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A New Transformation Formula Involving Derived WP-Bailey Pair and Its Applications

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Abstract. The main purpose of this paper is to obtain a new transformation formula involving the derived WP-Bailey pair. As applications, by using two ${}_{10}\Phi_9$ summation formulas, some transformation formulas in terms of generalized Lambert series are obtained.

1. Introduction

Throughout this paper, let q denote a complex number with $|q| < 1$. Here and in what follows, we adopt the standard q-series notation [8]. For any positive integer n,

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

and for convenience, we also adopt the following compact notation for the multiple q-shifted factorial:

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

where n ia an integer or ∞.

We shall make use of the basic (or q-) hypergeometric function ${}_r\Phi_s$ with r numerator and s denominator parameters, which is defined by (see [18, p. 347, Eq. 9.4 (272)]; see also [8] and [16])

$${}_r\Phi_s\left[\begin{matrix}a_1, a_2, \ldots, a_r\\ b_1, b_2, \ldots, b_s\end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \qquad (1)$$

The q-series in (1) converges for all values of z if $r < 1+s$ and for $| z |< 1$ if $r = 1+s$.

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Bailey [3] and Daum [7] independently discovered the summation formula, we also called the Bailey-Daum summation formula [8, Eq.(1.8.1)]

$$ {}_2\Phi_1\left[\begin{matrix} a, b, \\ \frac{aq}{b} \end{matrix}; q, -\frac{q}{b}\right] = \frac{(-q;q)_\infty(aq, \frac{aq^2}{b^2}; q^2)_\infty}{(\frac{aq}{b}, -\frac{q}{b}; q)_\infty}. \tag{2} $$

Following Andrews' work on the lemma and transform associated with the WP-Bailey pair (see, for details, [1]; see also [4, 5, 14, 16, 18, 19]), a WP-Bailey pair relative to the parameter a is a pair of sequences $(\alpha_n(a,k), \beta_n(a,k))$ satisfying $\alpha_0(a,k) = \beta_0(a,k) = 1$ and

$$ \begin{aligned} \beta_n(a,k) &= \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}}\alpha_r(a,k) \\ &= \frac{(k,k/a;q)_n}{(q,aq;q)_n}\sum_{r=0}^{n}\frac{(q^{-n},kq^n;q)_r}{(aq^{1-n}/k, aq^{1+n};q)_r}(\frac{aq}{k})^r\alpha_r(a,k). \end{aligned} \tag{3} $$

Bailey's definition of a conjugate Bailey pair can now be extended to define a conjugate WP-Bailey pair relative to the parameter a to be a pair of sequences $(\gamma_n(a,k), \delta_n(a,k))$ such that

$$ \begin{aligned} \gamma_n(a,k) &= \sum_{r=0}^{\infty} \frac{(k/a;q)_r(k;q)_{2n+r}}{(q;q)_r(aq;q)_{r+2n}}\delta_{r+n}(a,k) \\ &= \frac{(k;q)_{2n}}{(aq;q)_{2n}}\sum_{r=0}^{\infty}\frac{(k/a;q)_r(kq^{2n};q)_r}{(q;q)_r(aq^{2n+1};q)_r}\delta_{r+n}(a,k). \end{aligned} \tag{4} $$

The WP-Bailey pair and conjugate WP-Bailey pair play an essential role in the study of basic hypergeometric series. For example, Andrews [1] showed that there were two distinct ways to construct new WP-Bailey pairs from given pairs. This type of construction is termed a WP-Bailey chain, since the process may be iterated to deduce a chain of WP-Bailey pairs

$$ (\alpha_n, \beta_n) \to (\alpha_n^{'}, \beta_n^{'}) \to (\alpha_n^{''}, \beta_n^{''}) \to \cdots $$

from any initial pair. These two constructions allow a "tree" of WP-Bailey pairs to be generated from a single WP-Bailey pair. In [2], Andrews and Berkovich further investigated these two branches of the WP-Bailey tree, in the process obtaining many new transformations for basic hypergeometric series. Spiridonov [17] derived an elliptic generalization of Andrews first WP-Bailey chain. In [22], Warnaar added four new branches to the WP-Bailey tree, two of which had generalizations to the elliptic level. Liu and Ma [15] added one new branch to the WP-Bailey tree and Laughlin and Zimmer [12] added three new WP-Bailey chains. In [23], Zhang and Huang deduced a new WP-Bailey lattice and further application to q-series. Bhatnagar and Schlosser [6] deduced A_n, C_n and D_n extensions of the elliptic WP-Bailey transform and lemma.

Recently, analogous to the Bailey transform [5], Srivastava et al. [19, 20] considered WP-Bailey pair as well as conjugate WP-Bailey pair, and deduced the following result.

Lemma 1.1. *Let $(\alpha_n(a,k), \beta_n(a,k))$ be a WP-Bailey pair. Also let $(\gamma_n(a,k), \delta_n(a,k))$ be a conjugate WP-Bailey pair relative to the parameter a. Then, under suitable convergence conditions,*

$$ \sum_{n=0}^{\infty}\alpha_n(a,k)\gamma_n(a,k) = \sum_{n=0}^{\infty}\beta_n(a,k)\delta_n(a,k). $$

More applications of Lemma 1.1 to q-hypergeometric series can be found in [21]. For a WP-Bailey pair $(\alpha_n(a,k), \beta_n(a,k))$ and $n \in N$, define the derived WP-Bailey pairs by

$$ \alpha_n^{'}(a) = \lim_{k\to 1}\alpha_n(a,k), \tag{5} $$

and

$$\beta'_n(a) = \lim_{k\to 1} \frac{\beta_n(a,k)}{1-k}, \tag{6}$$

where suppose that each of the limits in (5) and (6) exists. For more other details, see[11, 13]. Many q-series have been represented by Ramanujan and others in terms of Lambert series of the types encountered earlier. Let $(\alpha_n(a,k), \beta_n(a,k))$ be a WP-Bailey pair and $(\alpha'_n(a), \beta'_n(a))$ the derived WP-Bailey pair and assuming suitable convergence conditions, Laughlin [10] considered limiting cases of the two WP-Bailey chains described by Andrews in [1] and obtained the following result.

$$\sum_{n=1}^{\infty} \beta'_n(a)(a^2q)^n - \sum_{n=1}^{\infty} \frac{(q;q)_{2n-1}(a^2q)^n}{(a^2q;q)_{2n}} \alpha'_n(a) = \sum_{r=1}^{\infty} \frac{a^2q^r}{1-a^2q^r} - \sum_{r=1}^{\infty} \frac{aq^r}{1-aq^r}, \tag{7}$$

where $\{|q|, |qa|, |qa^2| < 1\}$.

In [20], Srivastava et al. obtained others two new identities [20, Eq.(2.13), (2.18)].

$$\sum_{n=1}^{\infty} \beta'_n(a)a^{2n} - \frac{1}{1+a} \sum_{n=1}^{\infty} \frac{(q;q)_{2n-1}}{(a^2q;q)_{2n}} a^{2n}(1+aq^{2n})\alpha'_n(a)$$
$$= \sum_{r=1}^{\infty} \frac{a^2q^r}{1-a^2q^r} - \sum_{r=1}^{\infty} \frac{aq^r}{1-aq^r} - \frac{a}{1+a}, \tag{8}$$

where $(\alpha'_n(a), \beta'_n(a))$ is a derived WP-Bailey pair. And

$$\sum_{n=1}^{\infty} \beta'_n(a)(aq^{\frac{3}{2}})^n - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(\sqrt{q};\sqrt{q})_{n-1}}{(a\sqrt{q};\sqrt{q})_n} (a\sqrt{q})^n \alpha'_n(a)$$
$$+ \frac{1}{2} \frac{(\sqrt{q};\sqrt{q})_\infty(-a\sqrt{q};\sqrt{q})_\infty}{(a\sqrt{q};\sqrt{q})_\infty(-\sqrt{q};\sqrt{q})_\infty} \sum_{n=1}^{\infty} \frac{(-\sqrt{q};\sqrt{q})_{n-1}}{(-a\sqrt{q};\sqrt{q})_n} (a\sqrt{q})^n \alpha'_n(a)$$
$$= -\frac{1}{4} - \frac{1}{4} \frac{(\sqrt{q};\sqrt{q})_\infty(-a\sqrt{q};\sqrt{q})_\infty}{(-\sqrt{q};\sqrt{q})_\infty(a\sqrt{q};\sqrt{q})_\infty} - \sum_{r=1}^{\infty} \frac{q^r}{1-q^r}$$
$$+ \frac{1}{2} \sum_{r=1}^{\infty} \frac{q^{\frac{r}{2}}}{1-q^{\frac{r}{2}}} + \frac{1}{2} \sum_{r=1}^{\infty} \frac{aq^{\frac{r}{2}}}{1-aq^{\frac{r}{2}}} - \sum_{r=1}^{\infty} \frac{aq^{r-\frac{1}{2}}}{1-aq^{r-\frac{1}{2}}}. \tag{9}$$

where $(\alpha'_n(a), \beta'_n(a))$ is a derived WP-Bailey pair.

The main purpose of this paper is to obtain a new transformation formula from the derived WP-Bailey pair. We state our main result as follow.

Theorem 1.2. *We have*

$$\sum_{n=1}^{\infty} (-aq)^n \beta'_n(a) - \sum_{n=1}^{\infty} \frac{(q;q)_{2n-1}(-aq)^n}{(q, a^2q^2; q^2)_n} \alpha'_n(a) = \sum_{r=1}^{\infty} \frac{q^r}{1-q^{2r}} - \sum_{r=1}^{\infty} \frac{aq^r}{1-a^2q^{2r}}, \tag{10}$$

where $(\alpha'_n(a), \beta'_n(a))$ is a derived WP-Bailey pair.

The remainder of the paper is structured as follows. In Section 2, we prove Theorem 1.2. As applications, we obtain some transformation formulas from two known ${}_{10}\Phi_9$ summation formulas in Section 3.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We also believe that this method can be applied to discover and prove many others identities.

Upon setting

$$\delta_r(a,k) = (-\frac{aq}{k})^r,$$

in (4), and upon summing the series by using the Bailey-Daum summation (2), we have

$$\begin{aligned}\gamma_n(a,k) =&\frac{(k;q)_{2n}}{(aq;q)_{2n}}\sum_{r=0}^{\infty}\frac{(k/a,kq^{2n};q)_r}{(q,aq^{1+2n};q)_r}\delta_{r+n}(a,k)\\ =&\frac{(k;q)_{2n}}{(aq;q)_{2n}}(-\frac{aq}{k})^n{}_2\Phi_1\left[\begin{matrix}k/a,kq^{2n},\\ aq^{1+2n}\end{matrix};q,-\frac{aq}{k}\right]\\ =&\frac{(k;q)_{2n}}{(aq;q)_{2n}}\frac{(kq^{2n+1},a^2q^{2n+2}/k;q^2)_\infty(-q;q)_\infty}{(aq^{2n+1},-aq/k;q)_\infty}(-\frac{aq}{k})^n\\ =&\frac{(k;q)_{2n}}{(kq,a^2q^2/k;q^2)_n}\frac{(kq,a^2q^2/k;q^2)_\infty(-q;q)_\infty}{(aq,-aq/k;q)_\infty}(-\frac{aq}{k})^n.\end{aligned}$$

Substituting these values of $\delta_r(a,k)$ and $\gamma_n(a,k)$ into Lemma 1.1, we get that

$$\sum_{n=0}^{\infty}\beta_n(a,k)(-\frac{aq}{k})^n = \frac{(kq,a^2q^2/k;q^2)_\infty(-q;q)_\infty}{(aq,-aq/k;q)_\infty}\sum_{n=0}^{\infty}\frac{(k;q)_{2n}}{(kq,a^2q^2/k;q^2)_n}(-\frac{aq}{k})^n\alpha_n(a,k),$$

which can be rewritten as follow:

$$\begin{aligned}\sum_{n=1}^{\infty}\beta_n(a,k)(-\frac{aq}{k})^n-&\frac{(kq,a^2q^2/k;q^2)_\infty(-q;q)_\infty}{(aq,-aq/k;q)_\infty}\sum_{n=1}^{\infty}\frac{(k;q)_{2n}}{(kq,a^2q^2/k;q^2)_n}(-\frac{aq}{k})^n\alpha_n(a,k)\\ &=\frac{(kq,a^2q^2/k;q^2)(-q;q)_\infty}{(aq,-aq/k;q)_\infty}-1.\end{aligned}\tag{11}$$

Dividing both sides of (11) by $1-k$ and then taking the limit $k\to 1$, we obtain

$$\sum_{n=1}^{\infty}\beta_n^{'}(a)(-aq)^n-\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}(-aq)^n}{(q,a^2q^2;q^2)_n}\alpha_n^{'}(a)=\lim_{k\to 1}-\frac{d}{dk}\Big(\frac{(-q;q)_\infty(kq,a^2q^2/k;q^2)_\infty}{(aq,-aq/k;q)_\infty}\Big).\tag{12}$$

We next suppose that

$$y=\frac{(-q;q)_\infty(kq,a^2q^2/k;q^2)_\infty}{(aq,-aq/k;q)_\infty}=\frac{\prod_{r=0}^{\infty}(1+q^{r+1})\prod_{r=0}^{\infty}(1-kq^{2r+1})\prod_{r=0}^{\infty}(1-a^2q^{2r+2}/k)}{\prod_{r=0}^{\infty}(1-aq^{r+1})\prod_{r=0}^{\infty}(1+aq^{r+1}/k)}.\tag{13}$$

Taking the logarithm of both sides of (13), we get

$$\begin{aligned}\log y=&\sum_{r=0}^{\infty}\log(1+q^{r+1})+\sum_{r=0}^{\infty}\log(1-kq^{2r+1})+\sum_{r=0}^{\infty}\log(1-a^2q^{2r+2}/k)\\ &-\sum_{r=0}^{\infty}\log(1-aq^{r+1})-\sum_{r=0}^{\infty}\log(1+aq^{r+1}/k).\end{aligned}\tag{14}$$

Upon differentiating both sides of (14) ,with respect to k and then taking the limit as $k \to 1$, we have

$$\begin{aligned}\lim_{k\to1}\frac{dy}{dk} &= \lim_{k\to1}\frac{d}{dk}\Big(\frac{(-q;q)_\infty(kq,a^2q^2/k;q^2)_\infty}{(aq,-aq/k;q)_\infty}\Big)\\ &= -\sum_{r=0}^{\infty}\frac{q^{2r+1}}{1-q^{2r+1}}+\sum_{r=0}^{\infty}\frac{a^2q^{2r+2}}{1-a^2q^{2r+2}}+\sum_{r=0}^{\infty}\frac{aq^{r+1}}{1+aq^{r+1}}\\ &= -\sum_{r=1}^{\infty}\frac{q^{2r-1}}{1-q^{2r-1}}+\sum_{r=1}^{\infty}\frac{a^2q^{2r}}{1-a^2q^{2r}}+\sum_{r=1}^{\infty}\frac{aq^{r}}{1+aq^{r}}. \end{aligned}\tag{15}$$

Note from [9] that

$$\sum_{k=1}^{\infty}\frac{q^k}{1-q^{2k}}=\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1-q^{2k-1}},$$

consequently, the above identity (15) reduces to

$$\lim_{k\to1}\frac{dy}{dk} = -\sum_{r=1}^{\infty}\frac{q^{r}}{1-q^{2r}}+\sum_{r=1}^{\infty}\frac{aq^{r}}{1-a^2q^{2r}}.\tag{16}$$

Combining the above equations (12)–(16) and by a straightforward calculation, we obtain the desired identity (10).

This completes the proof.

3. Some applications

In this section, with the aid of the results that we proved two ${}_{10}\Phi_9$ summation formulae, we deduce some sporadic identities in terms of Lambert series for basic hypergeometric series.

The following summation formulas will be used from [2, Eq.(3.2)], [12, Lemma 2].

$${}_{10}\Phi_9\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},a\sqrt{\frac{q}{k}},-a\sqrt{\frac{q}{k}},\frac{a}{\sqrt{k}},-\frac{aq}{\sqrt{k}},\frac{k}{a},kq^n,q^{-n}\\ \sqrt{a},-\sqrt{a},\sqrt{kq},-\sqrt{kq},q\sqrt{k},-\sqrt{k},\frac{a^2q}{k},\frac{aq^{1-n}}{k},aq^{n+1}\end{matrix};q,q\right]=\frac{(aq,\sqrt{k},\frac{k^2}{a^2};q)_n}{(k,\frac{k}{a},q\sqrt{k};q)_n},\tag{17}$$

$${}_{10}\Phi_9\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},a\sqrt{\frac{q}{k}},-a\sqrt{\frac{q}{k}},\frac{a}{\sqrt{k}},-\frac{a}{\sqrt{k}},\frac{kq}{a},kq^n,q^{-n}\\ \sqrt{a},-\sqrt{a},\sqrt{kq},-\sqrt{kq},q\sqrt{k},-q\sqrt{k},\frac{a^2}{k},\frac{aq^{1-n}}{k},aq^{n+1}\end{matrix};q,q\right]=\frac{1-k}{1-kq^{2n}}\frac{(aq,\frac{qk^2}{a^2};q)_n}{(k,\frac{k}{a};q)_n}.\tag{18}$$

Firstly, setting

$$\alpha_n(a,k)=\frac{(a,q\sqrt{a},-q\sqrt{a},a\sqrt{\frac{q}{k}},-a\sqrt{\frac{q}{k}},\frac{a}{\sqrt{k}},-\frac{aq}{\sqrt{k}},\frac{k}{a};q)_n}{(q,\sqrt{a},-\sqrt{a},\sqrt{kq},-\sqrt{kq},q\sqrt{k},-\sqrt{k},\frac{a^2q}{k};q)_n}(\frac{k}{a})^n,$$

in (3), and using (17), then

$$\beta_n(a,k)=\frac{(k,\frac{k}{a};q)_n}{(q,aq;q)_n}\sum_{r=0}^{n}\frac{(q^{-n},kq^n;q)_r}{(\frac{aq^{1-n}}{k},aq^{1+n};q)_r}(\frac{aq}{k})^r\alpha_r(a,k)=\frac{(\sqrt{k},\frac{k^2}{a^2};q)_n}{(q,q\sqrt{k};q)_n}.$$

The derived WP-Bailey pair is obtained as follow.

Theorem 3.1. *We have*

$$\alpha_n^{'}(a) = \lim_{k\to 1} \alpha_n(a,k) = \frac{(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q}, -a\sqrt{q}, a, -aq, 1/a; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{q}, -\sqrt{q}, q, -1, a^2q; q)_n}(\frac{1}{a})^n, \tag{19}$$

$$\beta_n^{'}(a) = \lim_{k\to 1} \frac{\beta_n(a,k)}{1-k} = \frac{1}{2}\frac{(1/a^2; q)_n}{(q;q)_n(1-q^n)}, \tag{20}$$

which can be found in [10, Eq.(2.14)]. Substituting (19)–(20) into (8)–(10), respectively, we get the following three new identities.

Corollary 3.2. *We have*

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(1/a^2;q)_n a^{2n}}{(q;q)_n(1-q^n)} - \frac{1}{1+a}\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}(1+aq^{2n})}{(a^2q;q)_{2n}}\frac{(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q}, -a\sqrt{q}, a, -aq, 1/a; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{q}, -\sqrt{q}, q, -1, a^2q; q)_n}a^n$$
$$= \sum_{r=1}^{\infty}\frac{a^2q^r}{1-a^2q^r} - \sum_{r=1}^{\infty}\frac{aq^r}{1-aq^r} - \frac{a}{1+a},$$

$$\sum_{n=1}^{\infty}\frac{(1/a^2;q)_n(aq^{\frac{3}{2}})^n}{(q;q)_n(1-q^n)} - \sum_{n=1}^{\infty}\frac{(\sqrt{q};\sqrt{q})_{n-1}}{(a\sqrt{q};\sqrt{q})_n}\frac{(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q}, -a\sqrt{q}, a, -aq, 1/a; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{q}, -\sqrt{q}, q, -1, a^2q; q)_n}q^{\frac{n}{2}}$$
$$+ \frac{(\sqrt{q};\sqrt{q})_\infty(-a\sqrt{q};\sqrt{q})_\infty}{(a\sqrt{q};\sqrt{q})_\infty(-\sqrt{q};\sqrt{q})_\infty}\sum_{n=1}^{\infty}\frac{(-\sqrt{q};\sqrt{q})_{n-1}}{(-a\sqrt{q};\sqrt{q})_n}\frac{(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q}, -a\sqrt{q}, a, -aq, 1/a; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{q}, -\sqrt{q}, q, -1, a^2q; q)_n}q^{\frac{n}{2}}$$
$$= -\frac{1}{2} - \frac{1}{2}\frac{(\sqrt{q};\sqrt{q})_\infty(-a\sqrt{q};\sqrt{q})_\infty}{(-\sqrt{q};\sqrt{q})_\infty(a\sqrt{q};\sqrt{q})_\infty} - \sum_{r=1}^{\infty}\frac{2q^r}{1-q^r} + \sum_{r=1}^{\infty}\frac{q^{\frac{r}{2}}}{1-q^{\frac{r}{2}}} + \sum_{r=1}^{\infty}\frac{aq^{\frac{r}{2}}}{1-aq^{\frac{r}{2}}} - \sum_{r=1}^{\infty}\frac{2aq^{r-\frac{1}{2}}}{1-aq^{r-\frac{1}{2}}},$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(1/a^2;q)_n(-aq)^n}{(q;q)_n(1-q^n)} - \sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(q;a^2q^2;q^2)_n}\frac{(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q}, -a\sqrt{q}, a, -aq, 1/a; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{q}, -\sqrt{q}, q, -1, a^2q; q)_n}(-q)^n$$
$$= \sum_{r=1}^{\infty}\frac{q^r}{1-q^{2r}} - \sum_{r=1}^{\infty}\frac{aq^r}{1-a^2q^{2r}}.$$

Setting

$$\alpha_n(a,k) = \frac{(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{a}{\sqrt{k}}, -\frac{a}{\sqrt{k}}, \frac{kq}{a}; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{kq}, -\sqrt{kq}, q\sqrt{k}, -q\sqrt{k}, \frac{a^2}{k}; q)_n}(\frac{k}{a})^n,$$

in (3), and using (18), we have

$$\beta_n(a,k) = \frac{1-k}{1-kq^{2n}}\frac{(\frac{qk^2}{a^2};q)_n}{(q;q)_n}.$$

The derived WP-Bailey pair is obtained as follow.

Theorem 3.3. *We have*

$$\alpha_n^{'}(a) = \lim_{k\to 1} \alpha_n(a,k) = \frac{(a, q\sqrt{a}, -q\sqrt{a}, a\sqrt{q}, -a\sqrt{q}, a, -a, q/a; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{q}, -\sqrt{q}, q, -q, a^2; q)_n}(\frac{1}{a})^n, \tag{21}$$

$$\beta_n^{'}(a) = \lim_{k\to 1} \frac{\beta_n(a,k)}{1-k} = \frac{1}{1-q^{2n}}\frac{(q/a^2;q)_n}{(q;q)_n}. \tag{22}$$

Similarly, substituting (21)–(22) into (7)–(10), respectively, we can conclude four identities.

Corollary 3.4. *We have*

$$\sum_{n=1}^{\infty}\frac{(q/a^2;q)_n(a^2q)^n}{(1-q^{2n})(q;q)_n}-\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(a^2q;q)_{2n}}\frac{(a,q\sqrt{a},-q\sqrt{a},a\sqrt{q},-a\sqrt{q},a,-a,q/a;q)_n}{(q,\sqrt{a},-\sqrt{a},\sqrt{q},-\sqrt{q},q,-q,a^2;q)_n}(aq)^n$$
$$=\sum_{r=1}^{\infty}\frac{a^2q^r}{1-a^2q^r}-\sum_{r=1}^{\infty}\frac{aq^r}{1-aq^r}, \tag{23}$$

$$\sum_{n=1}^{\infty}\frac{(q/a^2;q)_na^{2n}}{(1-q^{2n})(q;q)_n}-\frac{1}{1+a}\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}(1+aq^{2n})}{(a^2q;q)_{2n}}\frac{(a,q\sqrt{a},-q\sqrt{a},a\sqrt{q},-a\sqrt{q},a,-a,q/a;q)_n}{(q,\sqrt{a},-\sqrt{a},\sqrt{q},-\sqrt{q},q,-q,a^2;q)_n}a^n$$
$$=\sum_{r=1}^{\infty}\frac{a^2q^r}{1-a^2q^r}-\sum_{r=1}^{\infty}\frac{aq^r}{1-aq^r}-\frac{a}{1+a}, \tag{24}$$

$$\sum_{n=1}^{\infty}\frac{2(q/a^2;q)_n(aq^{\frac{3}{2}})^n}{(1-q^{2n})(q;q)_n}-\sum_{n=1}^{\infty}\frac{(\sqrt{q};\sqrt{q})_{n-1}}{(a\sqrt{q};\sqrt{q})_n}\frac{(a,q\sqrt{a},-q\sqrt{a},a\sqrt{q},-a\sqrt{q},a,-a,q/a;q)_n}{(q,\sqrt{a},-\sqrt{a},\sqrt{q},-\sqrt{a},q,-q,a^2;q)_n}q^{\frac{n}{2}}$$
$$+\frac{(\sqrt{q};\sqrt{q})_\infty(-a\sqrt{q};\sqrt{q})_\infty}{(a\sqrt{q};\sqrt{q})_\infty(-\sqrt{q};\sqrt{q})_\infty}\sum_{n=1}^{\infty}\frac{(-\sqrt{q};\sqrt{q})_{n-1}}{(-a\sqrt{q};\sqrt{q})_n}\frac{(a,q\sqrt{a},-q\sqrt{a},a\sqrt{q},-a\sqrt{q},a,-a,q/a;q)_n}{(q,\sqrt{a},-\sqrt{a},\sqrt{q},-\sqrt{a},q,-q,a^2;q)_n}q^{\frac{n}{2}}$$
$$=-\frac{1}{2}-\frac{1}{2}\frac{(\sqrt{q};\sqrt{q})_\infty(-a\sqrt{q};\sqrt{q})_\infty}{(-\sqrt{q};\sqrt{q})_\infty(a\sqrt{q};\sqrt{q})_\infty}-\sum_{r=1}^{\infty}\frac{2q^r}{1-q^r}$$
$$+\sum_{r=1}^{\infty}\frac{q^{\frac{r}{2}}}{1-q^{\frac{r}{2}}}+\sum_{r=1}^{\infty}\frac{aq^{\frac{r}{2}}}{1-aq^{\frac{r}{2}}}-\sum_{r=1}^{\infty}\frac{2aq^{r-\frac{1}{2}}}{1-aq^{r-\frac{1}{2}}}, \tag{25}$$

$$\sum_{n=1}^{\infty}\frac{(q/a^2;q)_n(-aq)^n}{(1-q^{2n})(q;q)_n}-\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(q;q^2a^2;q^2)_n}\frac{(a,q\sqrt{a},-q\sqrt{a},a\sqrt{q},-a\sqrt{q},a,-a,q/a;q)_n}{(q,\sqrt{a},-\sqrt{a},\sqrt{q},-\sqrt{q},q,-q,a^2;q)_n}(-q)^n$$
$$=\sum_{r=1}^{\infty}\frac{q^r}{1-q^{2r}}-\sum_{r=1}^{\infty}\frac{aq^r}{1-a^2q^{2r}}. \tag{26}$$

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