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The Composition of Linear Canonical Wavelet Transforms on Generalized Function Spaces

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Abstract. The main goal of this paper is to study the continuity of composition of linear canonical wavelet transform (LCWTs) on generalized test function spaces $L^{p,A}, G^{p,A}$ and $B_A(\mathbb{R}^3)$. The boundedness result for composition of linear canonical wavelet transforms on $H^p_{s,A}$ is given .

1. Introduction

The comprehension of wavelets as a family of functions are constructed by using translation and dilation of a single function, called the mother wavelet. It was first introduced by Jean Morlet in 1982's as:

$$\psi_{\beta,\alpha}(x) = \frac{1}{\sqrt{\alpha}}\psi\Big(\frac{x-\beta}{\alpha}\Big), \quad \alpha > 0, \beta \in \mathbb{R}, \tag{1}$$

where α is called a scaling parameter which measures the degree of compression or scale, and β a translation parameter which determines the time location of the wavelet. Mathematically it is a square integrable function ψ on $\mathbb{R}$ which satisfies the condition

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty, \tag{2}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. Condition (2) is known as admissibility condition for mother wavelet ψ. The mother wavelet appear as a local oscillation or wave, in which most of the energy of the oscillation is located in a narrow region in the physical space. This localization in the physical space limits the localization in the frequency or wave number domain due to the uncertainty principle. The dilation parameter α controls the width and rate of this local oscillation. The translation parameter β ingenuously moves the wavelet throughout the domain. Wavelet analysis is about analyzing signal with short duration finite energy functions. They transform the signal under observation into another representation which presents the signal in a more useful form. This transformation of the signal is called wavelet transform. If scale and position are varied smoothly then the transform is called continuous wavelet transform (CWT).

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The continuous wavelet transform of a signal $f \in L^2(\mathbb{R})$ with respect to a mother wavelet $\psi_{\beta,\alpha}$ is defined as [8, 11, 12]

$$(W_\psi f)(\beta,\alpha) = \left\langle f, \psi_{\beta,\alpha} \right\rangle = \int_{-\infty}^{\infty} f(x)\overline{\psi_{\beta,\alpha}}(x)dx.$$

Using the Parseval's relation of the Fourier transform, the CWT can be rewritten as

$$(W_\psi f)(\beta,\alpha) = \sqrt{\alpha}\int_{-\infty}^{\infty} e^{i\beta\xi}\hat{f}(\xi)\overline{\hat{\psi}(\alpha\xi)}d\xi.$$

The continuous wavelet transform was developed as an alternative approach to the short time Fourier transform to overcome the resolution problem. Continuous wavelet transform has been regularly developed in the field of mathematics viz Yang et al. [45] developed the continuous wavelet transform in the framework of the local fractional calculus and Pandey et al. [28] extend the theory of classical continuous wavelet transform in multidimensional on the Schwartz tempered distribution space $\mathcal{S}'(\mathbb{R}^n)$. Moreover, Srivastava et al. [38] studied the fractional wavelet transformation and discussed some of its basic properties. Recently, continuous wavelet transform is one of the great achievements in the field of harmonic analysis [23, 42]. The continuous wavelet transform can effectively useful in the study of signal analysis, image processing, pattern recognition and it also can solve many difficult problems that cannot be solved by using Fourier transform and Laplace transform. The continuous wavelet transform involving many integral transform have emerged as one of the most useful function transform of this century with a huge application in the field of science and engineering like in signal processing [10, 20, 36], image processing [5, 7], computer vision [6], biomedical engineering [1] and geophysics [14] etc. Presently, Prasad et al. [30] and Guo et al. [15] constructed the continuous wavelet transform by using the theory of linear canonical transform (LCT) and discussed its some properties. Motivated from the previous progressive works, in this paper we proved the continuity of composition of LCWTs on some function spaces.

The LCT is a four parameter a, b, c, d class of linear integral transformation for studying the behavior of many useful transformations like Fourier transform [8, 11, 12], fractional Fourier transform [3, 24, 32–35], Fresnel transform [16], Laplace transform and scaling operator in physics and engineering in general. The LCT is a powerful mathematical tool in the understanding and solution of problems of classical mechanics [2, 3, 8, 9, 22, 24, 29, 46]. It also used to the study of Wigner-Ville distribution and ambiguity function [47, 48] as well as for windowed linear canonical transform [18]. The linear canonical transform [4, 16, 22, 46, 48] with four parameters in terms of (2×2) unimodular matrix (i.e, determinant is one) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^{-1}$ of a function $\varphi \in L^1(\mathbb{R})$ denoted by $(\mathcal{L}_A\varphi)(\xi)$ is given by

$$(\mathcal{L}_A\varphi)(\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}} \mathcal{K}_A(\xi, x)\varphi(x)dx, \tag{3}$$

where the kernel

$$\mathcal{K}_A(\xi,x) = \begin{cases} \frac{1}{\sqrt{2\pi ib}} e^{i\frac{a}{2b}x^2 - i\frac{1}{b}x\xi + i\frac{d}{2b}\xi^2}, & \text{if } b \neq 0, \\ \frac{1}{\sqrt{a}} e^{i(\frac{c}{2a})\xi^2}\delta(x - \frac{\xi}{a}), & \text{if } b = 0. \end{cases}$$

The LCT obeys the additivity and reversibility by $\mathcal{L}_A\mathcal{L}_B = \mathcal{L}_{AB}$ and $\mathcal{L}_A^{-1} = \mathcal{L}_{A^{-1}}$.
Inversion of LCT is given by

$$\varphi(x) = (\mathcal{L}_{A^{-1}}\hat{\varphi})(x) = \int_{\mathbb{R}} \mathcal{K}_{A^{-1}}(x,\xi)\hat{\varphi}(\xi)d\xi, \tag{4}$$

where $\mathcal{K}_{A^{-1}}(x,\xi)$ is the complex conjugate of $\mathcal{K}_A(\xi,x)$. For typographical convenience, we write the parameter matrix A as $A = (a,\ b\ ;\ c,\ d)$ in the text. Thought out the paper, we shall use $b \neq 0$.

1.1. Properties of linear canonical transform

Definition 1.1. *The Schwartz space $\mathcal{S}(\mathbb{R})$ is the set of rapidly decreasing complex-valued infinity differentiable functions φ on $\mathbb{R}$ such that for every choice of p and q of non-negative integers, it satisfies*

$$\gamma_{p,q}(\varphi) = \sup_{x\in\mathbb{R}} |x^p D_x^q \varphi(x)| < \infty, \quad D_x = \frac{d}{dx}.$$

Definition 1.2. *The Schwartz type space $\mathcal{S}_A(\mathbb{R})$ is defined as follows: φ is a members of $\mathcal{S}_A(\mathbb{R})$ iff is a complex-valued C^∞-function on $\mathbb{R}$ and for every choice of p and q of non-negative integers it satisfies*

$$\Gamma_{p,q}^A(\varphi) = \sup_{x\in\mathbb{R}} |x^p \Delta_x^q \varphi(x)| < \infty, \tag{5}$$

where $\Delta_x = -(\frac{d}{dx} + i\frac{a}{b}x)$ and a, b as above. If matrix $A = (0,\ 1;\ -1,\ 0)$, then it turus into Schwartz space $\mathcal{S}(\mathbb{R})$.
The space $\mathcal{S}_A(\mathbb{R})$ is equipped with the topology generated by the collection of semi-norm $\Gamma_{p,q}^A$; it is a Fréchet space. The dual of $\mathcal{S}_A$ is denoted by $\mathcal{S}_A'$; its element are called tempered distribution.
If φ is of polynomial growth and a locally integrable function on $\mathbb{R}$, then φ generates a distribution in $\mathcal{S}_A(\mathbb{R})$ as follows:

$$\left\langle \varphi, \psi \right\rangle = \int_{\mathbb{R}} \varphi(x)\psi(x)dx, \quad \psi(x) \in \mathcal{S}_A(\mathbb{R}).$$

Definition 1.3. *The canonical convolution of two functions $\varphi, \psi \in \mathcal{S}_A(\mathbb{R})$ is defined as[4]*

$$(\varphi \star_A \psi)(z) = \int_{\mathbb{R}} \varphi(x)\psi(z-x)e^{-i\frac{a}{2b}(z^2-x^2)}dx. \tag{6}$$

Lemma 1.4. *(Parseval's identity): Let $(\mathcal{L}_A\varphi)$ and $(\mathcal{L}_A\psi)$ denote the LCT of the functions $\varphi, \psi \in L^2(\mathbb{R})$ respectively, then from [4, 15, 30] we have*

$$\int_{\mathbb{R}} \varphi(x)\overline{\psi(x)}dx = \int_{\mathbb{R}} (\mathcal{L}_A\varphi)(\xi)\overline{(\mathcal{L}_A\psi)(\xi)}d\xi,$$

and

$$\int_{\mathbb{R}} |\varphi(x)|^2 dx = \int_{\mathbb{R}} |(\mathcal{L}_A\varphi)(\xi)|^2 d\xi.$$

Lemma 1.5. *Let $\varphi \in \mathcal{S}_A(\mathbb{R})$ and $\mathcal{K}_A(\xi, x)$, $\mathcal{K}_{A^{-1}}(x, \xi)$ be, respectively the kernel of the LCT and the inverse of LCT. Then*

(i) $(\Delta_x^*)^r \mathcal{K}_A(\xi, x) = \left(\frac{-i\xi}{b}\right)^r \mathcal{K}_A(\xi, x)$,

(ii) $\Delta_x^r \mathcal{K}_{A^{-1}}(x, \xi) = \left(\frac{-i\xi}{b}\right)^r \mathcal{K}_{A^{-1}}(x, \xi)$,

(iii) $\int_{\mathbb{R}} (\Delta_x^*)^r \mathcal{K}_A(\xi, x)\varphi(x)dx = \int_{\mathbb{R}} \mathcal{K}_A(\xi, x)\Delta_x^r\varphi(x)dx$,

(iv) $\int_{\mathbb{R}} \Delta_x^r \mathcal{K}_A(\xi, x)\varphi(x)dx = \int_{\mathbb{R}} \mathcal{K}_A(\xi, x)(\Delta_x^*)^r\varphi(x)dx$,

(v) $(\mathcal{L}_A(\Delta_x^r\varphi))(\xi) = \left(\frac{-i\xi}{b}\right)^r (\mathcal{L}_A\varphi)(\xi), \ \ \forall\ r \in \mathbb{N}_0$

(vi) $(\Delta_\xi^*)^r(\mathcal{L}_A\varphi)(\xi) = \mathcal{L}_A\left(\left(\frac{-ix}{b}\right)^r\varphi\right)(\xi), \ for\ \ d = a,$

(vii) $(\mathcal{L}_{A^{-1}}(\Delta_\xi^*)^r\varphi)(x) = \left(\frac{-ix}{b}\right)^r (\mathcal{L}_{A^{-1}}\varphi)(x), \ for\ \ d = a,$

where Δ_x is defined as above and $\Delta_x^* = (\frac{d}{dx} - i\frac{a}{b}x)$.

Proof. The proof of Lemma is state forward see [32]. □

Lemma 1.6. *If $\varphi(x), \psi(x) \in \mathcal{S}_A(\mathbb{R})$. Then*

$$(\Delta_x)^k[\varphi(x)\psi(x)] = \sum_{r=0}^{k} B_{k,r}(\Delta_x)^{k-r}\varphi(x)D_x^r\psi(x), \quad \forall\, k \in N_0,$$

where Δ_x defined as above and $B_{k,r}$ are constants.

Proof. Proof is straight forward as ([35], p. 207) and avoided. □

The article is organized as follows. Section 1, is introduction of LCT, in which different relation and properties of LCT are given. In section 2, we provide a brief review of the LCWT and its composition. In section 3, the generalized test function spaces $L^{p,A}$, $G^{p,A}$ and $B_A(\mathbb{R}^3)$ are introduced and proved the continuity of the linear canonical composition operator $(W^A\varphi)(\beta,\alpha,\gamma)$ on $L^{p,A}$, $G^{p,A}$ and $B_A(\mathbb{R}^3)$. Finally in section 4, boundedness result of $(W^A\varphi)(\beta,\alpha,\gamma)$ on the space $H^p_{s,A}(\mathbb{R})$ is given.

2. Linear canonical wavelet transform and its composition

Prasad et al. [30] and Guo et al. [15] defined the canonical wavelet as:

$$\psi_{\beta,\alpha,A}(x) = \frac{1}{\sqrt{\alpha}}\psi\Big(\frac{x-\beta}{\alpha}\Big)e^{-i\frac{a}{2b}(x^2-\beta^2)}, \tag{7}$$

and the linear canonical transform of $\psi_{\beta,\alpha,A}(x)$ is given by

$$\hat{\psi}_{\beta,\alpha,A}(\xi) = \sqrt{\alpha}e^{i\frac{d}{2b}\xi^2-\frac{i}{b}\beta\xi+i\frac{a}{2b}\beta^2-i\frac{d}{2b}\alpha^2\xi^2}\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi](\alpha\xi), \tag{8}$$

for all α, β and A as above.

Definition 2.1. *(Admissibility condition). A function $\psi \in L^2(\mathbb{R})$ is known as canonical wavelet. if it satisfies the following admissibility condition:*

$$C_{\psi,A} = \int_{\mathbb{R}} \frac{|\mathcal{L}_A(e^{-i\frac{a}{2b}(\cdot)^2}\psi)|^2}{|\xi|}d\xi < \infty. \tag{9}$$

As per [15, 30] the linear canonical wavelet transform (LCWT) of $\varphi \in L^2(\mathbb{R})$ with canonical wavelet $\psi_{\beta,\alpha,A}(x)$ is defined by

$$(W^A_\psi\varphi)(\beta,\alpha) = \int_{\mathbb{R}} \varphi(x)\overline{\psi_{\beta,\alpha,A}}(x)dx. \tag{10}$$

Now, using Parseval's relation for linear canonical transform and (8), the above expression for linear canonical wavelet transform $(W^A_\psi\varphi)(\beta,\alpha)$ can be written as:

$$(W^A_\psi\varphi)(\beta,\alpha) = \sqrt{\alpha}\int_{\mathbb{R}} e^{-i\frac{d}{2b}\xi^2+\frac{i}{b}\beta\xi-i\frac{a}{2b}\beta^2+i\frac{d}{2b}\alpha^2\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi]}(\alpha\xi)(\mathcal{L}_A\varphi)(\xi)d\xi, \tag{11}$$

and using inversion formula of linear canonical transform, we have

$$\mathcal{L}_A\{(W^A_\psi\varphi)(\beta,\alpha)\} = \sqrt{\frac{2\pi b\alpha}{i}}e^{i\frac{d}{2b}(\alpha\xi)^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi]}(\alpha\xi)(\mathcal{L}_A\varphi)(\xi). \tag{12}$$

The linear canonical wavelet transform is a generalization of continuous wavelet transform with the parameters of matrix A. If we put in the matrix $A = (0,\ 1;\ -1,\ 0)$, then it reduced to the conventional continuous wavelet transform. The earlier work on the linear canonical wavelet transform was published by authors

[15, 30, 43, 44]. The wavelet transform has major applications in the field of image and signal processing, mathematical analysis, communications, radar and other [7, 13, 19, 21, 37, 39, 40].

Let $(W^A_{\psi_1}\varphi)(\beta,\alpha)$ and $(W^A_{\psi_2}\varphi)(\rho,\gamma)$ be two LCWT of a function $\varphi \in L^2(\mathbb{R})$ w.r.t. canonical wavelets $\psi_{1,\beta,\alpha,A} \in L^2(\mathbb{R})$ and $\psi_{2,\rho,\gamma,A} \in L^1(\mathbb{R})$ respectively defined as (10), for all $\alpha,\gamma > 0, \beta,\rho \in \mathbb{R}$ and A is the unimodular matrix as above. Then their canonical composition operator is defined as [31]:

$$\begin{aligned}
\Big(W^A\varphi\Big)(\beta,\alpha,\gamma) &= W^A_{\psi_1}\Big(W^A_{\psi_2}\varphi\Big)(\beta,\alpha,\gamma)\\
&= \int_{\mathbb{R}} (W^A_{\psi_2}\varphi)(\rho,\gamma)\overline{\psi_{1,\beta,\alpha,A}}(\rho)d\rho \qquad (13)\\
&= \int_{\mathbb{R}} \mathcal{L}^\rho_A\{(W^A_{\psi_2}\varphi)\}(\xi,\gamma)\overline{\mathcal{L}_A\{\psi_{1,\beta,\alpha,A}\}(\xi)}d\xi\\
&= \sqrt{\alpha}\int_{\mathbb{R}} e^{-i\frac{d}{2b}\xi^2+\frac{i}{b}\beta\xi-i\frac{a}{2b}\beta^2+i\frac{d}{2b}\alpha^2\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\mathcal{L}^\rho_A\{(W^A_{\psi_2}\varphi)\}(\xi,\gamma)d\xi\\
&= \sqrt{\frac{2\pi b\alpha\gamma}{i}}\int_{\mathbb{R}} e^{-i\frac{d}{2b}\xi^2+\frac{i}{b}\beta\xi-i\frac{a}{2b}\beta^2+i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\\
&\quad\times\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi)d\xi, \qquad (14)
\end{aligned}$$

where $\mathcal{L}^\rho_A$ denotes the linear canonical transform w.r.t. the variable ρ.
Therefore,

$$\mathcal{L}_A\{(W^A\varphi)(\beta,\alpha,\gamma)\} = \frac{2\pi b\sqrt{\alpha\gamma}}{i}e^{i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi) \qquad (15)$$

Definition 2.2. *(Admissibility condition). Let $\psi_1 \in L^2(\mathbb{R})$, $\psi_2 \in L^1(\mathbb{R})$ be such that there exists a positive constant $C_{\psi_1,\psi_2,A} < \infty$ and for ξ almost every where on $\mathbb{R}$*

$$C_{\psi_1,\psi_2,A} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} |\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1](\alpha\xi)|^2|\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2](\gamma\xi)|^2\frac{d\alpha d\gamma}{\alpha\gamma}, \qquad (16)$$

where $\mathcal{L}_A$ denotes the LCT defined as (3).

3. Composition of LCWTs on generalized test function spaces

In this section, the generalized test function spaces $L^{p,A}$, $G^{p,A}$ and $B_A(\mathbb{R}^3)$ are introduced and proved the continuity of composition of LCWTs on that spaces.

Definition 3.1. *For $1 \le p < \infty$, we define the generalized test function space $L^{p,A}$ by [27]:*

$$L^{p,A} = \Big\{\varphi : \varphi \in C^\infty(\mathbb{R}) \text{ and } x^r\Delta^s_x\varphi(x) \in L^p,\ \forall r,s \in \mathbb{N}_0\Big\}, \qquad (17)$$

where Δ_x is defined as above with the usual point wise operation of addition and scalar multiplication, $L^{p,A}$ becomes a linear space for all $\varphi \in L^{p,A}$, its norm defined as

$$\big\|\varphi\big\|_{L^{p,A}} = \big\|x^r\Delta^s_x\varphi\big\|_{L^p}, \quad \forall r,s \in \mathbb{N}_0. \qquad (18)$$

The space of all continuous linear functionals (distributions) on $L^{p,A}$ is denoted $(L^{p,A})'$ and is called dual space of $L^{p,A}$.

Definition 3.2. *The generalized test function space $G^{p,A}(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$ is defined, for $1 \le p < \infty$, $\alpha,\gamma > 0$ and $\beta \in \mathbb{R}$, by*

$$\begin{aligned}
G^{p,A}(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+) &= \Big\{\varphi(\beta,\alpha,\gamma) \in C^\infty(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+) :\\
&\Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\alpha^p\beta^q\gamma^r D^s_\gamma D^t_\alpha\Delta^k_\beta\Big[\frac{\varphi(\beta,\alpha,\gamma)}{\sqrt{\alpha\gamma}}\Big]d\alpha d\gamma\Big|^p d\beta\Big)^{\frac{1}{p}} < \infty\Big\}, \qquad (19)
\end{aligned}$$

where $\Delta_\beta = -(\frac{d}{d\beta} + i\frac{a}{b}\beta)$ *and* $p, q, r, s, t, k \in \mathbb{N}_0$ *satisfying* $k + t + s - p - r - 2 \geq 0$. *For all* $(W^A\varphi)(\beta, \alpha, \gamma) \in G^{p,A}$, *its norm is defined by*

$$\left\|\varphi(\beta,\alpha,\gamma)\right\|_{G^{p,A}} = \Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\alpha^p\beta^q\gamma^r D^s_\gamma D^t_\alpha \Delta^k_\beta\Big[\frac{\varphi(\beta,\alpha,\gamma)}{\sqrt{\alpha\gamma}}\Big]d\alpha d\gamma\Big|^p d\beta\Big)^{\frac{1}{p}}. \tag{20}$$

Theorem 3.3. *If* $a = d$ *and assume* $\varphi \in L^{p,A}$ *and* $\psi_1, \psi_2 \in \mathcal{S}_A(\mathbb{R})$ *satisfies the condition:*

$$\begin{aligned} 0 < C'_{\psi_1,\psi_2,A} &= \Big|\Big(\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\tau^p\nu^r D^t_\tau D^s_\nu\Big[e^{i\frac{d}{2b}(\tau^2+\nu^2)}\overline{\mathcal{L}_A[(e^{-i\frac{a}{2b}(\cdot)^2}\psi_1)]}(\tau)\\ &\times \overline{\mathcal{L}_A[(e^{-i\frac{a}{2b}(\cdot)^2}\psi_2)]}(\nu)\Big]d\tau d\nu\Big)\Big| < \infty \end{aligned} \tag{21}$$

then, the canonical composition operator $(W^A\varphi)(\beta, \alpha, \gamma) : L^{p,A}(\mathbb{R}) \to G^{p,A}(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ *is linear and continuous.*

Proof. Let $p, q, r, s, t, k \in \mathbb{N}_0$ be such that $k + t + s - p - r - 2 \geq 0$ and using (14) then, we have

$$\begin{aligned} I &= \alpha^p\beta^q\gamma^r D^s_\gamma D^t_\alpha(\Delta_\beta)^k\Big[\frac{(W^A\varphi)(\beta,\alpha,\gamma)}{\sqrt{\alpha\gamma}}\Big]\\ &= \alpha^p\beta^q\gamma^r D^s_\gamma D^t_\alpha(\Delta_\beta)^k\int_{\mathbb{R}}\sqrt{\frac{2\pi b}{i}}e^{-i\frac{d}{2b}\xi^2+\frac{i}{b}\beta\xi-i\frac{a}{2b}\beta^2+i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\\ &\quad\times\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi)d\xi\\ &= \alpha^p\beta^q\gamma^r D^s_\gamma D^t_\alpha\int_{\mathbb{R}}\sqrt{\frac{2\pi b}{i}}e^{-i\frac{d}{2b}\xi^2+\frac{i}{b}\beta\xi-i\frac{a}{2b}\beta^2+i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\\ &\quad\times\Big(\frac{-i\xi}{b}\Big)^k\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi)d\xi. \end{aligned}$$

Using the change of order of integration and putting $\alpha\xi = \tau$ and $\gamma\xi = \nu$. Hence

$$\begin{aligned} \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} Id\alpha d\gamma &= \sqrt{\frac{2\pi b}{i}}\Big(\frac{-i}{b}\Big)^k\beta^q\int_{\mathbb{R}}e^{-i\frac{d}{2b}\xi^2+\frac{i}{b}\beta\xi-i\frac{a}{2b}\beta^2}\ \xi^k(\mathcal{L}_A\varphi)(\xi)\\ &\quad\times\Big(\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\alpha^p\gamma^r D^s_\gamma D^t_\alpha\Big[e^{i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)\Big]d\alpha d\gamma\Big)d\xi\\ &= \sqrt{\frac{2\pi b}{i}}\Big(\frac{-i}{b}\Big)^k\beta^q\int_{\mathbb{R}}e^{-i\frac{d}{2b}\xi^2+\frac{i}{b}\beta\xi-i\frac{a}{2b}\beta^2}\ \xi^{k+t+s-p-r-2}(\mathcal{L}_A\varphi)(\xi)\\ &\quad\times\Big(\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\tau^p\nu^r D^t_\tau D^s_\nu\Big[e^{i\frac{d}{2b}(\tau^2+\nu^2)}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\tau)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\nu)\Big]d\tau d\nu\Big)d\xi. \end{aligned}$$

Taking modulus both side and using (21), we have

$$\begin{aligned} \Big|\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} Id\alpha d\gamma\Big| &\leq \sqrt{2\pi}C'_{\psi_1,\psi_2,A}\Big|\frac{1}{b}\Big|^{k-\frac{1}{2}}\Big|\beta^q\int_{\mathbb{R}}e^{-i\frac{d}{2b}\xi^2+\frac{i}{b}\beta\xi-i\frac{a}{2b}\beta^2}\xi^{k+t+s-p-r-2}(\mathcal{L}_A\varphi)(\xi)d\xi\Big|\\ &\leq \sqrt{2\pi}C'_{\psi_1,\psi_2,A}\Big|\frac{1}{b}\Big|^{k-\frac{1}{2}}\Big|\beta^q\sqrt{\frac{2\pi b}{i}}\mathcal{L}_{A^{-1}}\Big[\xi^{k+t+s-p-r-2}(\mathcal{L}_A\varphi)(\xi)\Big](\beta)\Big|\\ &\leq 2\pi C'_{\psi_1,\psi_2,A}\Big|\frac{1}{b}\Big|^{k-q-1}\Big|\mathcal{L}_{A^{-1}}\Big[(\Delta^*_\xi)^q\xi^{k+t+s-p-r-2}(\mathcal{L}_A\varphi)(\xi)\Big](\beta)\Big|. \end{aligned}$$

Therefore,

$$\Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} Id\alpha d\gamma\Big|^p d\beta\Big)^{\frac{1}{p}} \leq 2\pi C'_{\psi_1,\psi_2,A}\Big|\frac{1}{b}\Big|^{k-q-1}\Big(\int_{\mathbb{R}}\Big|\mathcal{L}_{A^{-1}}\Big[(\Delta^*_\xi)^q\xi^{k+t+s-p-r-2}(\mathcal{L}_A\varphi)(\xi)\Big](\beta)\Big|^p d\beta\Big)^{\frac{1}{p}}.$$

Now, using the Riesz-Thorin interpolation formula [17, 41], we have

$$\begin{aligned}
&\Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} Id\alpha d\gamma\Big|^p d\beta\Big)^{\frac{1}{p}}\\
&\leq C_{1,A}C'_{\psi_1,\psi_2,A}\Big(\int_{\mathbb{R}}\Big|(\Delta^*_\xi)^q\xi^{k+t+s-p-r-2}(\mathcal{L}_A\varphi)(\xi)\Big|^q d\xi\Big)^{\frac{1}{q}}\\
&= C''_{\psi_1,\psi_2,A}|b|^{k+t+s-p-r-2}\Big(\int_{\mathbb{R}}\Big|(\Delta^*_\xi)^q\mathcal{L}_A\Big[(\Delta_x)^{k+t+s-p-r-2}\varphi(x)\Big](\xi)\Big|^q d\xi\Big)^{\frac{1}{q}}\\
&= C''_{\psi_1,\psi_2,A}|b|^{k+t+s-q-p-r-2}\Big(\int_{\mathbb{R}}\Big|\mathcal{L}_A\Big[x^q(\Delta_x)^{k+t+s-p-r-2}\varphi(x)\Big](\xi)\Big|^q d\xi\Big)^{\frac{1}{q}}\\
&\leq C_{2,A}C''_{\psi_1,\psi_2,A}|b|^{k+t+s-q-p-r-2}\Big(\int_{\mathbb{R}}\Big|x^q(\Delta_x)^{k+t+s-p-r-2}\varphi(x)\Big|^p dx\Big)^{\frac{1}{p}}\\
&= C'''_{\psi_1,\psi_2,A}\Big\|x^q(\Delta_x)^{k+t+s-p-r-2}\varphi(x)\Big\|_{L^p(\mathbb{R})}.\\
&= C'''_{\psi_1,\psi_2,A}\Big\|\varphi(x)\Big\|_{L^{p,A}(\mathbb{R})}.
\end{aligned}$$

This complete the proof of the theorem. □

In above we defined the test function spaces $L^{p,A}$ and $G^{p,A}$.
Now, we define the test function space $B_A(\mathbb{R}^3)$ and prove that the canonical composition operator $(W^A\varphi)(\beta,\alpha,\gamma)$ is continuous on that space corresponding to [25].

Definition 3.4. *A complex valued smooth function $\varphi(\beta,\alpha,\gamma)$ belong to the test function space $B_A(\mathbb{R}^3)$ iff*

$$\Gamma^A_{m,n,l,k,p}(\varphi) = \sup_{(\beta,\alpha,\gamma)\in\mathbb{R}^3}\Big|\Big(\frac{\beta}{1+|\alpha|}\Big)^l\gamma^n\Big(\gamma\frac{\partial}{\partial\gamma}+1\Big)^n\Big(\alpha\frac{\partial}{\partial\alpha}+1\Big)^m(\Delta_\beta)^{p+k}\Big[\frac{\varphi(\beta,\alpha,\gamma)}{\sqrt{\alpha\gamma}}\Big]\Big| < \infty,$$

for all $l,m,n,k,p\in\mathbb{N}_0$, A is the matrix defined as above and $\Delta_\beta = -\Big(\frac{d}{d\beta}+i\frac{a}{b}\beta\Big)$.

In order to prove the Theorem 3.7, we need the following Lemmas.

Lemma 3.5. *Let $\psi_{\rho,\gamma,A}(x)$ be a canonical wavelet. Then*

$$\text{(i)}\quad (\Delta^*_x)^r\Big(\frac{\overline{\psi_{\rho,\gamma,A}}(x)}{\sqrt{\gamma}}\Big) = D^r_x\Big(\frac{\overline{\psi_{\rho,\gamma}}(x)}{\sqrt{\gamma}}\Big)e^{i\frac{a}{2b}(x^2-\rho^2)},\ \forall r\in\mathbb{N}_0\ ,$$

$$\text{(ii)}\quad (\Delta_x)^r\Big(\frac{\overline{\psi_{\rho,\gamma,A}}(x)}{\sqrt{\gamma}}\Big) = (-1)^r D^r_x\Big(\frac{\psi_{\rho,\gamma}(x)}{\sqrt{\gamma}}\Big)e^{-i\frac{a}{2b}(x^2-\rho^2)},$$

$$\text{(iii)}\quad (\Delta_\rho)^r\Big(\frac{\overline{\psi_{\rho,\gamma,A}}(x)}{\sqrt{\gamma}}\Big) = (\Delta^*_x)^r\Big(\frac{\overline{\psi_{\rho,\gamma,A}}(x)}{\sqrt{\gamma}}\Big),$$

$$\text{(iv)}\quad \Big(\gamma\frac{\partial}{\partial\gamma}+1\Big)^r\Big(\frac{\overline{\psi_{\rho,\gamma,A}}(x)}{\sqrt{\gamma}}\Big) = (\rho-x)^r(\Delta^*_x)^r\Big(\frac{\overline{\psi_{\rho,\gamma,A}}(x)}{\sqrt{\gamma}}\Big),$$

where Δ_x and Δ^*_x are defined as Lemma 1.5.

Proof. The proof of Lemma 21 is straight forward as see [4, 15]. □

Lemma 3.6. *For all $\varphi \in \mathcal{S}_A(\mathbb{R})$, $r \in \mathbb{N}_0$ and $n \in \mathbb{R}$, we have*

$$\text{(i)} \quad \int_{\mathbb{R}} (\Delta_x^*)^r \psi_{\rho,\gamma,A}(x)\varphi(x)dx = \int_{\mathbb{R}} \psi_{\rho,\gamma,A}(x)(\Delta_x)^r \varphi(x)dx$$

$$\text{(ii)} \quad \int_{\mathbb{R}} (\Delta_x^*)^r \overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx = \int_{\mathbb{R}} \overline{\psi_{\rho,\gamma,A}}(x)(\Delta_x)^r \varphi(x)dx$$

$$\text{(iii)} \quad \int_{\mathbb{R}} (x\Delta_x^* + n)\overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx = \int_{\mathbb{R}} (x\Delta_x + (n-1))\varphi(x)\overline{\psi_{\rho,\gamma,A}}(x)dx$$

$$\text{(iv)} \quad \int_{\mathbb{R}} (x\Delta_x + n)\overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx = \int_{\mathbb{R}} (x\Delta_x^* + (n+1))\varphi(x)\overline{\psi_{\rho,\gamma,A}}(x)dx.$$

Proof. Using Lemma 1.5 (iii), the proof of part (i) and (ii) are straight forward. (iii) we have

$$\begin{aligned}
\int_{\mathbb{R}} (x\Delta_x^* + n)\overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx &= \int_{\mathbb{R}} (\Delta_x^*)\overline{\psi_{\rho,\gamma,A}}(x)(x\varphi(x))dx + n\int_{\mathbb{R}} \overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx \\
&= \int_{\mathbb{R}} \overline{\psi_{\rho,\gamma,A}}(x)\Delta_x(x\varphi(x))dx + n\int_{\mathbb{R}} \overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx \\
&= \int_{\mathbb{R}} \overline{\psi_{\rho,\gamma,A}}(x) - \Big(\frac{d}{dx} + i\frac{a}{b}x\Big)(x\varphi(x))dx + n\int_{\mathbb{R}} \overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx \\
&= \int_{\mathbb{R}} \overline{\psi_{\rho,\gamma,A}}(x)\Big(-\varphi(x) + x\Delta_x\varphi(x)\Big)dx + n\int_{\mathbb{R}} \overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx \\
&= \int_{\mathbb{R}} (x\Delta_x + (n-1))\varphi(x)\overline{\psi_{\rho,\gamma,A}}(x)dx.
\end{aligned}$$

Thus

$$\int_{\mathbb{R}} (x\Delta_x^* + n)\overline{\psi_{\rho,\gamma,A}}(x)\varphi(x)dx = \int_{\mathbb{R}} (x\Delta_x + (n-1))\varphi(x)\overline{\psi_{\rho,\gamma,A}}(x)dx, \ \forall\, n \in \mathbb{R}.$$

(iv) The proof is similar to that of (iii). □

Theorem 3.7. *Let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R})$ and $\varphi \in \mathcal{S}_A(\mathbb{R})$. Then the canonical composition operator $(W^A\varphi)(\beta,\alpha,\gamma)$ is a continuous linear mapping of $\mathcal{S}_A(\mathbb{R})$ into $B_A(\mathbb{R}^3)$.*

Proof. By applying Fubini's Theorem and using (13), Lemma 1.6, Lemma 3.5 (iii), (iv), Lemma 3.6 (ii) and integrating by parts, we have

$$\begin{aligned}
(\Delta_\beta)^{p+k}\frac{\Big((W^A\varphi)(\beta,\alpha,\gamma)\Big)}{\sqrt{\alpha\gamma}} &= \int_{\mathbb{R}} \frac{(W^A_{\psi_2}\varphi)(\rho,\gamma)}{\sqrt{\gamma}}(\Delta_\beta)^{p+k}\Big[\frac{\overline{\psi_{1,\beta,\alpha,A}}(\rho)}{\sqrt{\alpha}}\Big]d\rho \\
&= \int_{\mathbb{R}} \frac{(W^A_{\psi_2}\varphi)(\rho,\gamma)}{\sqrt{\gamma}}(\Delta_\rho^*)^{p+k}\Big[\frac{\overline{\psi_{1,\beta,\alpha,A}}(\rho)}{\sqrt{\alpha}}\Big]d\rho \\
&= \int_{\mathbb{R}} \Big((\Delta_\rho)^{p+k}\frac{(W^A_{\psi_2}\varphi)(\rho,\gamma)}{\sqrt{\gamma}}\Big)\frac{\overline{\psi_{1,\beta,\alpha,A}}(\rho)}{\sqrt{\alpha}}d\rho \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} (\Delta_\rho)^{p+k}\varphi(x)\frac{\overline{\psi_{2,\rho,\gamma,A}}(x)}{\sqrt{\gamma}}dx\Big)\frac{\overline{\psi_{1,\beta,\alpha,A}}(\rho)}{\sqrt{\alpha}}d\rho
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\varphi(x)(\Delta_\rho)^{p+k}\Big[\overline{\frac{\psi_{2,\rho,\gamma,A}}{\sqrt{\gamma}}}(x)\Big]dx\Big)\overline{\frac{\psi_{1,\beta,\alpha,A}}{\sqrt{\alpha}}}(\rho)\,d\rho\\
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\varphi(x)(\Delta_x^*)^{p+k}\Big[\overline{\frac{\psi_{2,\rho,\gamma,A}}{\sqrt{\gamma}}}(x)\Big]dx\Big)\overline{\frac{\psi_{1,\beta,\alpha,A}}{\sqrt{\alpha}}}(\rho)\,d\rho\\
&= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\Delta_x)^{p+k}\varphi(x)\Big[\overline{\frac{\psi_{2,\rho,\gamma,A}}{\sqrt{\gamma}}}(x)\Big]dx\Big)\overline{\frac{\psi_{1,\beta,\alpha,A}}{\sqrt{\alpha}}}(\rho)\,d\rho.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\Big(\gamma\frac{\partial}{\partial\gamma}+1\Big)^n\Big(\alpha\frac{\partial}{\partial\alpha}+1\Big)^m(\Delta_\beta)^{p+k}\frac{(W^A\varphi)(\beta,\alpha,\gamma)}{\sqrt{\alpha\gamma}}\\
&=\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\Delta_x)^{p+k}\varphi(x)\Big(\gamma\frac{\partial}{\partial\gamma}+1\Big)^n\Big(\overline{\frac{\psi_{2,\rho,\gamma,A}}{\sqrt{\gamma}}}(x)\Big)dx\Big)\Big(\alpha\frac{\partial}{\partial\alpha}+1\Big)^m\Big(\overline{\frac{\psi_{1,\beta,\alpha,A}}{\sqrt{\alpha}}}(\rho)\Big)d\rho\\
&=\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(\Delta_x)^{p+k}\varphi(x)(\rho-x)^n(\Delta_x^*)^n\Big(\overline{\frac{\psi_{2,\rho,\gamma,A}}{\sqrt{\gamma}}}(x)\Big)dx\Big)(\beta-\rho)^m(\Delta_\rho^*)^m\Big(\overline{\frac{\psi_{1,\beta,\alpha,A}}{\sqrt{\alpha}}}(\rho)\Big)d\rho\\
&=\int_{\mathbb{R}}\Big[(-1)^{m+n}\Big(\int_{\mathbb{R}}(\Delta_x)^{p+k}[\varphi(x)(x-\rho)^n]D_x^n\Big[\overline{\frac{\psi_{2,\rho,\gamma}}{\sqrt{\gamma}}}(x)\Big]e^{i\frac{a}{2b}(x^2-\rho^2)}dx\Big)\\
&\quad\times(\rho-\beta)^mD_\rho^m\Big[\overline{\frac{\psi_{1,\beta,\alpha}}{\sqrt{\alpha}}}(\rho)\Big]e^{i\frac{a}{2b}(\rho^2-\beta^2)}\Big]d\rho.
\end{aligned}
$$

Now, using the inequality,

$$
|\beta|^l\le(|\rho-\beta|+|\rho|)^l\le|\alpha|^l2^l\Big(\Big|\frac{(\beta-\rho)}{\alpha}\Big|^l+\Big|\frac{\rho}{\alpha}\Big|^l\Big),\ \ l>0.
$$

Therefore,

$$
\begin{aligned}
&\Big|\beta^l\gamma^n\Big(\gamma\frac{\partial}{\partial\gamma}+1\Big)^n\Big(\alpha\frac{\partial}{\partial\alpha}+1\Big)^m(\Delta_\beta)^{p+k}\frac{(W^A\varphi)(\beta,\alpha,\gamma)}{\sqrt{\alpha\gamma}}\Big|\\
&\le(2|\alpha|)^l\Big(\int_{\mathbb{R}}\Big(\Big|\frac{(\beta-\rho)}{\alpha}\Big|^l+\Big|\frac{\rho}{\alpha}\Big|^l\Big)\Big(\int_{\mathbb{R}}\Big|(\Delta_x)^{p+k}[\varphi(x)(x-\rho)^n]\Big|\\
&\quad\times\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)e^{i\frac{a}{2b}(x^2-\rho^2)}\Big|dx\Big)\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)e^{i\frac{a}{2b}(\rho^2-\beta^2)}\Big|d\rho\Big)\\
&\le(2|\alpha|)^l\Big(\int_{\mathbb{R}}\Big(\Big|\frac{(\beta-\rho)}{\alpha}\Big|^l+\Big|\frac{\rho}{\alpha}\Big|^l\Big)\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)e^{i\frac{a}{2b}(\rho^2-\beta^2)}\Big|d\rho\Big)\\
&\quad\times\Big(\int_{\mathbb{R}}\Big|(\Delta_x)^{p+k}[\varphi(x)(x-\rho)^n]\Big|\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)e^{i\frac{a}{2b}(x^2-\rho^2)}\Big|dx\Big)\\
&\le(2|\alpha|)^l\Big(\int_{\mathbb{R}}\Big|\frac{(\beta-\rho)}{\alpha}\Big|^l\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|d\rho\Big)\Big(\int_{\mathbb{R}}\Big|(\Delta_x)^{p+k}[\varphi(x)(x-\rho)^n]\Big|\\
&\quad\times\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)\Big|dx\Big)+(2|\alpha|)^l\Big(\int_{\mathbb{R}}\Big|\frac{\rho}{\alpha}\Big|^l\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|d\rho\Big)\\
&\quad\times\Big(\int_{\mathbb{R}}\Big|(\Delta_x)^{p+k}[\varphi(x)(x-\rho)^n]\Big|\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)\Big|dx\Big)
\end{aligned}
$$

$$\leq (2|\alpha|)^l\Big(\int_{\mathbb{R}}\Big|\frac{(\beta-\rho)}{\alpha}\Big|^l\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|d\rho\Big)$$
$$\times\Big(\int_{\mathbb{R}}\Big|\sum_{r=0}^{p+k}B'_{p+k,r}(x-\rho)^{n-r}(\Delta_x)^{p+k-r}\varphi(x)\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)\Big|dx\Big)$$
$$+2^l\Big(\int_{\mathbb{R}}|\rho|^l\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|d\rho\Big)$$
$$\times\Big(\int_{\mathbb{R}}\Big|\sum_{r=0}^{p+k}B'_{p+k,r}(x-\rho)^{n-r}(\Delta_x)^{p+k-r}\varphi(x)\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)\Big|dx\Big)$$
$$\leq 2^l(1+|\alpha|)^l\sum_{r=0}^{p+k}B'_{p+k,r}\Big[\sup_{x\in\mathbb{R}}\Big|(x-\rho)^{n-r}(\Delta_x)^{p+k-r}\varphi(x)\Big|$$
$$\times\Big(\int_{\mathbb{R}}|\rho|^l\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|d\rho\Big)\Big(\int_{\mathbb{R}}\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)\Big|dx\Big)$$
$$+\sup_{x\in\mathbb{R}}\Big|(x-\rho)^{n-r}(\Delta_x)^{p+k-r}\varphi(x)\Big|\Big(\int_{\mathbb{R}}\Big|\frac{(\beta-\rho)}{\alpha}\Big|^l\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|d\rho\Big)$$
$$\times\Big(\int_{\mathbb{R}}\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)\Big|dx\Big)\Big].$$

Hence,

$$\sup_{(\beta,\alpha,\gamma')\in\mathbb{R}^3}\Big|\Big(\frac{\beta}{1+|\alpha|}\Big)^l\gamma^n\Big(\gamma\frac{\partial}{\partial\gamma'}+1\Big)^n\Big(\alpha\frac{\partial}{\partial\alpha}+1\Big)^m(\Delta_\beta)^{p+k}\frac{(W^A\varphi)(\beta,\alpha,\gamma)}{\sqrt{\alpha\gamma}}\Big|$$
$$\leq 2^l\sum_{r=0}^{p+k}B'_{p+k,r}\Big[\sup_{x\in\mathbb{R}}\Big|(x-\rho)^{n-r}(\Delta_x)^{p+k-r}\varphi(x)\Big|$$
$$\times\Big(\int_{\mathbb{R}}|\rho|^l\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|d\rho\Big)\Big(\int_{\mathbb{R}}\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)\Big|dx\Big)$$
$$+\sup_{x\in\mathbb{R}}\Big|(x-\rho)^{n-r}(\Delta_x)^{p+k-r}\varphi(x)\Big|\Big(\int_{\mathbb{R}}\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|^l\Big|\Big(\frac{\rho-\beta}{\alpha}\Big)^m\frac{1}{\alpha}\overline{\psi_1}^{(m)}\Big(\frac{\rho-\beta}{\alpha}\Big)\Big|d\rho\Big)$$
$$\times\Big(\int_{\mathbb{R}}\Big|\frac{1}{\gamma}\overline{\psi_2}^{(n)}\Big(\frac{x-\rho}{\gamma}\Big)\Big|dx\Big)\Big]$$
$$\leq 2^l\sum_{r=0}^{p+k}B'_{p+k,r}\Big[\Gamma^A_{n-r,p+k-r}(\varphi)\Big(\int_{\mathbb{R}}|u|^m\Big|\Big(u+\frac{\beta}{\alpha}\Big)^l\Big|\overline{\psi_1}^{(m)}(u)\Big|du\Big)\Big(\int_{\mathbb{R}}|\overline{\psi_2}^{(n)}(v)\Big|dv\Big)$$
$$+\Gamma^A_{n-r,p+k-r}(\varphi)\Big(\int_{\mathbb{R}}|u|^{l+m}\Big|\overline{\psi_1}^{(m)}(u)\Big|du\Big)\Big(\int_{\mathbb{R}}|\overline{\psi}_2^{(n)}(v)\Big|dv\Big)\Big]$$
$$\leq 2^l\sum_{r=0}^{p+k}B'_{p+k,r}\Big[\sum_{q=0}^{l}\binom{l}{q}\Big(\frac{\beta}{\alpha}\Big)^{l-q}\Gamma^A_{n-r,p+k-r}(\varphi)\sup_{u\in\mathbb{R}}\Big|(1+u^2)u^{m+q}D_u^m\psi_1(u)\Big|$$
$$\times\sup_{v\in\mathbb{R}}\Big|(1+v^2)D_v^n\psi_2(v)\Big|\Big(\int_{\mathbb{R}}\frac{1}{1+u^2}du\Big)\Big(\int_{\mathbb{R}}\frac{1}{1+v^2}dv\Big)$$
$$+\Gamma^A_{n-r,p+k-r}(\varphi)\sup_{u\in\mathbb{R}}\Big|u^{l+m}(1+u^2)D_u^m\psi_1(u)\Big|\sup_{v\in\mathbb{R}}\Big|(1+v^2)D_v^n\psi_2(v)\Big|$$
$$\times\Big(\int_{\mathbb{R}}\frac{1}{1+u^2}du\Big)\Big(\int_{\mathbb{R}}\frac{1}{1+v^2}dv\Big)\Big]$$

$$\leq \pi^2 2^l \sum_{r=0}^{p+k} B'_{p+k,r}\Bigg[\sum_{q=0}^{l}\binom{l}{q}\Big(\frac{\beta}{\alpha}\Big)^{l-q}\Gamma^A_{n-r,p+k-r}(\varphi)\sup_{u\in\mathbb{R}}\Big|(1+u^2)u^{m+q}D^m_u\psi_1(u)\Big|$$
$$\times\sup_{v\in\mathbb{R}}\Big|(1+v^2)D^n_v\psi_2(v)\Big| + \Gamma^A_{n-r,p+k-r}(\varphi)$$
$$\times\sup_{u\in\mathbb{R}}\Big|u^{l+m}(1+u^2)D^m_u\psi_1(u)\Big|\sup_{v\in\mathbb{R}}\Big|(1+v^2)D^n_v\psi_2(v)\Big|\Bigg],$$

where $B'_{p+k,r} > 0$. Since $\varphi \in \mathcal{S}_A(\mathbb{R})$ and $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R})$ each term of the right-hand side is finite. Hence, the canonical composition operator $(W^A\varphi)$ is a continuous linear mapping of $\mathcal{S}_A(\mathbb{R})$ into $B_A(\mathbb{R}^3)$. □

4. Composition of LCWTs on generalized Sobolev space $H^p_{s,A}(\mathbb{R})$

Pathak [26] obtained certain boundedness results for continuous wavelet transform on L^p-Sobolev space, Motivated by this work we have defined the generalized Sobolev space $H^p_{s,A}$ and discuss the boundedness for composition of linear canonical wavelet transform on $H^p_{s,A}$.

Definition 4.1. *For $s \in \mathbb{R}$ and $p \in [1,\infty)$, the generalized L^p-Sobolev space $H^p_{s,A}$ is defined as*

$$H^p_{s,A} = \Big\{\varphi \;\Big|\; \varphi(x) \in \mathcal{S}'_A(\mathbb{R}) \text{ and } |\xi|^s(\mathcal{L}_A\varphi)(\xi) \in L^p(\mathbb{R})\Big\}.$$

Its norm is defined by

$$\big\|\varphi\big\|_{H^p_{s,A}} = \big\||\xi|^s(\mathcal{L}_A\varphi)(\xi)\big\|_{L^p}.$$

Definition 4.2. *The generalized Sobolev space $H^{q,s}_{p,A}$ of all measurable function $\varphi(\beta,\alpha,\gamma)$ defined on $(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$ satisfying*

$$\big\|\varphi(\beta,\alpha,\gamma)\big\|_{H^{q,s}_{p,A}} = \Big(\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\Big(\int_{\mathbb{R}}\big|\varphi(\beta,\alpha,\gamma)\big|^p d\beta\Big)^{\frac{q}{p}}(\alpha\gamma)^{-s-1}d\alpha d\gamma\Big)^{\frac{1}{q}} < \infty, \tag{22}$$

where $1 \leq p,\ q < \infty,\ s \in \mathbb{R}$.

Theorem 4.3. *Assume that the canonical wavelets ψ_1, ψ_2 satisfies the following admissibility condition:*

$$C^{s,q}_{\psi_1,\psi_2,A} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{\big|\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1](\alpha\xi)\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2](\gamma\xi)\big|^q}{(\alpha\gamma|\xi|)^s}\frac{d\alpha d\gamma}{\alpha\gamma} < \infty. \tag{23}$$

Then the canonical composition operator $(W^A\varphi)(\beta,\alpha,\gamma)$ is a bounded linear operator from $H^p_{s,A}(\mathbb{R})$ into $H^{q,s}_{p,A}(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$ for $1 \leq p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$ and $s \in \mathbb{R}$.

Proof. From (15), we have

$$\mathcal{L}_A\{(W^A\varphi)(\beta,\alpha,\gamma)\} = \frac{2\pi b\sqrt{\alpha\gamma}}{i}e^{i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi). \tag{24}$$

Using the Riesz-Thorin interpolation formula [17, 41] for LCT, we have

$$\Big(\int_{\mathbb{R}}\Big|\frac{2\pi b\sqrt{\alpha\gamma}}{i}e^{i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi)\Big|^q d\xi\Big)^{\frac{1}{q}}$$
$$= \Big(\int_{\mathbb{R}}\big|\mathcal{L}_A\{\big(W^A\varphi\big)(\beta,\alpha,\gamma)\}\big|^q d\xi\Big)^{\frac{1}{q}}$$
$$\leq C_{p,A}\Big(\int_{\mathbb{R}}\big|\big(W^A\varphi\big)(\beta,\alpha,\gamma)\big|^p d\beta\Big)^{\frac{1}{p}},$$

where $C_{p,A} > 0$ is a constant. Multiplying by $(\alpha\gamma)^{-s-1}d\alpha d\gamma$ and integrating over $\mathbb{R}^+ \times \mathbb{R}^+$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}(\alpha\gamma)^{-s-1}d\alpha d\gamma \int_{\mathbb{R}}\Big|\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi)\Big|^q d\xi\\
&\leq \Big(\frac{C_{p,A}}{2\pi b\sqrt{\alpha\gamma}}\Big)^q\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\Big(\int_{\mathbb{R}}\Big|\big(W^A\varphi\big)(\beta,\alpha,\gamma)\Big|^p d\beta\Big)^{\frac{q}{p}}(\alpha\gamma)^{-s-1}d\alpha d\gamma.
\end{aligned}$$

Now, using (23) the above expansion can be written as

$$\begin{aligned}
\big(C^{s,q}_{\psi_1,\psi_2,A}\big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}}\big|\xi\big|^s\big|(\mathcal{L}_A\varphi)(\xi)\big|^q d\xi\Big)^{\frac{1}{q}} &\leq \frac{C_{p,A}}{2\pi b\sqrt{\alpha\gamma}}\Big(\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\Big(\int_{\mathbb{R}}\Big|\big(W^A\varphi\big)(\beta,\alpha,\gamma)\Big|^p d\beta\Big)^{\frac{q}{p}}(\alpha\gamma)^{-s-1}d\alpha d\gamma\Big)^{\frac{1}{q}}\\
&\leq \frac{C_{p,A}}{2\pi b\sqrt{\alpha\gamma}}\big\|\big(W^A\varphi\big)(\beta,\alpha,\gamma)\big\|_{H^{q,s}_{p,A}(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)}.
\end{aligned}$$

Moreover, we have

$$\big\|\varphi\big\|_{H^q_{s,A}(\mathbb{R})} \leq \frac{C_{p,A}}{2\pi b\sqrt{\alpha\gamma}(C^{s,q}_{\psi_1,\psi_2,A})^{\frac{1}{q}}}\big\|\big(W^A\varphi\big)(\beta,\alpha,\gamma)\big\|_{H^{q,s}_{p,A}(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)}. \tag{25}$$

Furthermore, based on the Riesz-Thorin interpolation formula [17, 41] for LCT and using (24), then

$$\begin{aligned}
&\Big(\int_{\mathbb{R}}\Big|\big(W^A\varphi\big)(\beta,\alpha,\gamma)\Big|^q d\beta\Big)^{\frac{1}{q}}\\
&=\Big(\int_{\mathbb{R}}\Big|\frac{2\pi b\sqrt{\alpha\gamma}}{i}\mathcal{L}_{A^{-1}}\Big[e^{i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)\\
&\times(\mathcal{L}_A\varphi)(\xi)\Big](\beta)\Big|^q d\beta\Big)^{\frac{1}{q}}\\
&\leq D_{p,A}(2\pi b\sqrt{\alpha\gamma})\Big(\int_{\mathbb{R}}\Big|e^{i\frac{d}{2b}(\alpha^2+\gamma^2)\xi^2}\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi)\Big|^p d\xi\Big)^{\frac{1}{p}},
\end{aligned}$$

where $D_{p,A} > 0$ is a constant. Therefore

$$\begin{aligned}
&\int_{\mathbb{R}+}\int_{\mathbb{R}+}(\alpha\gamma)^{-s-1}\Big(\int_{\mathbb{R}}\Big|\big(W^A\varphi\big)(\beta,\alpha,\gamma)\Big|^q d\beta\Big)^{\frac{p}{q}}d\alpha d\gamma\\
&\leq\Big(D_{p,A}2\pi b\sqrt{\alpha\gamma}\Big)^p\int_{\mathbb{R}+}\int_{\mathbb{R}+}(\alpha\gamma)^{-s-1}\Big(\int_{\mathbb{R}}\Big|\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_1]}(\alpha\xi)\overline{\mathcal{L}_A[e^{-i\frac{a}{2b}(\cdot)^2}\psi_2]}(\gamma\xi)(\mathcal{L}_A\varphi)(\xi)\Big|^p d\xi\Big)d\alpha d\gamma\\
&\leq\Big(D_{p,A}2\pi b\sqrt{\alpha\gamma}\Big)^p C^{s,p}_{\psi_1,\psi_2,A}\int_{\mathbb{R}}\big|\xi\big|^s\big|(\mathcal{L}_A\varphi)(\xi)\big|^p d\xi,
\end{aligned}$$

where $C^{s,p}_{\psi_1,\psi_2,A}$ is given by (24). Thus

$$\big\|\big(W^A\varphi\big)(\beta,\alpha,\gamma)\big\|_{H^{p,s}_{q,A}(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)} \leq \Big(D_{p,A}2\pi b\sqrt{\alpha\gamma}\Big)(C^{s,p}_{\psi_1,\psi_2,A})^{\frac{1}{p}}\big\|\varphi\big\|_{H^p_{s,A}(\mathbb{R})}. \tag{26}$$

From (25) and (26), we conclude that canonical composition operator $(W^A\varphi)$ is a bounded linear operator from $H^p_{s,A}(\mathbb{R})$ into $H^{q,s}_{p,A}(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$. □

5. Conclusion

In this work, we defined the canonical composition operator and proof the continuity of this operator on the generalized test function spaces $L^{p,A}$, $G^{p,A}$ and $B_A(\mathbb{R}^3)$. Also, we proved that the canonical composition operator as a bounded linear operator from the generalized Sobolev space $H^p_{s,A}$ into $H^{q,s}_{p,A}$.

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