i) If $n \equiv 0 \pmod 4$, by Theorem 3.1 (i), the maximal energy is $E(C_2^+) + E(C_{n-2}^+) = 2\csc\frac{\pi}{2} + 2\csc\frac{\pi}{n-2}$, and by Theorem 3.4 (i), the maximal energy is $E(C_2^+) + E(C_{n-2}^-) = 2\csc\frac{\pi}{2} + 2\cot\frac{\pi}{n-2}$. Obviously, $2\csc\frac{\pi}{2} + 2\csc\frac{\pi}{n-2} > 2\csc\frac{\pi}{2} + 2\cot\frac{\pi}{n-2}$, so S has maximal energy if the two signed directed even cycles C_{r_1} and C_{r_2} are C_2^+ and C_{n-2}^+.

The proofs of (ii)-(v) are analogous. □

Acknowledgments

The authors would like to express their gratitude to the referee for his/her helpful comments.

References

[1] N. Alawiah, N.J. Rad, A. Jahanbani, H. Kamarulhaili, New upper bounds on the energy of a graph, MATCH Commun. Math. Comput. Chem. 79 (2) (2018) 287–301.
[2] F. Belardo, M. Brunetti, A. Ciampella, On the multiplicity of α as an $A_\alpha(\Gamma)$-eigenvalue of signed graphs with pendant vertices, Discrete Math. 342 (2019) 2223–2233.
[3] F. Belardo, E.M. Li Marzi, S.K. Simić, Signed line graphs with least eigenvalue −2: the star complement technique, Discrete Appl. Math. 207 (2016) 29–38.
[4] M.A. Bhat, S. Pirzada, Spectra and energy of bipartite signed digraphs, Linear Multilinear Algebra, 64 (2016) 1863–1877.
[5] K.C. Das, S.A. Mojallal, I. Gutman, Improving McClelland's lower bound for energy, MATCH Commun. Math. Comput. Chem. 70 (2) (2013) 663–668.
[6] R. Farooq, S. Chand, M. Khan, Iota energy of bicyclic signed digraphs, Asian-Eur. J. Math. 12 (5) (2019), 1950078, 14 pp.
[7] R. Farooq, M. Khan, F. Ahmad, Extremal iota energy of bicyclic digraphs, Appl. Math. Comput. 303 (2017) 24–33.
[8] R. Farooq, M. Khan, S. Chand, On iota energy of signed digraphs, Linear Multilinear Algebra, 67 (2019) 705–724.
[9] K.A. Germina, K.S. Hameed, T. Zaslavsky, On products and line graphs of signed graphs, their eigenvalues and energy, Linear Algebra Appl. 435 (10) (2011) 2432–2450.
[10] I. Gutman, The energy of a graph, Ber. Math. Stat. Sekt. Forschungszentrum Graz, 103 (1978) 1–22.
[11] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer, Berlin, (2001) pp. 196–211.
[12] I. Gutman, X.L. Li (Eds.), Energies of Graphs – theory and applications, in: Mathematical Chemistry Monographs, No.17. University of Kragujevac, Kragujevac, 2016.
[13] J.M. Guo, H. Qian, H.Y. Shan, Z.W. Wang, Ordering of connected bipartite unicyclic graphs with large energies, MATCH Commun. Math. Comput. Chem. 82 (2) (2019) 443–458.
[14] S. Hafeez, R. Farooq, M. Khan, Bicyclic signed digraphs with maximal energy, Appl. Math. Comput. 347 (2019) 702–711.
[15] Y.P. Hou, Z.K. Tang, D.J. Wang, On signed graphs with just two distinct adjacency eigenvalues, Discrete Math. 342 (2019) 111615.
[16] M. Khan, R. Farooq, A.A. Siddiqui, On the extremal energy of bicyclic digraphs, J. Math. Inequal. 9 (2015) 799–810.
[17] M. Khan, R. Farooq, On the energy of bicyclic signed digraphs, J. Math. Inequal. 11 (2017) 845–862.
[18] W.W. Lang, L.G. Wang, Energy of generalized line graphs, Linear Algebra Appl. 437 (9) (2012) 2386–2396.
[19] X.L. Li, Y.T. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.
[20] S.C. Li, S.J. Wang, The energy of random signed graph, Linear Algebra Appl. 585 (2020) 227–1240.
[21] J. Monsalve, J. Rada, Bicyclic digraphs with maximal energy, Appl. Math. Comput. 280 (2016) 124–131.
[22] S. Pirzada, M.A. Bhat, Energy of signed digraphs, Discrete Appl. Math. 169 (2014) 195–205.
[23] I. Peña, J. Rada, Energy of digraphs, Linear Multilinear Algebra, 56 (2008) 565–578.
[24] J. Rada, I. Gutman, R. Cruz, The energy of directed hexagonal systems, Linear Algebra Appl. 439 (2013) 1825–1833.
[25] D.J. Wang, Y.P. Hou, Bicyclic signed graphs with at most one odd cycle and maximal energy, Discrete Appl. Math. 260 (2019) 244–255.
[26] X.W. Yang, L.G. Wang, On the ordering of bicyclic digraphs with respect to energy and iota energy, Appl. Math. Comput. 339 (2018) 768–778.
[27] X.W. Yang, L.G. Wang, Extremal Laplacian energy of directed trees, unicyclic digraphs and bicyclic digraphs, Appl. Math. Comput. 366 (2020) 124737.
[28] H. Yusuke, S. Iwao, A balanced signed digraph, Graphs Combin. 31 (2015) 2215–2230.