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Controlled b-Branciari Metric Type Spaces and Related Fixed Point Theorems with Applications

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Abstract. In this manuscript, we present and develop different F-contraction methods using new kinds of contractions, namely F_g-contraction and extended F_g-contraction in the context of controlled b-Branciari metric type space. We then suggest an easy and effective solution for Fredholm integral equations using the fixed point method in the framework of controlled b-Branciari metric type space. We also provide an illustrative example for the existence of solution to second order boundary value problem to demonstrate the efficiency of the work that has been developed.

1. Introduction

Since 1922, with the admired Banach fixed point theorem, fixed point theory has inspired many researchers. Czerwik [12] unveiled the idea of b-metric space as a generalization of metric space by altering the triangle inequality by inserting a constant multiple $s \geq 1$ on the right side of the equation.

Recently, Kamran et al. [17] initiated the concept of extended b-metric space in which the constant s was replaced by a non-negative function $\theta(x, y)$, where the variables x and y depends on the left-hand side of the triangle inequality. Followed by Kamran et al. many authors have dealt with extended b-metric space and proved fixed point theorems for different type of contractions. For further information about extended b-metric space, extended Branciari b-distance space, extended hexagonal b-metric space, readers can therefore refer to [2, 3, 5–10, 14, 15, 18, 23, 25, 26].

In [21], Nabil Mlaiki et al. established the banach contraction principle on new type of metric space, namely controlled metric type space, which is an expansion of b-metric space by replacing the constant s with a control function $\theta(x, y)$ to act independently on each term of the triangle inequality on the right side of the equation. In [1], the same authors established the concept of double controlled metric type space by modifying controlled metric type space through two control functions $\alpha(x, y)$ and $\mu(x, y)$, the parameters

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of which depend on the equation's right side. Recent research in exploring new generalized metric spaces (and/or its relevant results) has stimulated huge attention in metric fixed point theory, see [4, 19, 20, 22].

Inspired by the aforementioned facts, we demonstrate certain fixed point theorems using F_g-contraction and extended F_g-contraction in the context of newly established metric space, namely controlled b-Branciari metric type space, which provide solutions for fredholm integral equations using the fixed point approach.

2. Preliminaries

We begin with some basic definitions which will be applied in the sequel.

Nabil Mlaiki et al. [21], recently presented a new type of generalized b-metric space and named a controlled metric type space, which is as follows:

Definition 2.1. *Given a non-empty set X and $\alpha : X \times X \to [1, \infty)$. A function $d_\alpha : X \times X \to [0, \infty)$ is called a controlled metric type if:*

(1) $d_\alpha(t, u) = 0$ *if and only if* $t = u$;

(2) $d_\alpha(t, u) = d_\alpha(u, t)$;

(3) $d_\alpha(t, u) \leq \alpha(t, w) d_\alpha(t, w) + \alpha(w, u) d_\alpha(w, u)$

for all $t, u, w \in X$. The pair (X, d_α) is called a controlled metric type space.

Very recently, Thabet Abdeljawad et al. [3] revealed the idea of an extended Branciari b-distance, that is:

Definition 2.2. *For a non-empty set X and a mapping $\omega : X \times X \to [1, \infty)$, we say that a function $d_\omega : X \times X \to [0, \infty)$ is called an extended Branciari b-distance if:*

(1) $d_\omega(t, u) = 0$ *if and only if* $t = u$;

(2) $d_\omega(t, u) = d_\omega(u, t)$;

(3) $d_\omega(t, u) \leq \omega(t, u)[d_\omega(t, r) + d_\omega(r, s) + d_\omega(s, u)]$

for all $t, u \in X$ and all distinct $r, s \in X \backslash \{t, u\}$. The pair (X, d_ω) is called an extended Branciari b-distance space.

There was an incredible research called F-contraction, one of the most significant work in metric fixed point theory. It was implemented in 2012 by an author named Wardkowski, and with his ideological touch, he brought this growth to the mathematical society. The notion of F-contraction defined by Wardowski [27] as follows.

Definition 2.3. *Let (X, d) be a metric space. A mapping $\mathcal{G} : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $t, u \in X$,*

$$d(\mathcal{G}t, \mathcal{G}u) > 0 \Rightarrow \tau + F(d(\mathcal{G}t, \mathcal{G}u)) \leq F(d(t, u))$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying:

(1) *F is strictly increasing, i.e., for all $\mathfrak{x}, \mathfrak{y} \in \mathbb{R}^+$ such that $\mathfrak{x} < \mathfrak{y}$ implies $F(\mathfrak{x}) < F(\mathfrak{y})$;*

(2) *For each sequence $\{\mathfrak{x}_n\}_{n \in \mathbb{N}}$ of positive numbers $\lim\limits_{n \to \infty} \mathfrak{x}_n = 0$ iff $\lim\limits_{n \to \infty} F(\mathfrak{x}_n) = -\infty$;*

(3) *There exists $k \in (0, 1)$ such that $\lim\limits_{\mathfrak{x} \to 0^+} \mathfrak{x}^k F(\mathfrak{x}) = 0$.*

Later on, many researchers have extended F-contraction mappings to Reich, Geraghty and Suzuki type mappings. For instance, see [11, 22, 24, 28, 29].

In [13], Nawab Hussain et al. introduced the following new family of functions.

Definition 2.4. *Let Δ_η be the set of all functions $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following:*

(η_1) $\liminf\limits_{i \to \infty} \eta(t_i) > 0$ *for all real sequences (t_i) with $t_i > 0$;*

It is worth noting that (η_1) indicates:

(η_2) $\sum\limits_{i=0}^{\infty} \eta(t_i) = +\infty$, *for each sequence (t_i) with $t_i > 0$.*

3. Main Results

The notions, controlled metric type space and extended Branciari b-distance will be combined by the following description underneath the form of a controlled b-Branciari metric type space.

Definition 3.1. *Let X be a non-empty set and $g : X \times X \to [1, \infty)$. A function $d_g : X \times X \to [0, \infty)$ is called a controlled b-Branciari metric type if it satisfies:*

(1) $d_g(t, u) = 0$ *if and only if* $t = u$ *for all* $t, u \in X$;

(2) $d_g(t, u) = d_g(u, t)$ *for all* $t, u \in X$;

(3) $d_g(t, u) \leq g(t, r)d_g(t, r) + g(r, s)d_g(r, s) + g(s, u)d_g(s, u)$

for all $t, u \in X$ and for all distinct points $r, s \in X$, each distinct from t and u respectively. The pair (X, d_g) is called a controlled b-Branciari metric type space.

Example 3.2. *Let $X = \{1, 2, 3, 4\}$. Define $d_g : X \times X \to [0, \infty)$ as follows:*

$d_g(t, t) = 0,\ \forall t \in X,\ d_g(1, t) = d_g(t, 1) = 50,\ \forall t \in X - \{1\},$

$d_g(2, 3) = d_g(3, 2) = d_g(2, 4) = d_g(4, 2) = 200,$

$d_g(4, 3) = d_g(3, 4) = 800.$

Let $g : X \times X \to [1, \infty)$ be symmetric and can be defined as follows:

$g(t, t) = 1,\ \forall t \in X,$

$g(1, 2) = 3,\ g(1, 3) = 4,\ g(1, 4) = g(2, 3) = 5, g(2, 4) = 6,\ g(3, 4) = 2.$

Hence (X, d_g) is a controlled b-Branciari metric type space. Although, we can see that

(i) $d_g(3, 4) = 800 > g(3, 4)[d_g(3, 1) + d_g(1, 2) + d_g(2, 4)] = 600.$
(ii) $d_g(3, 4) = 800 > g(3, 1)d_g(3, 1) + g(1, 4)d_g(1, 4) = 450.$

Thus (X, d_g) is neither a controlled metric type space nor an extended Branciari b-distance space.

Now in the sense of controlled b-Branciari metric type space, we implement the following significant definitions.

Definition 3.3. *Let (X, d_g) be a controlled b-Branciari metric type space. Let $\{t_n\}$ be a sequence in X. We say that*

1. $\{t_n\}$ *is convergent, if* $\lim_{n\to\infty} d_g(t_n, t) = 0$ *for some* $t \in X$.
2. $\{t_n\}$ *is Cauchy, if* $\lim_{n,m\to\infty} d_g(t_m, t_n) = 0$.
3. (X, d_g) *is a complete controlled b-Branciari metric type space if every Cauchy sequence is convergent in X.*

Definition 3.4. *Let (X, d_g) be a controlled b-Branciari metric type space. A mapping $\mathcal{G} : X \to X$ is called an F_g-contraction if there exists function $\eta \in \Delta_\eta$ such that*

$$d_g(\mathcal{G}t, \mathcal{G}u) > 0 \Rightarrow \eta(d_g(t, u)) + F_g(d_g(\mathcal{G}t, \mathcal{G}u)) \leq F_g(d_g(t, u)),\ \forall t, u \in X \tag{1}$$

such that for each $t_0 \in X$, $\sup_{m\geq 1} \lim_{i\to\infty} g(t_{i+1}, t_{i+2})g(t_{i+1}, t_m) < \frac{1}{\lambda}$, where $t_n = \mathcal{G}^n t_0$, $n = 0, 1, \ldots$, $\lambda \in (0, 1)$ and $F_g : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying:

(F_1) *F_g is strictly increasing, i.e., for all $\mathfrak{x}, \mathfrak{y} \in \mathbb{R}^+$ such that $\mathfrak{x} < \mathfrak{y}$ implies $F_g(\mathfrak{x}) < F_g(\mathfrak{y})$;*

(F_2) *For each sequence $\{\mathfrak{x}_\mathfrak{n}\}_{\mathfrak{n}\in\mathbb{N}}$ of positive numbers $\lim_{\mathfrak{n}\to\infty} \mathfrak{x}_\mathfrak{n} = 0$ iff $\lim_{\mathfrak{n}\to\infty} F_g(\mathfrak{x}_\mathfrak{n}) = -\infty$;*

(F_3) *There exists $\lambda \in (0, 1)$ such that $\lim_{\mathfrak{x}\to 0^+} \mathfrak{x}^\lambda F_g(\mathfrak{x}) = 0$.*

We denote by $\mathcal{F}_g$, the set of all functions satisfying (F_1)-(F_3).

Theorem 3.5. *Let (X, d_g) be a complete controlled b-Branciari metric type space such that d_g is a continuous functional and $\mathcal{G} : X \to X$ is an F_g-contraction. Moreover, if*

$$\lim_{n\to\infty} g(t_n, t) \text{ and } \lim_{n\to\infty} g(t, t_n) \text{ exist and are finite, for every } t \in X. \tag{2}$$

Then, $\mathcal{G}$ has a unique fixed point in X.

Proof. Let $t_0 \in X$ be arbitrary. Construct the sequence $\{t_n\}$ by

$$t_0,\ \mathcal{G}t_0 = t_1,\ \mathcal{G}t_1 = t_2 \Rightarrow t_2 = \mathcal{G}^2 t_0,\ \ldots,\ t_{n+1} = \mathcal{G}^{n+1} t_0.$$

If there is an $k_0 \in \mathbb{N}$ such that $t_{k_0} = t_{k_0+1}$, then t_{k_0} is a fixed point of $\mathcal{G}$. We now presume that $t_n \neq t_{n+1}$ for all $n \geq 0$. This yields $d_g(t_n, t_{n+1}) > 0$, i.e., $d_g(\mathcal{G}t_{n-1}, \mathcal{G}t_n) > 0$. We shall now divide the proof into 4 steps.
***Step 1**: The first step is prove*

$$\lim_{n\to\infty} d_g(t_n, t_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d_g(t_n, t_{n+2}) = 0.$$

Taking $t = t_{n-1}$ and $u = t_n$ in (1), we get

$$\eta(d_g(t_{n-1}, t_n)) + F_g(d_g(t_n, t_{n+1})) \leq F_g(d_g(t_{n-1}, t_n)). \tag{3}$$

Consequently, we have

$$\begin{aligned} F_g(d_g(t_n, t_{n+1})) &\leq F_g(d_g(t_{n-1}, t_n)) - \eta(d_g(t_{n-1}, t_n)) \\ &\leq F_g(d_g(t_{n-2}, t_{n-1})) - \eta(d_g(t_{n-2}, t_{n-1})) - \eta(d_g(t_{n-1}, t_n)) \\ &= F_g(d_g(t_{n-2}, t_{n-1})) - [\eta(d_g(t_{n-1}, t_n)) + \eta(d_g(t_{n-2}, t_{n-1}))] \\ &\ \ \vdots \\ &\leq F_g(d_g(t_0, t_1)) - \sum_{i=1}^{n} \eta(d_g(t_{i-1}, t_i)). \end{aligned} \tag{4}$$

By using (η_2), we get

$$\lim_{n\to\infty} F_g(d_g(t_n, t_{n+1})) = -\infty \tag{5}$$

which implies

$$\lim_{n\to\infty} d_g(t_n, t_{n+1}) = 0. \tag{6}$$

From (F_3), there exists $\lambda \in (0, 1)$ such that

$$\lim_{n\to\infty} (d_g(t_n, t_{n+1}))^\lambda F_g(d_g(t_n, t_{n+1})) = 0. \tag{7}$$

By (4), we have

$$(d_g(t_n, t_{n+1}))^\lambda F_g(d_g(t_n, t_{n+1})) - (d_g(t_n, t_{n+1}))^\lambda F_g(d_g(t_0, t_1)) \leq -(d_g(t_n, t_{n+1}))^\lambda \sum_{i=1}^{n} \eta(d_g(t_{i-1}, t_i)). \tag{8}$$

By (η_1), there exists $\mathfrak{C} > 0$ such that $\eta(d_g(t_n, t_{n+1})) > \mathfrak{C}$, $\forall n > n_0$. Subsequently, we get

$$\begin{aligned} (d_g(t_n, t_{n+1}))^\lambda F_g(d_g(t_n, t_{n+1})) - (d_g(t_n, t_{n+1}))^\lambda F_g(d_g(t_0, t_1)) &\leq -(d_g(t_n, t_{n+1}))^\lambda \sum_{i=1}^{n} \eta(d_g(t_{i-1}, t_i)) \\ &= (d_g(t_n, t_{n+1}))^\lambda \Big(-[\eta(d_g(t_0, t_1)) + \eta(d_g(t_1, t_2)) + \ldots + \eta(d_g(t_{n_0-1}, t_{n_0}))] \\ &\quad - [\eta(d_g(t_{n_0}, t_{n_0+1})) + \ldots + \eta(d_g(t_{n-1}, t_n))] \Big) \\ &\leq -(d_g(t_n, t_{n+1}))^\lambda (n - n_0)\mathfrak{C}. \end{aligned} \tag{9}$$

Letting $n \to \infty$ in (9), we obtain

$$\lim_{n\to\infty} n(d_g(t_n, t_{n+1}))^\lambda = 0. \tag{10}$$

Then there exists $n_1 \in \mathbb{N}$ such that $n[d_g(t_n, t_{n+1})]^\lambda \leq 1$ for all $n \geq n_1$. Thus, we acquire

$$d_g(t_n, t_{n+1}) \leq \frac{1}{n^{\frac{1}{\lambda}}}. \tag{11}$$

Again taking $t = t_{n-1}$ and $u = t_{n+1}$ in (1), we get

$$\eta(d_g(t_{n-1}, t_{n+1})) + F_g(d_g(t_n, t_{n+2})) \leq F_g(d_g(t_{n-1}, t_{n+1})). \tag{12}$$

Accordingly, we have

$$F_g(d_g(t_n, t_{n+2})) \leq F_g(d_g(t_0, t_2)) - \sum_{i=1}^{n} \eta(d_g(t_{i-1}, t_{i+1})). \tag{13}$$

By using (η_2), we get

$$\lim_{n\to\infty} F_g(d_g(t_n, t_{n+2})) = -\infty \tag{14}$$

which implies

$$\lim_{n\to\infty} d_g(t_n, t_{n+2}) = 0. \tag{15}$$

***Step 2**: Now, we will demonstrate that $t_n \neq t_m$, for $n \neq m$. Suppose, we take $t_n = t_m$ for some $n = m + l > m$, we have $t_{n+1} = \mathcal{G}t_n = \mathcal{G}t_m = t_{m+1}$. Inequality (1), therefore implies that*

$$\begin{aligned} F_g(d_g(t_m, t_{m+1})) = F_g(d_g(t_n, t_{n+1})) &= F_g(d_g(\mathcal{G}t_{n-1}, \mathcal{G}t_n)) \\ &\leq F_g(d_g(t_{n-1}, t_n)) - \eta(d_g(t_{n-1}, t_n)) \\ &< F_g(d_g(t_{n-1}, t_n)) \\ &= F_g(d_g(\mathcal{G}t_{n-2}, \mathcal{G}t_{n-1})) \\ &\leq F_g(d_g(t_{n-2}, t_{n-1})) - \eta(d_g(t_{n-2}, t_{n-1})) \\ &\ \ \vdots \\ &< F_g(d_g(t_m, t_{m+1})) \end{aligned}$$

which is a contradiction. Hence, we conclude that $t_n \neq t_m$, for all $n \neq m$.

***Step 3**: In this step, we will attempt to demonstrate that $\{t_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence i.e., $\lim_{n\to\infty} d_g(t_n, t_{n+q}) = 0$, for $q \in \mathbb{N}$. We have already proved for the cases $q = 1$ and $q = 2$, respectively. Now choose $q \geq 1$ arbitrary. We discern between the two cases.*

Case 1: *Let $q = 2m$, where $m \geq 2$. Thereafter, we get*

$$\begin{aligned}
d_g(t_n, t_{n+2m}) &\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + g(t_{n+2}, t_{n+3})d_g(t_{n+2}, t_{n+3}) + g(t_{n+3}, t_{n+2m}) \\
&\quad d_g(t_{n+3}, t_{n+2m}) \\
&\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + g(t_{n+2}, t_{n+3})d_g(t_{n+2}, t_{n+3}) + g(t_{n+3}, t_{n+2m}) \\
&\quad [g(t_{n+3}, t_{n+4})d_g(t_{n+3}, t_{n+4}) + g(t_{n+4}, t_{n+5})d_g(t_{n+4}, t_{n+5}) + g(t_{n+5}, t_{n+2m}) \\
&\quad d_g(t_{n+5}, t_{n+2m})] \\
&\quad \vdots \\
&\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + g(t_{n+2}, t_{n+3})d_g(t_{n+2}, t_{n+3}) + \\
&\quad g(t_{n+3}, t_{n+2m})[g(t_{n+3}, t_{n+4})d_g(t_{n+3}, t_{n+4}) + g(t_{n+4}, t_{n+5})d_g(t_{n+4}, t_{n+5})] + \\
&\quad \vdots \\
&\quad g(t_{n+3}, t_{n+2m})g(t_{n+5}, t_{n+2m}) \ldots g(t_{n+2m-3}, t_{n+2m})[g(t_{n+2m-3}, t_{n+2m-2}) \\
&\quad d_g(t_{n+2m-3}, t_{n+2m-2}) + g(t_{n+2m-2}, t_{n+2m-1})d_g(t_{n+2m-2}, t_{n+2m-1})] + \\
&\quad g(t_{n+3}, t_{n+2m})g(t_{n+5}, t_{n+2m}) \ldots g(t_{n+2m-1}, t_{n+2m})d_g(t_{n+2m-1}, t_{n+2m}) \\
&\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + \sum_{i=n+2}^{n+2m-2} d_g(t_i, t_{i+1}) \prod_{j=1}^{i} g(t_j, t_{n+2m})g(t_i, t_{i+1}) \\
&\quad + \prod_{i=1}^{n+2m-1} g(t_i, t_{n+2m})d_g(t_{n+2m-1}, t_{n+2m}) \\
&\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + \sum_{i=n+2}^{n+2m-1} d_g(t_i, t_{i+1}) \prod_{j=1}^{i} g(t_j, t_{n+2m})g(t_i, t_{i+1}).
\end{aligned}$$

We observe that the series $\sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m})g(t_i, t_{i+1})$ *converges. Since,*

$$\begin{aligned}
\sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m})g(t_i, t_{i+1}) &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}} \prod_{i=1}^{n} g(t_i, t_{n+2m})g(t_i, t_{i+1}) \\
&< \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}}, \text{ which is convergent.}
\end{aligned}$$

Let

$$Y = \sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m})g(t_n, t_{n+1})$$

$$Y_n = \sum_{j=1}^{n} d_g(t_j, t_{j+1}) \prod_{i=1}^{j} g(t_i, t_{n+2m})g(t_j, t_{j+1}).$$

The aforementioned inequality therefore indicates:

$$d_g(t_n, t_{n+2m}) \leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + Y_{n+2m-1} - Y_{n+1}.$$

Letting $n \to \infty$ and using equation (15), we deduce that

$$\lim_{n\to\infty} d_g(t_n, t_{n+2m}) = 0. \tag{16}$$

Case 2: *Let $q = 2m+1$, where $m \geq 1$. Then, we find*

$$\begin{aligned}
d_g(t_n, t_{n+2m+1}) &\leq g(t_n, t_{n+1})d_g(t_n, t_{n+1}) + g(t_{n+1}, t_{n+2})d_g(t_{n+1}, t_{n+2}) + g(t_{n+2}, t_{n+2m+1}) \\
&\quad d_g(t_{n+2}, t_{n+2m+1}) \\
&\leq g(t_n, t_{n+1})d_g(t_n, t_{n+1}) + g(t_{n+1}, t_{n+2})d_g(t_{n+1}, t_{n+2}) + g(t_{n+2}, t_{n+2m+1}) \\
&\quad [g(t_{n+2}, t_{n+3})d_g(t_{n+2}, t_{n+3}) + g(t_{n+3}, t_{n+4})d_g(t_{n+3}, t_{n+4}) + g(t_{n+4}, t_{n+2m+1}) \\
&\quad d_g(t_{n+4}, t_{n+2m+1})] \\
&\quad \vdots \\
&\leq g(t_n, t_{n+1})d_g(t_n, t_{n+1}) + g(t_{n+1}, t_{n+2})d_g(t_{n+1}, t_{n+2}) + \\
&\quad g(t_{n+2}, t_{n+2m+1})[g(t_{n+2}, t_{n+3})d_g(t_{n+2}, t_{n+3}) + g(t_{n+3}, t_{n+4})d_g(t_{n+3}, t_{n+4})] + \\
&\quad \vdots \\
&\quad g(t_{n+2}, t_{n+2m+1})g(t_{n+4}, t_{n+2m+1}) \ldots g(t_{n+2m-2}, t_{n+2m+1})[g(t_{n+2m-2}, t_{n+2m-1}) \\
&\quad d_g(t_{n+2m-2}, t_{n+2m-1}) + g(t_{n+2m-1}, t_{n+2m})d_g(t_{n+2m-1}, t_{n+2m}) + \\
&\quad g(t_{n+2m}, t_{n+2m+1})d_g(t_{n+2m}, t_{n+2m+1}) \\
&\leq \sum_{i=n}^{n+2m-1} d_g(t_i, t_{i+1}) \prod_{j=1}^{i} g(t_j, t_{n+2m+1}) g(t_i, t_{i+1}) + \prod_{i=1}^{n+2m} g(t_i, t_{n+2m+1}) d_g(t_{n+2m}, t_{n+2m+1}) \\
&\leq \sum_{i=n}^{n+2m} d_g(t_i, t_{i+1}) \prod_{j=1}^{i} g(t_j, t_{n+2m+1}) g(t_i, t_{i+1}).
\end{aligned}$$

Note that the series $\sum\limits_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m+1}) g(t_i, t_{i+1})$ *converges. Since*

$$\begin{aligned}
\sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m+1}) g(t_i, t_{i+1}) &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}} \prod_{i=1}^{n} g(t_i, t_{n+2m+1}) g(t_i, t_{i+1}) \\
&< \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}}, \text{ which is convergent.}
\end{aligned}$$

Let

$$Z = \sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m+1}) g(t_n, t_{n+1})$$

$$Z_n = \sum_{j=1}^{n} d_g(t_j, t_{j+1}) \prod_{i=1}^{j} g(t_i, t_{n+2m+1}) g(t_j, t_{j+1}).$$

Thereby, the preceding inequality clearly indicates:

$$d_g(t_n, t_{n+2m+1}) \leq Z_{n+2m} - Z_{n-1}.$$

Letting $n \to \infty$ in the inequality above, we deduce that

$$\lim_{n\to\infty} d_g(t_n, t_{n+2m+1}) = 0. \tag{17}$$

Consequentially, by incorporating equations (16) and (17), we obtain

$$\lim_{n\to\infty} d_g(t_n, t_{n+q}) = 0, \text{ for all } q \in \mathbb{N}. \tag{18}$$

Hence, we infer that $\{t_n\}$ is a Cauchy sequence i.e., $\{\mathcal{G}^n t\}$ is a Cauchy sequence. Since (X, d_g) is a complete controlled b-Branciari metric type space, let $t_n \to t \in X$. We will now reveal that t is a fixed point of $\mathcal{G}$. Consider

$$d_g(t, t_{n+2}) \leq g(t, t_n)d_g(t, t_n) + g(t_n, t_{n+1})d_g(t_n, t_{n+1}) + g(t_{n+1}, t_{n+2})d_g(t_{n+1}, t_{n+2}).$$

Using (2) and (18), we obtain

$$\lim_{n\to\infty} d_g(t, t_{n+2}) = 0. \tag{19}$$

Consider

$$\begin{aligned} d_g(t, \mathcal{G}t) &\leq g(t, t_{n+2})d_g(t, t_{n+2}) + g(t_{n+2}, t_{n+1})d_g(t_{n+2}, t_{n+1}) + g(t_{n+1}, \mathcal{G}t)d_g(t_{n+1}, \mathcal{G}t) \\ &= g(t, t_{n+2})d_g(t, t_{n+2}) + g(t_{n+2}, t_{n+1})d_g(t_{n+2}, t_{n+1}) + g(t_{n+1}, \mathcal{G}t)d_g(\mathcal{G}^{n+1}t, \mathcal{G}t). \end{aligned}$$

Letting $n \to \infty$, we obtain $d_g(t, t_{n+2}) \to 0$ by (19). Since $\mathcal{G}^n t \to t$ and from the continuity of $\mathcal{G}$, $\lim_{n\to\infty} d_g(\mathcal{G}^{n+1}t, \mathcal{G}t) = 0$. Thus $d_g(t, \mathcal{G}t) = 0$, which yields $t = \mathcal{G}t$. Hence t is a fixed point of $\mathcal{G}$.

***Step 4**: Now, we will attempt to prove that t is a unique fixed point of $\mathcal{G}$. Let us assume that $\mathcal{G}$ has atmost one fixed point. Let u be an another fixed point of $\mathcal{G}$, then $\mathcal{G}u = u \neq t = \mathcal{G}t$. So, we get $d_g(t, u) > 0$ i.e., $d_g(\mathcal{G}t, \mathcal{G}u) > 0$. Now equation (1), implies*

$$\eta(d_g(t, u)) + F_g(d_g(\mathcal{G}t, \mathcal{G}u)) \leq F_g(d_g(t, u)).$$

Therefore

$$\begin{aligned} &\eta(d_g(t, u)) + F_g(d_g(t, u)) \leq F_g(d_g(t, u)) \\ &\qquad \eta(d_g(t, u)) \leq F_g(d_g(t, u)) - F_g(d_g(t, u)) = 0 \end{aligned}$$

which is a contradiction. Hence, $\mathcal{G}$ has a unique fixed point in X. □

Definition 3.6. *Let (X, d_g) be a controlled b-Branciari metric type space. A mapping $\mathcal{G} : X \to X$ is called an extended F_g-contraction if there exists function $\eta \in \Delta_\eta$ such that*

$$\begin{aligned} d_g(\mathcal{G}t, \mathcal{G}u) > 0 \Rightarrow \eta(d_g(t, u)) + F_g(d_g(\mathcal{G}t, \mathcal{G}u)) \leq F_g\Big(\gamma_1\, d_g(t, u) + \gamma_2\, \frac{d_g(t, \mathcal{G}t)}{1 + d_g(t, \mathcal{G}t)} \\ \gamma_3\, \frac{d_g(u, \mathcal{G}u)}{1 + d_g(u, \mathcal{G}u)} + \gamma_4\, \frac{d_g(t, \mathcal{G}t)\, d_g(u, \mathcal{G}u)}{d_g(t, u) + d_g(t, \mathcal{G}u) + d_g(u, \mathcal{G}t)}\Big), \ \forall t, u \in X \end{aligned} \tag{20}$$

where $F_g \in \mathcal{F}_g$, $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0$ satisfying $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 < 1$. In addition, for each $t_0 \in X$, we have

$$\sup_{m\geq 1} \lim_{i\to\infty} g(t_{i+1}, t_{i+2})g(t_{i+1}, t_m) < \frac{1}{\gamma}, \text{ here } t_n = \mathcal{G}^n t_0,\ n = 0, 1, \ldots.$$

Theorem 3.7. *Let (X, d_g) be a complete controlled b-Branciari metric type space such that d_g is a continuous functional and $\mathcal{G} : X \to X$ be an extended F_g-contraction. Then, $\mathcal{G}$ has a unique fixed point in X.*

Proof. *Let $t_0 \in X$ be arbitrary and define the sequence $\{t_n\}$ by*

$$t_0,\ \mathcal{G}t_0 = t_1,\ \mathcal{G}t_1 = t_2 \Rightarrow t_2 = \mathcal{G}^2 t_0,\ \ldots,\ t_{n+1} = \mathcal{G}^{n+1}t_0.$$

If there is an $l_0 \in \mathbb{N}$ such that $t_{l_0} = t_{l_0+1}$, then t_{l_0} is a fixed point of $\mathcal{G}$. We therefore suppose that $t_n \neq t_{n+1}$ for all $n \geq 0$. This yields $d_g(t_n, t_{n+1}) > 0$, i.e., $d_g(\mathcal{G}t_{n-1}, \mathcal{G}t_n) > 0$.

***Step 1**: In the first step, we will attempt to prove*

$$\lim_{n\to\infty} d_g(t_n, t_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d_g(t_n, t_{n+2}) = 0.$$

By using (20), for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\eta(d_g(t_{n-1},t_n)) + F_g(d_g(t_n,t_{n+1})) &\leq F_g\Big(\gamma_1\, d_g(t_{n-1},t_n) + \gamma_2\, \frac{d_g(t_{n-1},\mathcal{G}t_{n-1})}{1+d_g(t_{n-1},\mathcal{G}t_{n-1})} \\ &\qquad \gamma_3\, \frac{d_g(t_n,\mathcal{G}t_n)}{1+d_g(t_n,\mathcal{G}t_n)} + \gamma_4\, \frac{d_g(t_{n-1},\mathcal{G}t_{n-1})\, d_g(t_n,\mathcal{G}t_n)}{d_g(t_{n-1},t_n)+d_g(t_{n-1},\mathcal{G}t_n)+d_g(t_n,\mathcal{G}t_{n-1})}\Big) \\ &\leq F_g\Big(\gamma_1\, d_g(t_{n-1},t_n) + \gamma_2\, d_g(t_{n-1},t_n) \\ &\qquad \gamma_3\, d_g(t_n,t_{n+1}) + \gamma_4\, \frac{d_g(t_{n-1},t_n)\, d_g(t_n,t_{n+1})}{d_g(t_{n-1},t_n)}\Big) \\ &= F_g\Big(d_g(t_{n-1},t_n)\,(\gamma_1+\gamma_2) + d_g(t_n,t_{n+1})\,(\gamma_3+\gamma_4)\Big).\end{aligned} \tag{21}$$

This yields

$$d_g(t_n,t_{n+1}) < d_g(t_{n-1},t_n)\,(\gamma_1+\gamma_2) + d_g(t_n,t_{n+1})\,(\gamma_3+\gamma_4)$$

i.e.,

$$(1-\gamma_3-\gamma_4)d_g(t_n,t_{n+1}) \leq (\gamma_1+\gamma_2)d_g(t_{n-1},t_n).$$

As $\gamma_1+\gamma_2+\gamma_3+\gamma_4 < 1$, we have

$$d_g(t_n,t_{n+1}) \leq \frac{\gamma_1+\gamma_2}{1-\gamma_3-\gamma_4} d_g(t_{n-1},t_n) < d_g(t_{n-1},t_n).$$

From (21), we obtain

$$\eta(d_g(t_{n-1},t_n)) + F_g(d_g(t_n,t_{n+1})) \leq F(d_g(t_{n-1},t_n)).$$

Resultantly, we get

$$F_g(d_g(t_n,t_{n+1})) \leq F_g(d_g(t_0,t_1)) - \sum_{i=1}^{n} \eta(d_g(t_{i-1},t_i)).$$

By using (η_2), we get

$$\lim_{n\to\infty} F_g(d_g(t_n,t_{n+1})) = -\infty \tag{22}$$

which implies

$$\lim_{n\to\infty} d_g(t_n,t_{n+1}) = 0. \tag{23}$$

It tends to follow from the same reasoning as in the proof of Theorem (3.5) that there exist $n_1 \in \mathbb{N}$ and $\lambda \in (0,1)$ such that

$$d_g(t_n,t_{n+1}) \leq \frac{1}{n^{\frac{1}{\lambda}}}, \text{ for all } n \geq n_1.$$

Taking $t = t_{n-1}$ and $u = t_{n+1}$ in (20), we have

$$\begin{aligned}
\eta(d_g(t_{n-1}, t_{n+1})) + F_g(d_g(t_n, t_{n+2})) &\leq F_g\Big(\gamma_1\, d_g(t_{n-1}, t_{n+1}) + \gamma_2\, \frac{d_g(t_{n-1}, \mathcal{G}t_{n-1})}{1 + d_g(t_{n-1}, \mathcal{G}t_{n-1})} \\
&\quad \gamma_3\, \frac{d_g(t_{n+1}, \mathcal{G}t_{n+1})}{1 + d_g(t_{n+1}, \mathcal{G}t_{n+1})} + \gamma_4\, \frac{d_g(t_{n-1}, \mathcal{G}t_{n-1})\, d_g(t_{n+1}, \mathcal{G}t_{n+1})}{d_g(t_{n-1}, t_{n+1}) + d_g(t_{n-1}, \mathcal{G}t_{n+1}) + d_g(t_{n+1}, \mathcal{G}t_{n-1})}\Big) \\
&\leq F_g\Big(\gamma_1\, d_g(t_{n-1}, t_n) + \gamma_2\, d_g(t_{n-1}, t_n) \\
&\quad \gamma_3\, d_g(t_{n+1}, t_{n+2}) + \gamma_4\, \frac{d_g(t_{n-1}, t_n)\, d_g(t_n, t_{n+1})}{d_g(t_{n-1}, t_{n+1}) + d_g(t_{n-1}, t_{n+2}) + d_g(t_{n+1}, t_n)}\Big) \\
&\leq F_g\Big(\gamma_1\, d_g(t_{n-1}, t_n) + (\gamma_2 + \gamma_4) d_g(t_{n-1}, t_n) + \gamma_3 d_g(t_{n+1}, t_{n+2})\Big).
\end{aligned} \tag{24}$$

This gives

$$\begin{aligned}
d_g(t_n, t_{n+2}) &\leq \gamma_1\, d_g(t_{n-1}, t_n) + (\gamma_2 + \gamma_4) d_g(t_{n-1}, t_n) + \gamma_3 d_g(t_{n+1}, t_{n+2}) \\
&\leq \gamma_1 [g(t_{n-1}, t_{n+3}) d_g(t_{n-1}, t_{n+3}) + g(t_{n+3}, t_{n+2}) d_g(t_{n+3}, t_{n+2}) \\
&\quad + g(t_{n+2}, t_{n+1}) d_g(t_{n+2}, t_{n+1})] + (\gamma_2 + \gamma_4) d_g(t_{n-1}, t_n) + \gamma_3 d_g(t_{n+1}, t_{n+2}) \\
&\leq \gamma_1 [g(t_{n-1}, t_{n+3})\, [g(t_{n-1}, t_n) d_g(t_{n-1}, t_n) + g(t_n, t_{n+2}) d_g(t_n, t_{n+2}) \\
&\quad + g(t_{n+2}, t_{n+1}) d_g(t_{n+2}, t_{n+1})] + g(t_{n+3}, t_{n+2}) d_g(t_{n+3}, t_{n+2}) \\
&\quad + g(t_{n+2}, t_{n+1}) d_g(t_{n+2}, t_{n+1})] + (\gamma_2 + \gamma_4) d_g(t_{n-1}, t_n) + \gamma_3 d_g(t_{n+1}, t_{n+2}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
d_g(t_n, t_{n+2})[1 - \gamma_1 g(t_{n-1}, t_{n+3}) g(t_n, t_{n+2})] &\leq [\gamma_2 + \gamma_4 + \gamma_1 g(t_{n-1}, t_{n+3}) g(t_{n-1}, t_n)] d_g(t_{n-1}, t_n) \\
&\quad + [\gamma_1 g(t_{n+2}, t_{n+1})(1 + g(t_{n-1}, t_{n-3})] d_g(t_{n+1}, t_{n+2}) \\
&\quad + \gamma_1 g(t_{n+2}, t_{n+3}) d_g(t_{n+2}, t_{n+3}).
\end{aligned}$$

Taking into account $\lim_{n\to\infty} g(t_{n-1}, t_{n+3}) g(t_n, t_{n+2}) < \frac{1}{\gamma} < \frac{1}{\gamma_1}$ and by employing equation (23), we obtain

$$\lim_{n\to\infty} d_g(t_n, t_{n+2}) = 0. \tag{25}$$

***Step 2**: The next step is to affirm $t_n \neq t_m$, for $n \neq m$. Suppose, we claim that $t_n = t_m$ for some $n = m + k > m$, then we have $t_{n+1} = \mathcal{G}t_n = \mathcal{G}t_m = t_{m+1}$. Inequality (20), signifies that*

$$\begin{aligned}
F_g(d_g(t_m, t_{m+1})) = F_g(d_g(t_n, t_{n+1})) &= F_g(d_g(\mathcal{G}t_{n-1}, \mathcal{G}t_n)) \\
&\leq F_g((\gamma_1 + \gamma_2) d_g(t_{n-1}, t_n) + (\gamma_3 + \gamma_4) d_g(t_n, t_{n+1})) \\
&\quad - \eta(d_g(t_{n-1}, t_n)) \\
&< F_g((\gamma_1 + \gamma_2) d_g(t_{n-1}, t_n) + (\gamma_3 + \gamma_4) d_g(t_n, t_{n+1})).
\end{aligned}$$

By the property of $\mathcal{F}_g$, the above equation has been changed as

$$\begin{aligned}
d_g(t_m, t_{m+1}) = d_g(t_n, t_{n+1}) &\leq \frac{\gamma_1 + \gamma_2}{1 - \gamma_3 - \gamma_4} d_g(t_{n-1}, t_n) \\
&\leq \Big(\frac{\gamma_1 + \gamma_2}{1 - \gamma_3 - \gamma_4}\Big)^2 d_g(t_{n-2}, t_{n-1}) \\
&\quad \vdots \\
&\leq \Big(\frac{\gamma_1 + \gamma_2}{1 - \gamma_3 - \gamma_4}\Big)^n d_g(t_m, t_{m+1}) < d_g(t_m, t_{m+1})
\end{aligned}$$

which is impossible. Thus, we conclude that $t_n \neq t_m$, for all $n \neq m$.
***Step 3**: In this step, we will prove $\{t_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence i.e., $\lim_{n\to\infty} d_g(t_n, t_{n+q}) = 0$, for $q \in \mathbb{N}$. We have already done for the cases $q = 1$ and $q = 2$, respectively. Now, choose $q \geq 1$ arbitrary. We delineate between two cases.*
***Case 1:** Let $q = 2m$, where $m \geq 2$. Thereafter, we get*

$$
\begin{aligned}
d_g(t_n, t_{n+2m}) &\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + g(t_{n+2}, t_{n+3})d_g(t_{n+2}, t_{n+3}) + g(t_{n+3}, t_{n+2m})\\
&\quad d_g(t_{n+3}, t_{n+2m})\\
&\quad \vdots\\
&\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + g(t_{n+2}, t_{n+3})d_g(t_{n+2}, t_{n+3}) +\\
&\quad g(t_{n+3}, t_{n+2m})[g(t_{n+3}, t_{n+4})d_g(t_{n+3}, t_{n+4}) + g(t_{n+4}, t_{n+5})d_g(t_{n+4}, t_{n+5})] +\\
&\quad \vdots\\
&\quad g(t_{n+3}, t_{n+2m})g(t_{n+5}, t_{n+2m}) \ldots g(t_{n+2m-3}, t_{n+2m})[g(t_{n+2m-3}, t_{n+2m-2})\\
&\quad d_g(t_{n+2m-3}, t_{n+2m-2}) + g(t_{n+2m-2}, t_{n+2m-1})d_g(t_{n+2m-2}, t_{n+2m-1})] +\\
&\quad g(t_{n+3}, t_{n+2m})g(t_{n+5}, t_{n+2m}) \ldots g(t_{n+2m-1}, t_{n+2m})d_g(t_{n+2m-1}, t_{n+2m})\\
&\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + \sum_{i=n+2}^{n+2m-2} d_g(t_i, t_{i+1}) \prod_{j=1}^{i} g(t_j, t_{n+2m})g(t_i, t_{i+1}) +\\
&\quad \prod_{i=1}^{n+2m-1} g(t_i, t_{n+2m})d_g(t_{n+2m-1}, t_{n+2m})\\
&\leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + \sum_{i=n+2}^{n+2m-1} d_g(t_i, t_{i+1}) \prod_{j=1}^{i} g(t_j, t_{n+2m})g(t_i, t_{i+1}).
\end{aligned}
$$

Notice that the series $\sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m})g(t_i, t_{i+1})$ converges. Since

$$
\begin{aligned}
\sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m})g(t_i, t_{i+1}) &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}} \prod_{i=1}^{n} g(t_i, t_{n+2m})g(t_i, t_{i+1})\\
&< \frac{1}{\gamma_1} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}}, \text{ which is convergent.}
\end{aligned}
$$

Let

$$
Y = \sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m})g(t_n, t_{n+1})
$$

$$
Y_n = \sum_{j=1}^{n} d_g(t_j, t_{j+1}) \prod_{i=1}^{j} g(t_i, t_{n+2m})g(t_j, t_{j+1}).
$$

From the above inequality, it follows that

$$
d_g(t_n, t_{n+2m}) \leq g(t_n, t_{n+2})d_g(t_n, t_{n+2}) + Y_{n+2m-1} - Y_{n+1}.
$$

Letting $n \to \infty$ and using equation (25), we deduce that

$$
\lim_{n\to\infty} d_g(t_n, t_{n+2m}) = 0. \tag{26}
$$

***Case 2**: Let $q = 2m+1$, where $m \geq 1$. Then, we find*

$$\begin{aligned}
d_g(t_n, t_{n+2m+1}) &\leq g(t_n, t_{n+1})d_g(t_n, t_{n+1}) + g(t_{n+1}, t_{n+2})d_g(t_{n+1}, t_{n+2}) + g(t_{n+2}, t_{n+2m+1}) \\
&\quad d_g(t_{n+2}, t_{n+2m+1}) \\
&\quad \vdots \\
&\leq g(t_n, t_{n+1})d_g(t_n, t_{n+1}) + g(t_{n+1}, t_{n+2})d_g(t_{n+1}, t_{n+2}) + \\
&\quad g(t_{n+2}, t_{n+2m+1})[g(t_{n+2}, t_{n+3})d_g(t_{n+2}, t_{n+3}) + g(t_{n+3}, t_{n+4})d_g(t_{n+3}, t_{n+4})] \\
&\quad \vdots \\
&\quad g(t_{n+2}, t_{n+2m+1})g(t_{n+4}, t_{n+2m+1}) \ldots g(t_{n+2m-2}, t_{n+2m+1})[g(t_{n+2m-2}, t_{n+2m-1}) \\
&\quad d_g(t_{n+2m-2}, t_{n+2m-1}) + g(t_{n+2m-1}, t_{n+2m})d_g(t_{n+2m-1}, t_{n+2m}) + \\
&\quad g(t_{n+2m}, t_{n+2m+1})d_g(t_{n+2m}, t_{n+2m+1})] \\
&\leq \sum_{i=n}^{n+2m-1} d_g(t_i, t_{i+1}) \prod_{j=1}^{i} g(t_j, t_{n+2m+1})g(t_i, t_{i+1}) + \prod_{i=1}^{n+2m} g(t_i, t_{n+2m+1})d_g(t_{n+2m}, t_{n+2m+1}) \\
&\leq \sum_{i=n}^{n+2m} d_g(t_i, t_{i+1}) \prod_{j=1}^{i} g(t_j, t_{n+2m+1})g(t_i, t_{i+1}).
\end{aligned}$$

We observe that the series $\sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m+1})g(t_i, t_{i+1})$ converges. Since,

$$\begin{aligned}
\sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m+1})g(t_i, t_{i+1}) &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}} \prod_{i=1}^{n} g(t_i, t_{n+2m+1})g(t_i, t_{i+1}) \\
&< \frac{1}{\gamma_1} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}}, \text{ which is convergent.}
\end{aligned}$$

Let

$$Z = \sum_{n=1}^{\infty} d_g(t_n, t_{n+1}) \prod_{i=1}^{n} g(t_i, t_{n+2m+1})g(t_n, t_{n+1})$$

$$Z_n = \sum_{j=1}^{n} d_g(t_j, t_{j+1}) \prod_{i=1}^{j} g(t_i, t_{n+2m+1})g(t_j, t_{j+1}).$$

Eventually, the above inequality yields:

$$d_g(t_n, t_{n+2m+1}) \leq Z_{n+2m} - Z_{n-1}.$$

Letting $n \to \infty$, we deduce that

$$\lim_{n\to\infty} d_g(t_n, t_{n+2m+1}) = 0. \tag{27}$$

Therefore, by combining equations (26) and (27), we get

$$\lim_{n\to\infty} d_g(t_n, t_{n+q}) = 0, \text{ for all } q \in \mathbb{N}. \tag{28}$$

Hence, we arrive at the conclusion that $\{t_n\}$ is a Cauchy sequence i.e., $\{\mathcal{G}^n t\}$ is a Cauchy sequence. Since X is complete, let $t_n \to t \in X$. By continuity of $\mathcal{G}$, we obtain

$$t = \lim_{n\to\infty} t_{n+1} = \lim_{n\to\infty} \mathcal{G}t_n = \mathcal{G} \lim_{n\to\infty} t_n = \mathcal{G}t.$$

i.e., t is a fixed point of $\mathcal{G}$.

Step 4: *Let $u \neq t$ be an another fixed point of $\mathcal{G}$ i.e., $\mathcal{G}u = u$. It follows from equation (20) that,*

$$\begin{aligned}\eta(d_g(t,u)) + F_g(d_g(t,u)) &= \eta(d_g(t,u)) + F_g(d_g(\mathcal{G}t,\mathcal{G}u))\\ &\leq F_g\Big(\gamma_1\, d_g(t,u) + \gamma_2 \frac{d_g(t,\mathcal{G}t)}{1+d_g(t,\mathcal{G}t)} + \gamma_3 \frac{d_g(u,\mathcal{G}u)}{1+d_g(u,\mathcal{G}u)} +\\ &\quad \gamma_4 \frac{d_g(t,\mathcal{G}t)\, d_g(u,\mathcal{G}u)}{d_g(t,u)+d_g(t,\mathcal{G}u)+d_g(u,\mathcal{G}t)}\Big)\\ &\leq F_g(\gamma_1\, d_g(t,u)) < F_g(d_g(t,u))\end{aligned}$$

i.e., $\eta(d_g(t,u)) < 0$, which is a contradiction. Hence, $\mathcal{G}$ has a unique fixed point in X. □

Example 3.8. *Let $X = \{0,1,2,3\}$. Define $d_g : X \times X \to [0,\infty)$ as follows:*
$d_g(t,t) = 0,\ \forall t \in X, d_g(0,1) = d_g(1,0) = 2,$
$d_g(0,2) = d_g(2,0) = d_g(0,3) = d_g(3,0) = 3,$
$d_g(1,2) = d_g(2,1) = d_g(3,1) = d_g(1,3) = 5,$
$d(2,3) = d(3,2) = 15.$
Let $g : X \times X \to [1,\infty)$ be symmetric and can be defined as follows:
$g(t,t) = 1,\ \forall t \in X,$
$g(0,1) = g(0,2) = g(0,3) = \frac{3}{2}$
$g(1,2) = 3,\ g(1,3) = 2,\ g(2,3) = \frac{5}{4}.$
Then (X,d_g) is a complete controlled b-Branciari metric type space. Note that
(1) *(X,d_g) is not an extended Branciari b-distance space. Since*

$$d_g(3,2) = 15 > g(3,2)[d_g(3,0) + d_g(0,1) + d_g(1,2)] = 12.5$$

(2) *(X,d_g) is not a controlled metric type space. Since*

$$d_g(3,2) = 15 > g(3,0)d_g(3,0) + g(0,2)d_g(0,2) = 9.$$

Let $\mathcal{G} : X \to X$ given by $\mathcal{G}0 = \mathcal{G}1 = 0$, $\mathcal{G}2 = \mathcal{G}3 = 1$. Define $F_g : \mathbb{R}^+ \to \mathbb{R}$ by $F_g(t) = t - \frac{1}{2}$, $\forall t \in \mathbb{R}^+$ and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\eta(t) = \frac{t+1}{t+2}$, $\forall t \in \mathbb{R}^+$.

***Case 1**: Let $t = 0$. Now $d_g(\mathcal{G}0,\mathcal{G}1) = d_g(0,0) = 0$. Therefore, we only need to consider $u = 2,3$. Consider*

$$\begin{aligned}\eta(d_g(0,2)) + F_g(d_g(\mathcal{G}0,\mathcal{G}2)) &= \frac{d_g(0,2)+1}{d_g(0,2)+2} + d_g(\mathcal{G}0,\mathcal{G}2) - \frac{1}{2}\\ &= \frac{4}{5} + 2 - \frac{1}{2} = \frac{23}{10}.\end{aligned}$$

Therefore $\eta(d_g(0,2)) + F_g(d_g(\mathcal{G}0,\mathcal{G}2)) < 2.5 = F_g(d_g(0,2))$. Similarly, we can prove for $u = 3$.

Case 2: *Let $t = 2$. Now $d_g(\mathcal{G}2,\mathcal{G}3) = d_g(1,1) = 0$. Therefore, we only need to consider for $u = 1$. Consider*

$$\begin{aligned}\eta(d_g(2,1)) + F_g(d_g(\mathcal{G}2,\mathcal{G}1)) &= \frac{d_g(2,1)+1}{d_g(2,1)+2} + d_g(\mathcal{G}2,\mathcal{G}1) - \frac{1}{2}\\ &= \frac{6}{7} + 2 - \frac{1}{2} = \frac{33}{14}.\end{aligned}$$

Therefore $\eta(d_g(2,1) + F_g(d_g(\mathcal{G}2,\mathcal{G}1)) < 4.5 = F_g(d_g(2,1))$. For $t = 3$, the proof is similar as above cases. In addition, for each $t \in X$, we have $\sup_{m\geq 1} \lim_{i\to\infty} g(t_{i+1},t_{i+2})g(t_{i+1},t_m) < \frac{1}{\lambda}$, with $\lambda = \frac{1}{2}$. Furthermore, we can easily verify that $\lim_{n\to\infty} g(t_n,t)$ and $\lim_{n\to\infty} g(t,t_n)$, exist and are finite, for every $t \in X$. Thus, $\mathcal{G}$ satifies all the conditions of Theorem (3.5) and hence it has a unique fixed point, which is $t = 0$.

4. Application

In this last segment, we are attempting to apply Theorem 3.5 to prove the existence and uniqueness of the solution of the given fredholm integral equation

$$t(\mathfrak{x}) = \int_{\mathfrak{c}}^{\mathfrak{d}} \tau(\mathfrak{x}, \mathfrak{s}, t(\mathfrak{s}))d\mathfrak{s} + f(\mathfrak{x}), \ \forall \mathfrak{x}, \mathfrak{s} \in [\mathfrak{c}, \mathfrak{d}] \tag{29}$$

where $\tau, f \in C([\mathfrak{c}, \mathfrak{d}], \mathbb{R})$ (say $X = C([\mathfrak{c}, \mathfrak{d}], \mathbb{R})$). Define $d_g : X \times X \to \mathbb{R}^+$ and $g : X \times X \to [1, \infty)$ by:

$$d_g(t, u) = \sup_{\mathfrak{x} \in [\mathfrak{c}, \mathfrak{d}]} \left| t(\mathfrak{x}) - u(\mathfrak{x}) \right|^2$$

and

$$g(t, u) = \begin{cases} 1 + \sup\limits_{\mathfrak{x} \in [\mathfrak{c}, \mathfrak{d}]} |t(\mathfrak{x}) - u(\mathfrak{x})|, \text{ if } t(\mathfrak{x}) \neq u(\mathfrak{x}) \\ 1, \text{ if } t(\mathfrak{x}) = u(\mathfrak{x}) \end{cases}$$

It is clear that (X, d_g) is a complete controlled b-Branciari metric type space. We now state and prove our main result as follows:

Theorem 4.1. *Assume that for all $t, u \in C([\mathfrak{c}, \mathfrak{d}], \mathbb{R})$*

$$|\tau(\mathfrak{x}, \mathfrak{s}, t(\mathfrak{s})) - \tau(\mathfrak{x}, \mathfrak{s}, u(\mathfrak{s}))| \leq \frac{e^{-\frac{1}{|t(\mathfrak{s}) - u(\mathfrak{s})|}}}{\mathfrak{d} - \mathfrak{c}} |t(\mathfrak{s}) - u(\mathfrak{s})|, \ \forall \mathfrak{x}, \ \mathfrak{s} \in [\mathfrak{c}, \mathfrak{d}]. \tag{30}$$

Then, the integral equation (29) has a solution.

Proof. *Define $\mathcal{G} : X \to X$ by*

$$\mathcal{G}t(\mathfrak{x}) = \int_{\mathfrak{c}}^{\mathfrak{d}} \tau(\mathfrak{x}, \mathfrak{s}, t(\mathfrak{s}))d\mathfrak{s} + f(\mathfrak{x}), \ \forall \mathfrak{x}, \mathfrak{s} \in [\mathfrak{c}, \mathfrak{d}]. \tag{31}$$

We will show that the operator $\mathcal{G}$ meets the requirements of Theorem 3.5. For all $t, u \in X$, we have

$$\begin{aligned} |\mathcal{G}t(\mathfrak{x}) - \mathcal{G}u(\mathfrak{x})|^2 &\leq \Big(\int_{\mathfrak{c}}^{\mathfrak{d}} |\tau(\mathfrak{x}, \mathfrak{s}, t(\mathfrak{s})) - \tau(\mathfrak{x}, \mathfrak{s}, u(\mathfrak{s}))| \, d\mathfrak{s} \Big)^2 \\ &\leq \Big(\int_{\mathfrak{c}}^{\mathfrak{d}} \frac{e^{-\frac{1}{|t(\mathfrak{s}) - u(\mathfrak{s})|}}}{\mathfrak{d} - \mathfrak{c}} |t(\mathfrak{s}) - u(\mathfrak{s})| \, d\mathfrak{s} \Big)^2 \\ &\leq \frac{1}{(\mathfrak{d} - \mathfrak{c})^2} \, e^{-\frac{1}{\sup_{r \in [\mathfrak{c}, \mathfrak{d}]} |t(\mathfrak{x}) - u(\mathfrak{x})|^2}} \sup_{r \in [\mathfrak{c}, \mathfrak{d}]} |t(\mathfrak{x}) - u(\mathfrak{x})|^2 \Big(\int_{\mathfrak{c}}^{\mathfrak{d}} d\mathfrak{s} \Big)^2 \\ &= e^{\frac{-1}{d_g(t,u)}} d_g(t, u) \end{aligned}$$

which implies

$$d_g(\mathcal{G}t, \mathcal{G}u) \leq e^{\frac{-1}{d_g(t,u)}} d_g(t, u).$$

Taking logarithms on both sides, we acquire

$$ln(d_g(\mathcal{G}t, \mathcal{G}u)) \leq \frac{-1}{d_g(t, u)} + ln(d_g(t, u)).$$

Resultantly, we get

$$\frac{1}{d_g(t,u)} + ln(d_g(\mathcal{G}t,\mathcal{G}u)) \leq ln(d_g(t,u)). \tag{32}$$

Let us define $F_g : \mathbb{R}^+ \to \mathbb{R}$ *and* $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $F_g(\mathfrak{z}) = ln(\mathfrak{z}),\ \mathfrak{z} > 0$ *and* $\eta(\mathsf{t}) = \frac{1}{\mathsf{t}},\ \mathsf{t} \in \mathbb{R}^+$. *Thereby further, from the inequality above we get*

$$\eta(d_g(t,u)) + F_g(d_g(\mathcal{G}t,\mathcal{G}u)) \leq F_g(d_g(t,u)). \tag{33}$$

Henceforth, all the requirements of Theorem 3.5 are fulfilled. Operator $\mathcal{G}$*, therefore has a unique fixed point i.e., the fredholm integral equation has a solution.* □

Example 4.2. *The existence of solution for the succeeding second order boundary value problem is established in this section:*

$$\begin{aligned} &\mathfrak{u}''(\mathsf{r}) = \mathfrak{g}(\mathsf{r},\mathfrak{u}(\mathsf{r})), \mathsf{r} \in [0,1];\\ &\mathfrak{u}(0) = 0, \mathfrak{u}(1) = 1 \end{aligned} \tag{34}$$

where $\mathfrak{g} : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function. Let* $X = C([0,1],\mathbb{R})$*. Take into account the controlled b-Branciari metric* $'d_g'$ *and the control function* $'g'$ *specified in Theorem (4.1). Then* (X, d_g) *is a complete controlled b-Branciari metric type space.*
Problem (34) is analogous to the Fredholm integral equation of second kind which is given by

$$\mathfrak{u}(\mathsf{r}) = \mathsf{f}(\mathsf{r}) + \int_0^1 \mathsf{G}^*(\mathsf{r},\mathfrak{s})\mathfrak{u}(\mathfrak{s})d\mathfrak{s},\ \forall \mathsf{r},\mathfrak{s} \in [0,1] \tag{35}$$

where $\mathsf{f}(\mathsf{r}) = \frac{\mathsf{r}(\mathsf{r}+1)}{2}$ *and* $\mathsf{G}^*(\mathsf{r},\mathfrak{s})$ *is the Green's function given by*

$$\mathsf{G}^*(\mathsf{r},\mathfrak{s}) = \begin{cases} \mathfrak{s}^2(1-\mathsf{r}), if\ \ \mathfrak{s} \leq \mathsf{r} \\ \mathfrak{s}\mathsf{r}(1-\mathfrak{s}), if\ \ \mathsf{r} \leq \mathfrak{s} \end{cases}$$

Let $\mathcal{G} : X \to X$ *be the mapping defined by*

$$\mathfrak{u}(\mathsf{r}) = \mathsf{f}(\mathsf{r}) + \int_0^1 \mathsf{G}^*(\mathsf{r},\mathfrak{s})d\mathfrak{s},\ \forall \mathsf{r},\mathfrak{s} \in [0,1]. \tag{36}$$

Notice that if $\mathfrak{u} \in C([0,1],\mathbb{R})$ *is a fixed point of* $\mathcal{G}$*, then* $\mathfrak{u}$ *is a solution of the given boundary value problem. Suppose, we assume that* $\frac{\mathsf{r}^2}{36} \leq e^{-\frac{1}{1+|\mathfrak{u}-\mathfrak{v}|^2}}$*. In order to prove* $\mathcal{G}$ *is a* F_g*-contraction, we consider the following:*

$$\begin{aligned} |\mathcal{G}\mathfrak{u}(\mathsf{r}) - \mathcal{G}\mathfrak{v}(\mathsf{r})|^2 &\leq \Big(\int_0^1 \mathsf{G}^*(\mathsf{r},\mathfrak{s})|\mathfrak{u}(\mathfrak{s}) - \mathfrak{v}(\mathfrak{s})|\, d\mathfrak{s}\Big)^2 \\ &\leq \sup_{\mathsf{r}\in[0,1]} |\mathfrak{u}(\mathsf{r}) - \mathfrak{v}(\mathsf{r})|^2 \Big(\int_0^1 \mathsf{G}^*(\mathsf{r},\mathfrak{s})\, d\mathfrak{s}\Big)^2 \\ &= \sup_{\mathsf{r}\in[0,1]} |\mathfrak{u}(\mathsf{r}) - \mathfrak{v}(\mathsf{r})|^2 \Big(\int_0^{\mathsf{r}} \mathfrak{s}^2(1-\mathsf{r})d\mathfrak{s} + \int_{\mathsf{r}}^1 \mathfrak{s}\mathsf{r}(1-\mathfrak{s})d\mathfrak{s}\Big)^2 \\ &= \sup_{\mathsf{r}\in[0,1]} |\mathfrak{u}(\mathsf{r}) - \mathfrak{v}(\mathsf{r})|^2 \Big(\frac{\mathsf{r}}{6} - \frac{\mathsf{r}^3}{6}\Big)^2 \\ &\leq \frac{\mathsf{r}^2}{36} \sup_{\mathsf{r}\in[0,1]} |\mathfrak{u}(\mathsf{r}) - \mathfrak{v}(\mathsf{r})|^2 \leq e^{\frac{-1}{1+|\mathfrak{u}-\mathfrak{v}|^2}} \sup_{\mathsf{r}\in[0,1]} |\mathfrak{u}(\mathsf{r}) - \mathfrak{v}(\mathsf{r})|^2 \end{aligned}$$

which implies

$$\sup_{\mathrm{r}\in[0,1]} |\mathcal{G}\mathfrak{u}(\mathrm{r}) - \mathcal{G}\mathfrak{v}(\mathrm{r})|^2 \leq e^{\frac{-1}{1+d_g(\mathfrak{u},\mathfrak{v})}} \sup_{\mathrm{r}\in[0,1]} |\mathfrak{u}(\mathrm{r}) - \mathfrak{v}(\mathrm{r})|^2$$

$$\Leftrightarrow \eta(d_g(\mathfrak{u},\mathfrak{v})) + F_g(d_g(\mathcal{G}\mathfrak{u},\mathcal{G}\mathfrak{v})) \leq F_g(d_g(\mathfrak{u},\mathfrak{v}))$$

where $F_g : \mathbb{R}^+ \to \mathbb{R}$ *and* $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ *given by* $F_g(\mathfrak{z}) = ln(\mathfrak{z})$, $\mathfrak{z} > 0$ *and* $\eta(\mathrm{t}) = \frac{1}{1+\mathrm{t}}$, $\mathrm{t} \in \mathbb{R}^+$. *Thus,* $\mathcal{G}$ *has a unique fixed point in* $C([0,1],\mathbb{R})$*, which is the solution of the integral equation. Accordingly, the differential equation (34) has a solution.*

5. Conclusion

In this manuscript, we proposed the concept of controlled b-Branciari metric type space as an extension of controlled metric type space and an extended Branciari b-distance space. Thereafter, we established certain fixed point theorems pertaining F_g-contraction and extended F_g-contraction in the context of controlled b-Branciari metric type space. In addition, we have utilized our fixed point results to demonstrate the existence of solution to ordinary boundary value problem of second order.

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References

[1] T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double controlled metric type spaces and some fixed point results, Mathematics 2018 6(12) 320.

[2] T. Abdeljawad, R. P Agarwal, E. Karapinar, P. Sumati Kumari, Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space, Symmetry 2019 11(5) 686.

[3] T. Abdeljawad, E. Karapinar, S. K. Panda, N. Mlaiki, Solutions of boundary value problems on extended-Branciari b-distance, Journal of Inequalities and Applications 103 (2020).

[4] J. Ahmed, D. Lateef, Fixed point theorems for rational type $(\alpha - \Theta)$-contractions in controlled metric spaces, Journal of Nonlinear Sciences and Applications 13 (2020) 163–170.

[5] B. Alqahtani, A. Fulga, E. Karapinar, Non-unique fixed point results in extended b-metric space, Mathematics 2018 6(5) 68.

[6] B. Alqahtani, A. Fulga, E. Karapinar, Common fixed point results on extended b-metric space, Journal of Inequalities and Applications 158 (2018).

[7] B. Alqahtani, E. Karapinar, A. Ozturk, On (α, ψ)-K-contractions in the extended b-metric space, Filomat 32(15) (2018).

[8] B. Alqahtani, A. Fulga, E. Karapinar, V. Rakočević, Contractions with rational inequalities in the extended b-metric space, Journal of Inequalities and Applications 220 (2019).

[9] B. Alqahtani, E. Karapinar, F. Khojasteh, On some new fixed point results in extended strong b-metric spaces, Bulletin of Mathematical Analysis and Applications 10(3) (2018) 25–35.

[10] H. Aydi, A. Felhi, T. Kamran, E. Karapinar, M. Usman Ali, On nonlinear contractions in new extended b-metric spaces, Applications and Applied Mathematics 14(1) (2019) 537–547.

[11] H. Aydi, E. Karapinar, H. Yazidi, Modified F-contractions via α-admissible mappings and application to integral equations, Filomat 31(5) (2017) 1141–1148.

[12] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis 1 (1993) 5–11.

[13] N. Hussain, V. Parvaneh, B. A. S. Alamri, Z. Kadelburg, F-HR-type contractions on (α, η)-complete rectangular b-metric spaces, Journal of Nonlinear Sciences and Applications 10 (2017) 1030–1043.

[14] M. Jleli, B. Samet, A new generalization of the banach contraction principle, Journal of Inequalities and Applications 38 (2014).

[15] G. Kalpana, Z. Sumaiya Tasneem, Some fixed point results in extended hexagonal b-metric spaces approach to the existence of a solution to fredholm integral equations, Journal of Mathematical Analysis 11(2) (2020) 1–17.

[16] G. Kalpana, Z. Sumaiya Tasneem, On fixed points in the setting of C^*-algebra-valued controlled F_c-metric type spaces, arXiv preprint arXiv:1910.00414 (2019).

[17] T. Kamran, M. Samreen, Q. UL Ain, A generalization of b-metric space and some fixed point theorems, Mathematics 2017 5(12) 19.

[18] E. Karapinar, P. Sumati Kumari, D. Lateef, A New approach to the solution of the fredholm integral equation via a fixed point on extended b-metric spaces, Symmetry 2018 10(10) 512.

[19] D. Lateef, Fisher type fixed point results in controlled metric spaces, Journal of Mathematics and Computer Science 20 (2020) 234–240.

[20] N. Mlaiki, M. Hajji, T. Abdeljawad, A new extension of the rectangular b-metric spaces, Advances in Mathematical Physics, Article ID 8319584.

[21] N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle, Mathematics 2018 6(10) 194.

[22] S. K. Panda, T. Abdeljawad, K. Kumara Swamy, New numerical scheme for solving integral equations via fixed point method using distinct $(\omega - F)$-contractions, Alexandria Engineering Journal (2020), https://doi.org/10.1016/j.aej.2019.12.034.

[23] S. K. Panda, A. Tassaddiq, R. P Agarwal, A New approach to the solution of the non-linear integral equations via various $\mathbb{F}_{B_e}$-contractions, Symmetry 2019 11(2) 206.

[24] H. Piri, P. Kumam, Fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces, Fixed Point Theory and Applications, 90 (2016).

[25] M. Samreen, T. Kamran, M. Postolache, Extended b-metric space, extended b-comparison function and nonlinear contractions, University Politehnica of Bucharest Scientific Bulletin, Series A 80(4) (2018) 21–28.

[26] W. Shatanawi, K. Abodayeh, A. Mukheimer, Some fixed point theorems in extended b-metric spaces, University Politehnica of Bucharest Scientific Bulletin, Series A 80(4) (2018) 71–78.

[27] D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, Fixed Point Theory and Applications 94 (2012).

[28] M. Younis, D. Singh, A. Goyal, Solving existence problems via F-reich contraction, C. Constanda, P. Harris (eds.), Integral Methods in Science and Engineering, Springer Nature Switzerland AG 2019, https://doi.org/10.1007/978-3-030-16077-7_34.

[29] M. Younis, D. Singh, D. Gopal, A. Goyal, R. Mahendra Singh, On applications of generalized F-contraction to differential equations, Nonlinear Functional Analysis and Applications 24(1) (2019) 155–174.