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Cosmological Meaning of Geometric Curvatures

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Abstract. In this paper, we analyzed the physical meaning of scalar curvatures for a generalized Riemannian space. It is developed the Madsen's formulae for pressures and energy-densities with respect to the corresponding energy-momentum tensors. After that, the energy-momentum tensors, pressures, energy-densities and state-parameters are analyzed with respect to different concepts of generalized Riemannian spaces. At the end of this paper, linearities of the energy-momentum tensor, pressure, energy-density and the state-parameter are examined.

1. Introduction

The main purpose of this paper is to find a physical meaning of scalar curvatures for a generalized Riemannian space [5] and complex or anti-symmetric metrics as well.

1.1. *Physical motivation for differential geometry: basics of cosmology*

Many geometric papers start with the motivation from General Relativity. In the paper (Ivanov, Zlatanović [6]), the physical motivation with respect to the Einstein's works [2–4] is well explained. Some other papers where these Einstein's works are cited as the motivations for further researches about the spaces with torsion are [15, 19–23] and many others.

Einstein involved the concept of a complex metric whose real part corresponds to the gravity but the imaginer part suits to the electromagnetism. Moreover, the affine connection coefficients Γ^i_{jk} of spaces in Einstein's works are determined by *the Einstein Metricity Condition*.

The Einstein's Theory of General Relativity is a cosmological model which was developed. The Kaluza-Klein cosmological model [7, 8] is one of commonly used models in the theory of cosmology. In this model, unlike in the Einstein's one, the electromagnetism is covered by the additional dimension of the symmetric (real) part of metrics.

The question that arises is whether the anti-symmetric parts of metric tensors are important for any physical application or they are excessive. We will physically and geometrically answer to this question in this paper.

The Kaluza-Klein model will not be studied here. The computational methodology applied in the book [1] but also in the article [17] combined with torsion tensors will be used in this study.

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1.2. Geometrical motivation: Generalized Riemannian space

An N-dimensional manifold $\mathcal{M}_N$, equipped with a non-symmetric metric tensor $\hat{g}$ of the type $(0, 2)$ whose components are g_{ij} is the generalized Riemannian space $\mathbb{GR}_N$ $\Big($in the sense of Eisenhart's definition [5]$\Big)$. S. M. Minčić [13–15], M. S. Stanković [19, 20, 22, 23], Lj. S. Velimirović [15, 19, 22], M. Lj. Zlatanović [6, 22, 23] and many others have continued the research about these spaces, the mappings between them and their generalizations.

The symmetric and anti-symmetric part of the tensor $\hat{g}$ are the tensors $\underline{\hat{g}} = \dfrac{1}{2}(\hat{g} + \hat{g}^T)$ and $\underset{\vee}{\hat{g}} = \frac{1}{2}(\hat{g} - \hat{g}^T)$, respectively. Their components are

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}) \quad \text{and} \quad g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}). \tag{1.1}$$

For our research, the matrix $\left[g_{\underline{ij}}\right]_{N\times N}$ should be non-singular. The metric determinant for the space $\mathbb{R}_4$ is $g = \det[g_{\underline{ij}}]$ and it is different of 0 because of the non-singularity of the matrix $\left[g_{\underline{ij}}\right]$. The components $g^{\underline{ij}}$ for the contravariant symmetric part of the metric tensor $\hat{g}$ are the corresponding elements of the inverse matrix $\left[g_{\underline{ij}}\right]^{-1}_{N\times N}$.

The components of the affine connection coefficients for the space $\mathbb{GR}_N$ are the components of the generalized Christoffel symbols of the second kind

$$\Gamma^i_{jk} = \frac{1}{2} g^{\underline{i\alpha}}\Big(g_{j\alpha,k} - g_{jk,\alpha} + g_{\alpha k,j}\Big), \tag{1.2}$$

for partial derivative $\partial/\partial x^i$ denoted by comma.

The components of the symmetric and anti-symmetric parts of the generalized Christoffel symbol of the second kind are

$$\Gamma^i_{\underline{jk}} = \frac{1}{2}\Big(\Gamma^i_{jk} + \Gamma^i_{kj}\Big) \quad \text{and} \quad \Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}\Big(\Gamma^i_{jk} - \Gamma^i_{kj}\Big). \tag{1.3}$$

After some computing, one gets

$$\Gamma^i_{\underline{jk}} = \frac{1}{2} g^{\underline{i\alpha}}\Big(g_{\underline{j\alpha},k} - g_{\underline{jk},\alpha} + g_{\underline{\alpha k},j}\Big) \quad \text{and} \quad \Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2} g^{\underline{i\alpha}}\Big(g_{\underset{\vee}{j\alpha},k} - g_{\underset{\vee}{jk},\alpha} + g_{\underset{\vee}{\alpha k},j}\Big). \tag{1.4}$$

The doubled components of the anti-symmetric parts $\Gamma^i_{\underset{\vee}{jk}}$ are the components of the torsion tensor $\hat{T}$ for the space $\mathbb{GR}_N$, i.e. the components of the torsion tensor are $T^i_{jk} = 2\Gamma^i_{\underset{\vee}{jk}}$. The components of the covariant torsion tensor are $T_{ijk} = g_{\underline{i\alpha}} T^\alpha_{jk}$.

The manifold $\mathcal{M}_N$ equipped with the tensor $\underline{\hat{g}}$ is the associated space $\mathbb{R}_N$ of the space $\mathbb{GR}_N$. The components of the symmetric part (1.3) of the generalized Christoffel symbols are the Christoffel symbols of the second kind. Hence, they are the affine connection coefficients of the associated space $\mathbb{R}_N$.

The associated space $\mathbb{R}_N$ is the Riemannian space $\Big($in the sense of Eisenhart's definition [5] $\Big)$. N. S. Sinyukov [18], Josef Mikeš with his research group [10–12] and many other authors have developed the theory of Riemannian spaces.

With respect to the affine connection of the associated space $\mathbb{R}_N$ and a tensor $\hat{a}$ of the type $(1,1)$, it is defined one kind of covariant derivative [10–12, 18]

$$a^i_{j|k} = a^i_{j,k} + \Gamma^i_{\underline{\alpha k}} a^\alpha_j - \Gamma^\alpha_{\underline{jk}} a^i_\alpha. \tag{1.5}$$

Based on this covariant derivative, it is founded one identity of the Ricci Type. From this identity, it is obtained one curvature tensor $\hat{R}$ of the space $\mathbb{R}_N$ $\big($see [10–12, 18]$\big)$. The components of this tensor are

$$R^i_{jmn} = \Gamma^i_{\underline{jm},n} - \Gamma^i_{\underline{jn},m} + \Gamma^\alpha_{\underline{jm}} \Gamma^i_{\underline{\alpha n}} - \Gamma^\alpha_{\underline{jn}} \Gamma^i_{\underline{\alpha m}}. \tag{1.6}$$

The components of the corresponding tensor of the Ricci curvature are $R_{ij} = R^\alpha_{ij\alpha}$. The scalar curvature of the associated space $\mathbb{R}_N$ is $R = g^{\underline{\alpha\beta}} R_{\alpha\beta}$.

A. Einstein studied the spaces whose affine connection coefficients are not functions of the metric tensor [2–4]. With respect to the Einstein Metricity Condition

$$g_{ij,k} - \Gamma^\alpha_{ik} g_{\alpha j} - \Gamma^\alpha_{kj} g_{i\alpha} = 0, \tag{1.7}$$

as the system of differential equations which generate the affine connection coefficients for the affine connection space, two kinds of covariant derivatives are defined

$$a^i_{j\underset{1}{|}k} = a^i_{j,k} + \Gamma^i_{\alpha k} a^\alpha_j - \Gamma^\alpha_{jk} a^i_\alpha \quad \text{and} \quad a^i_{j\underset{2}{|}k} = a^i_{j,k} + \Gamma^i_{k\alpha} a^\alpha_j - \Gamma^\alpha_{kj} a^i_\alpha. \tag{1.8}$$

M. Prvanović [16] obtained the fourth curvature tensor for a non-symmetric affine connection space. S. M. Minčić [13, 14] defined four kinds of covariant derivatives. These four kinds are the covariant derivatives (1.8) and two novel ones

$$a^i_{j\underset{3}{|}k} = a^i_{j,k} + \Gamma^i_{\alpha k} a^\alpha_j - \Gamma^\alpha_{kj} a^i_\alpha \quad \text{and} \quad a^i_{j\underset{4}{|}k} = a^i_{j,k} + \Gamma^i_{k\alpha} a^\alpha_j - \Gamma^\alpha_{jk} a^i_\alpha. \tag{1.9}$$

With respect to these four kinds of covariant derivatives, S. M. Minčić obtained four curvature tensors, eight derived curvature tensors and fifteen curvature pseudotensors of the space $\mathbb{GR}_N$. The components of curvature tensors for the space $\mathbb{GR}_N$ are elements of the family

$$K^i_{jmn} = R^i_{jmn} + u T^i_{jm|n} + u' T^i_{jn|m} + v T^\alpha_{jm} T^i_{\alpha n} + v' T^\alpha_{jn} T^i_{\alpha m} + w T^\alpha_{mn} T^i_{\alpha j}, \tag{1.10}$$

for the corresponding coefficients u, u', v, v', w. Six of them are linearly independent.

The components of Ricci-curvatures for the space $\mathbb{GR}_N$ are

$$K_{ij} = R_{ij} + u T^\alpha_{ij|\alpha} - (v' + w) T^\alpha_{i\beta} T^\beta_{j\alpha}. \tag{1.11}$$

Three of these tensors are linearly independent.

The family of scalar curvatures $K = g^{\underline{\gamma\delta}} K_{\gamma\delta}$ for the space $\mathbb{GR}_N$ is

$$K = R - (v' + w) g^{\underline{\gamma\delta}} g^{\underline{\alpha\epsilon}} g^{\underline{\beta\zeta}} T_{\alpha\gamma\beta} T_{\epsilon\delta\zeta}. \tag{1.12}$$

Two of these curvatures are linearly independent.

In this paper, we will stay focused on the space-time $\mathbb{GR}_4$ equipped with a non-symmetric metric tensor $\hat{g}$ whose symmetric part is diagonal.

1.3. Motivation

In the Theory of General Relativity, the Einstein-Hilbert action is the action that yields the Einstein field equations through the principle of least action. Let the full action of the theory be

$$S=\int d^4x\,\sqrt{|g|}\,(R+\mathcal{L}_M), \tag{1.13}$$

for the scalar curvature R of the associated space $\mathbb{R}_4$. The term $\mathcal{L}_M$ in the last equation is describing matter fields.

The Ricci tensor R_{ij} and the scalar curvature R for the space $\mathbb{GR}_4$ satisfy the Einstein's equations of motion

$$R_{ij}-\frac{1}{2}Rg_{\underline{ij}}=T_{ij}, \tag{1.14}$$

where T_{ij} are the components of the energy-momentum tensor $\hat{T}$. The last equations are generalized by the cosmological constant Λ as [1, 17]

$$R_{ij}-\frac{1}{2}Rg_{\underline{ij}}+\Lambda g_{\underline{ij}}=T_{ij}. \tag{1.15}$$

Remark 1.1. *The equation* (1.14) *is obtained from the Einstein-Hilbert action* $S=\int d^4x\,\sqrt{|g|}\big(R+\mathcal{L}_M\big)$ *but the equation* (1.15) *holds from the Einstein-Hilbert action* $S_\Lambda=\int d^4x\,\sqrt{|g|}\big(R-2\Lambda+\mathcal{L}_M\big)$, $S=S_0$.

In general, the components T_{ij} of the corresponding energy-momentum tensor $\hat{T}$ $\Big($*at the right sides of the equations* (1.14, 1.15)$\Big)$ *are multiplied by the constant κ,* $\kappa=8\pi Gc^{-4}$ *for the speed of light c and Newton's gravitational constant G but we will chose such coordinates to be* $\kappa=1$ *in further research.*

The Friedman-Lemaitre-Robertson-Walker (FLRW) and the Bianchi Type-I cosmological models are solutions of the Einstein's equations of motion.

This article is consisted of five sections plus conclusion. The purposes of this paper are:

1. To recall and develop the Madsen's formulae [9] for pressure, energy-density and state parameter,
2. To correlate the space-time model caused from the article [6] with the Einstein's equations of motion,
3. To analyze the linearity of energy-momentum tensors, pressures, energy-densities and state-parameters under summings of fields.

2. Pressure, density, state parameter and Madsen's formulae

Based on the action $I=\int d^4x\,\sqrt{|g|}\Big(\frac{1}{2}\Big(\frac{M_P^2}{8\pi}-\xi\phi^2\Big)R+\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi-V[\phi]\Big)$, the energy-momentum tensor is $\Big($see [9]$\Big)$

$$T_{ij}=\Big(1-\xi\phi^2\Big)^{-1}\Big[S_{ij}+\xi\Big\{g_{\underline{ij}}\Box(\phi^2)-(\phi^2)_{|i|j}\Big\}\Big], \tag{2.1}$$

for the constant ξ, the time-like scalar field ϕ which has unit magnitude, the operator $\Box$ defined as $\Box a^i_j=g^{\underline{\alpha\beta}}a^i_{j|\alpha|\beta}$ and the tensor $S_{ij}=\phi_{,i}\phi_{,j}-\Big(\frac{1}{2}g^{\underline{\alpha\beta}}\phi_{,\alpha}\phi_{,\beta}-V[\phi]\Big)g_{\underline{ij}}$. Madsen also chosen the units such that $\hbar=c=1$ and $M_P^2=8\pi$ for the Planck mass M_P.

In the second section of the paper [9], Madsen deals with the problem of finding a unique vector $\hat{u}$ related to the scalar field ϕ, appears in the energy-momentum tensor (2.1). The components for this vector are u^i and they satisfy the equation

$$u^\alpha u_\alpha = 1. \tag{2.2}$$

The components u^i are

$$u^i = \left(\partial^\alpha \phi \partial_\alpha \phi\right)^{-1/2} \partial^i \phi. \tag{2.3}$$

For the symmetric tensor $\hat{h}$ of the type $(0,2)$ whose components are [9]

$$h_{ij} = g_{\underline{ij}} - u_i u_j, \tag{2.4}$$

the energy-density ρ, the pressure p, the vector-field $\hat{q}$ such that $u_\alpha q^\alpha = 0$ and the tensor $\hat{\pi}$ of the type $(0,2)$ whose components satisfy the equalities $\pi_{i\alpha} u^\alpha = 0$ and $\pi^\alpha_\alpha = 0$, the components of the energy-momentum tensor $\hat{T}$ of the type $(0,2)$ are [9]

$$T_{ij} = \rho u_i u_j + q_i u_j + q_j u_i - \left(p h_{ij} + \pi_{ij}\right). \tag{2.5}$$

It holds $\big($see [9]$\big)$ $p = \frac{1}{3}\Pi^\alpha_\alpha$ and $\rho = T_{\alpha\beta} u^\alpha u^\beta$, for $\Pi_{ij} \equiv p h_{ij} + \pi_{ij} = -T_{\alpha\beta} h^\alpha_i h^\beta_j$. After composing the equation (2.4) by $g^{\underline{jk}}$, one obtains $h^k_i = \delta^k_i - u_i u^k$. If compose the equation (2.5) by the tensor $g^{\underline{ij}}$, use the symmetry $T_{ij} = T_{ji}$, the relation $u^\alpha u_\alpha = 1$ and the previously founded components h^i_j, we will acquire the following expressions

$$\Pi_{ij} = -T_{ij} + T_{i\alpha} u^\alpha u_j + T_{j\alpha} u^\alpha u_i - T_{\alpha\beta} u^\alpha u^\beta u_i u_j \quad \text{and} \quad \Pi^\alpha_\alpha = -T^\alpha_\alpha + T_{\alpha\beta} u^\alpha u^\beta. \tag{2.6}$$

Hence, the energy-momentum tensor $\hat{T}$, the pressure p, the energy-density ρ and the state parameter ω satisfy the next equalities

$$p = -\frac{1}{3} T^\alpha_\alpha + \frac{1}{3} T_{\alpha\beta} u^\alpha u^\beta, \quad \rho = T_{\alpha\beta} u^\alpha u^\beta, \quad \omega = -\frac{1}{3} T^\alpha_\alpha \left(T_{\beta\gamma} u^\beta u^\gamma\right)^{-1} + \frac{1}{3}. \tag{2.7}$$

In the comoving reference frame, $u^i = \delta^i_0$, the equalities (2.7) reduce to

$$p_0 = -\frac{1}{3} T^\alpha_\alpha + \frac{1}{3} T_{00}, \quad \rho_0 = T_{00}, \quad \omega_0 = -\frac{1}{3} T^\alpha_\alpha \left(T_{00}\right)^{-1} + \frac{1}{3}. \tag{2.8}$$

After composing the equation (1.15) by $u^i u^j$ and $g^{\underline{ij}}$ but using the equation (2.2) as well, we get

$$T_{\alpha\beta} u^\alpha u^\beta = R_{\alpha\beta} u^\alpha u^\beta - \frac{1}{2} R + \Lambda \quad \text{and} \quad T^\alpha_\alpha = -R + 4\Lambda. \tag{2.9}$$

In the case of $i = j = 0$, one obtains

$$T_{00} \overset{(1.15)}{=} R_{00} - \frac{1}{2} R g_{\underline{00}} + \Lambda g_{\underline{00}}. \tag{2.10}$$

If substitute the equations (2.9, 2.10) into the expressions (2.7, 2.8), we will find

In a reference system u^i	In the comoving reference system $u^i = \delta^i_0$
$p = \frac{1}{3}R_{\alpha\beta}u^\alpha u^\beta + \frac{1}{6}R - \Lambda$	$p_0 = \frac{1}{3}R_{00} + \frac{1}{6}R - \Lambda$
$\rho = R_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}R + \Lambda$	$\rho_0 = R_{00} - \frac{1}{2}R + \Lambda$
$\omega = p\rho^{-1}$	$\omega_0 = p_0\rho_0^{-1}$

Table 1: Pressures, energy-densities and the state parameters

3. Generalized Einstein's equations of motion I

Let us consider the Einstein-Hilbert action

$$S = \int d^4x\,\sqrt{|g|}\Big(K - 2\Lambda\Big), \tag{3.1}$$

with respect to the Shapiro's cosmological model [17].

Based on the equation (1.12), we transform the Einstein-Hilbert action (3.1) to

$$S = \int d^4x\,\sqrt{|g|}\Big(R - 2\Lambda - (v' + w)g^{\underline{\gamma\delta}}T^\alpha_{\gamma\beta}T^\beta_{\alpha\delta}\Big). \tag{3.2}$$

After lowering the contravariant indices into the last equation, we get

$$S = \int d^4x\,\sqrt{|g|}\Big(R - 2\Lambda - (v' + w)g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\epsilon\gamma\beta}T_{\alpha\delta\zeta}\Big). \tag{3.3}$$

If compare the variations of the functional (3.3) and the Einstein-Hilbert action $S = \int d^4x\,\sqrt{|g|}\Big(R - 2\Lambda + \mathcal{L}_M\Big)$, one finds

$$\mathcal{L}_M = -(v' + w)g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\epsilon\gamma\beta}T_{\alpha\delta\zeta}. \tag{3.4}$$

The variation of the functional $S_1 = S_1[\underline{\hat{g}}] = \int d^4x\,\sqrt{|g|}\Big(R - 2\Lambda\Big)$ is [1]

$$\delta S_1 = \int d^4x\,\sqrt{|g|}\Big(R_{\alpha\beta} - \frac{1}{2}Rg_{\underline{\alpha\beta}} + \Lambda g_{\underline{\alpha\beta}}\Big)\delta g^{\underline{\alpha\beta}}. \tag{3.5}$$

The variation of the functional $S_2 = S_2[\underline{\hat{g}}] = \int d^4x\,\sqrt{|g|}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\epsilon\gamma\beta}T_{\alpha\delta\zeta}$ is

$$\delta S_2 = \int d^4x\,\sqrt{|g|}\Big(3\tau_{\alpha\beta} + 2\mathcal{W}_{\alpha\beta} - \frac{1}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}g_{\underline{ij}}\Big)\delta g^{\underline{\alpha\beta}}, \tag{3.6}$$

for $\tau_{ij} = \dfrac{\delta g^{\underline{\gamma\delta}}}{\delta g^{\underline{ij}}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\epsilon\gamma\beta}T_{\alpha\delta\zeta}$ and $\mathcal{W}_{ij} = g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\epsilon\delta\zeta}\dfrac{\delta T_{\alpha\gamma\beta}}{\delta g^{\underline{ij}}}$.

Based on the variation rule, the variational derivatives $\delta T_{\alpha\gamma\beta}/\delta g^{\underline{ij}}$ are the components $v_{\alpha\gamma\beta ij}$ for the tensor $\hat{v}$ of the type $(0,5)$.

With respect to the equations (3.5, 3.6), we obtain

$$\delta S=\int d^4x\sqrt{|g|}\Big\{R_{\alpha\beta}-\frac{1}{2}Rg_{\underline{\alpha\beta}}+\Lambda g_{\underline{\alpha\beta}}-(v'+w)\Big[3\tau_{\alpha\beta}+2\mathcal{W}_{\alpha\beta}-\frac{1}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\eta}}g^{\underline{\theta\zeta}}T_{\eta\gamma\theta}T_{\epsilon\delta\zeta}g_{\underline{\alpha\beta}}\Big]\Big\}g^{\underline{\alpha\beta}}.$$

The right side of the last equation vanishes if and only if

$$R_{ij}-\frac{1}{2}Rg_{\underline{ij}}+\Lambda g_{\underline{ij}}=(v'+w)\Big(3\tau_{ij}+2\mathcal{W}_{ij}-\frac{1}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}g_{\underline{ij}}\Big). \tag{3.7}$$

The family of Einstein's equations of motion is presented by the equation (3.7).
We proved the following theorem in this way.

Theorem 3.1. *With respect to the equations of motion* (3.7)*, the families of the energy-momentum tensors and its traces are*

$$\begin{aligned}T_{ij}&=(v'+w)\Big(3\tau_{ij}+2\mathcal{W}_{ij}-\frac{1}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}g_{\underline{ij}}\Big),\\ T^{\alpha}_{\alpha}&=(v'+w)\Big(3\tau^{\alpha}_{\alpha}+2\mathcal{W}^{\alpha}_{\alpha}-2g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}\Big),\end{aligned} \tag{3.8}$$

respectively, for the coefficients v', w and the above defined tensors $\hat{\tau}$ and $\hat{\mathcal{W}}$ in the analyzed cosmological model.

With respect to the equations (2.7, 2.8) *and the equalities* (3.8)*, the families of the pressures and energy-densities are*

$$Pressure:\begin{cases}p=-\frac{1}{3}(v'+w)\Big[3\tau^{\alpha}_{\alpha}+2\mathcal{W}^{\alpha}_{\alpha}-\dfrac{3}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}-\Big(3\tau_{\alpha\beta}+2\mathcal{W}_{\alpha\beta}\Big)u^{\alpha}u^{\beta}\Big],\\ p_0=-\frac{1}{3}(v'+w)\Big(3\tau^{\alpha}_{\alpha}+2\mathcal{W}^{\alpha}_{\alpha}-\dfrac{3}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}-3\tau_{00}-2\mathcal{W}_{00}\Big),\end{cases} \tag{3.9}$$

$$Energy-density:\begin{cases}\rho=(v'+w)\Big[\Big(3\tau_{\alpha\beta}+2\mathcal{W}_{\alpha\beta}\Big)u^{\alpha}u^{\beta}-\dfrac{3}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}\Big],\\ \rho_0=(v'+w)\Big[\Big(3\tau_{00}+2\mathcal{W}_{00}\Big)-\dfrac{3}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}\Big].\end{cases} \tag{3.10}$$

The state-parameters $\omega=p\cdot\rho^{-1}$ and $\omega_0=p_0\cdot\rho_0^{-1}$ do not depend of the coefficients u, u', v, v', w which generate curvature tensors of the generalized Riemannian space $\mathbb{GR}_4$. □

Corollary 3.2. *The next equations hold*

$$\frac{1}{3}R_{\alpha\beta}u^{\alpha}u^{\beta}+\frac{1}{6}R-\Lambda=-\frac{1}{3}(v'+w)\Big[3\tau^{\alpha}_{\alpha}+2\mathcal{W}^{\alpha}_{\alpha}-\frac{3}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}-\Big(3\tau_{\alpha\beta}+2\mathcal{W}_{\alpha\beta}\Big)u^{\alpha}u^{\beta}\Big], \tag{3.11}$$

$$R_{\alpha\beta}u^{\alpha}u^{\beta}-\frac{1}{2}R+\Lambda=(v'+w)\Big[\Big(3\tau_{\alpha\beta}+2\mathcal{W}_{\alpha\beta}\Big)u^{\alpha}u^{\beta}-\frac{3}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}\Big], \tag{3.12}$$

in the reference frame u^i and

$$\frac{1}{3}R_{00}+\frac{1}{6}R-\Lambda=-\frac{1}{3}(v'+w)\Big[3\tau^{\alpha}_{\alpha}+2\mathcal{W}^{\alpha}_{\alpha}-\frac{3}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}-3\tau_{00}-2\mathcal{W}_{00}\Big], \tag{3.13}$$

$$R_{00}-\frac{1}{2}R+\Lambda=(v'+w)\Big[3\tau_{00}+2\mathcal{W}_{00}-\frac{3}{2}g^{\underline{\gamma\delta}}g^{\underline{\epsilon\alpha}}g^{\underline{\beta\zeta}}T_{\alpha\gamma\beta}T_{\epsilon\delta\zeta}\Big], \tag{3.14}$$

in the comoving reference frame.

Proof. After equalizing the expressions of the pressures p, p_0 and the energy-densities ρ, ρ_0 from the Table 1 and the equations (3.9, 3.10), we complete the proof for this corollary. □

The equations (3.11, 3.13) are *the equations of equilibrium for metric with respect to the pressure* p (pEQM). The equations (3.12, 3.14) are *the equations of equilibrium for metric with respect to the energy-density* ρ (ρEQM).

With respect to the equations (1.4, 3.4) and the meaning of the term $\mathcal{L}_M$, the torsion-free affine connection spaces (the Riemannian spaces $\mathbb{R}_4$ are the special ones) describe spaces without matter. Hence, the anti-symmetric part of the metric tensor $\hat{g}$ and the torsion tensor of the space $\mathbb{GR}_4$ as well are correlated to a matter.

3.1. Non-symmetric metrics and lagrangian

In this part of the paper, we will examine what are components for the anti-symmetric part $\underset{\vee}{\hat{g}}$ of the metric tensor $\hat{g}$ which correspond to the summand $\mathcal{L}_M$ in the Einstein-Hilbert action $\int d^4x\,\sqrt{|g|}(R+\mathcal{L}_M)$.

Let us consider the non-symmetric metric $\hat{g}$ whose components are

$$g=\begin{bmatrix} s_0(t) & n_0(t) & n_1(t) & n_2(t)\\ -n_0(t) & s_1(t) & n_3(t) & n_4(t)\\ -n_1(t) & -n_3(t) & s_2(t) & n_5(t)\\ -n_2(t) & -n_4(t) & -n_5(t) & s_3(t)\end{bmatrix}. \tag{3.15}$$

The components of the symmetric and anti-symmetric parts for this metric are

$$\underline{g}=\begin{bmatrix} s_0(t) & 0 & 0 & 0\\ 0 & s_1(t) & 0 & 0\\ 0 & 0 & s_2(t) & 0\\ 0 & 0 & 0 & s_3(t)\end{bmatrix} \quad\text{and}\quad \underset{\vee}{g}=\begin{bmatrix} 0 & n_0(t) & n_1(t) & n_2(t)\\ -n_0(t) & 0 & n_3(t) & n_4(t)\\ -n_1(t) & -n_3(t) & 0 & n_5(t)\\ -n_2(t) & -n_4(t) & -n_5(t) & 0\end{bmatrix}. \tag{3.16}$$

The components of the corresponding Christoffel symbols of the first kind are

$$\begin{aligned}
&\Gamma_{0.\underline{00}}=\tfrac{1}{2}s_0'(t),\\
&\Gamma_{0.\underline{11}}=-\tfrac{1}{2}s_1'(t),\quad \Gamma_{1.\underline{01}}=\Gamma_{1.\underline{10}}=\tfrac{1}{2}s_1'(t),\\
&\Gamma_{0.\underline{22}}=-\tfrac{1}{2}s_2'(t),\quad \Gamma_{2.\underline{02}}=\Gamma_{2.\underline{20}}=\tfrac{1}{2}s_2'(t),\\
&\Gamma_{0.\underline{33}}=-\tfrac{1}{2}s_3'(t),\quad \Gamma_{3.\underline{03}}=\Gamma_{3.\underline{30}}=\tfrac{1}{2}s_3'(t),
\end{aligned} \tag{3.17}$$

but $\Gamma_{i.\underline{jk}}=0$ in all other cases.

The components of the covariant torsion tensor are

$$\begin{aligned}
&T_{012}=-T_{021}=-T_{102}=T_{120}=T_{201}=-T_{210}=-n_3'(t),\\
&T_{013}=-T_{031}=-T_{103}=T_{130}=T_{301}=-T_{310}=-n_4'(t),\\
&T_{023}=-T_{032}=-T_{203}=T_{230}=T_{302}=-T_{320}=-n_5'(t),
\end{aligned} \tag{3.18}$$

and $T_{ijk}=0$ otherwise. As we may see, the components of the torsion tensor $\hat{T}$ do not depend of the components $n_0(t)$, $n_1(t)$, $n_2(t)$ of the anti-symmetric part of the metric tensor $\hat{g}$.

With respect to the equation (3.3), we obtain that the term $\mathcal{L}_M$ is

$$\mathcal{L}_M=-\frac{3}{2}(v'+w)\tilde{g}^{-1}s_0(t)\Big\{s_3(t)\big(n_3'(t)\big)^2+s_2(t)\big(n_4'(t)\big)^2+s_1(t)\big(n_5'(t)\big)^2\Big\}. \tag{3.19}$$

After basic computing, one gets

$$\left\{\begin{array}{l}\tau_{00}=6\big(s_1(t)\big)^{-1}\big(s_2(t)\big)^{-1}\big(s_3(t)\big)^{-1}\Big(\big(n_3'(t)\big)^2 s_3(t)+\big(n_4'(t)\big)^2 s_2(t)+\big(n_5'(t)\big)^2 s_1(t)\Big),\\ \tau_{11}=6\big(s_0(t)\big)^{-1}\big(s_2(t)\big)^{-1}\big(s_3(t)\big)^{-1}\Big(\big(n_3'(t)\big)^2 s_3(t)+\big(n_4'(t)\big)^2 s_2(t)\Big),\\ \tau_{22}=6\big(s_0(t)\big)^{-1}\big(s_1(t)\big)^{-1}\big(s_3(t)\big)^{-1}\Big(\big(n_3'(t)\big)^2 s_3(t)+\big(n_5'(t)\big)^2 s_1(t)\Big),\\ \tau_{33}=6\big(s_0(t)\big)^{-1}\big(s_1(t)\big)^{-1}\big(s_2(t)\big)^{-1}\Big(\big(n_4'(t)\big)^2 s_2(t)+\big(n_5'(t)\big)^2 s_1(t)\Big),\\ \tau_{12}=\tau_{21}=6\big(s_0(t)\big)^{-1}\big(s_3(t)\big)^{-1}n4'(t)n_5'(t),\quad \tau_{23}=\tau_{32}=6\big(s_0(t)\big)^{-1}\big(s_1(t)\big)^{-1}n_3'(t)n_4'(t),\\ \tau_{13}=\tau_{31}=-6\big(s_0(t)\big)^{-1}\big(s_2(t)\big)^{-1}n_3'(t)n_5'(t),\\ \tau^\alpha_\alpha=\big(s_0(t)\big)^{-1}\big(s_1(t)\big)^{-1}\big(s_2(t)\big)^{-1}\big(s_3(t)\big)^{-1}\Big\{\big(n_3'(t)\big)^2 s_3(t)\big(s(t)-s_3(t)\big)\\ \quad+\big(n_4'(t)\big)^2 s_2(t)\big(s(t)-s_2(t)\big)+\big(n_5'(t)\big)^2 s_1(t)\big(s(t)-s_1(t)\big)\Big\},\end{array}\right. \tag{3.20}$$

$$\mathcal{W}_{ij}=\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{\gamma=1}^{4}\sum_{\delta=1}^{4}\sum_{\epsilon=1}^{4}\sum_{\zeta=1}^{4}\delta^{(\gamma)}_{(\delta)}\delta^{(\epsilon)}_{(\alpha)}\delta^{(\beta)}_{(\zeta)}\big(s_\alpha(t)\big)^{-1}\big(s_\beta(t)\big)^{-1}\big(s_\gamma(t)\big)^{-1}T_{(\epsilon).(\delta)(\zeta)}v_{(\alpha)(\gamma)(\beta)ij}, \tag{3.21}$$

for $s(t)=s_0(t)+s_1(t)+s_2(t)+s_3(t)$, the above defined tensor $\hat{v}$ and $\tau_{ij}=0$ in all other cases. The brackets about the indices in the equation (3.21) mean that the Einstein's Summation Convention should not be applied to them.

It holds the next theorem.

Theorem 3.3. *The functions $s_0(t)$, $s_1(t)$, $s_2(t)$, $s_3(t)$, $n_3(t)$, $n_4(t)$, $n_5(t)$ are the components of a metric tensor* (3.15) *which corresponds to the Einstein-Hilbert action $S=\int d^4x\sqrt{|g|}\big(R+\mathcal{L}_M\big)$, for $\mathcal{L}_M$ given by the equation* (3.19) *if and only if they satisfy all of the equations of motion* (3.7). □

In the sense of the research in this subsection, the equations of motion (3.7) are differential equations by the functions $n_3(t)$, $n_4(t)$, $n_5(t)$ with respect to the known functions $s_0(t)$, $s_1(t)$, $s_2(t)$, $s_3(t)$. They express the correlation between the energy-momentum tensor with respect to the symmetric and anti-symmetric parts of metrics.

Let us consider a functional proportion $n_3'(t):n_4'(t):n_5'(t)=\alpha_3:\alpha_4:\alpha_5$. Hence, it exists a non-trivial function $n(t)$ such that $n_k'(t)=\alpha_k n(t)$, $k=3,4,5$. With respect to these changes, the equation (3.19) transforms to

$$\mathcal{L}_M=-\frac{3}{2}(v'+w)g^{-1}s_0(t)\Big[\alpha_3^2 s_3(t)+\alpha_4^2 s_2(t)+\alpha_5^2 s_1(t)\Big]\big(n(t)\big)^2. \tag{3.19'}$$

If $\mathcal{L}_M\neq 0$, the last equation involving $n(t)$ as the unknown has two solutions if and only if $(v'+w)\Big[\alpha_3^2 s_3(t)+\alpha_4^2 s_2(t)+\alpha_5^2 s_1(t)\Big]\neq 0$. In the case of $(v'+w)\Big[\alpha_3^2 s_3(t)+\alpha_4^2 s_2(t)+\alpha_5^2 s_1(t)\Big]\mathcal{L}_M\neq 0$ and with respect to the Existence and Uniqueness Theorem, the differential equation (3.19) has two solutions by the functions $\big(n_3(t),n_4(t),n_5(t)\big)$. These solutions are

$$n_{k_0}(t)=\int\alpha_k\sqrt{-\frac{2}{3(v'+w)}\big(s_0(t)\big)^{-1}\Big[\alpha_3^2 s_3(t)+\alpha_4^2 s_2(t)+\alpha_5^2 s_1(t)\Big]^{-1}g\mathcal{L}_M}dt,\quad n_{k_1}(t)=-n_{k_0}(t). \tag{3.22}$$

In other words, the 3-tuples $\big(n_{3_0}(t),n_{4_0}(t),n_{5_0}(t)\big)$ and $\big(n_{3_1}(t),n_{4_1}(t),n_{5_1}(t)\big)$ are the corresponding components of the anti-symmetric part of the metric tensor $\hat{g}$.

After substituting the expressions (3.20, 3.21) into the equations (3.9, 3.10), one gets the corresponding pressures p and p_0 and the densities ρ and ρ_0 as well. The proportions $p\cdot\rho^{-1}$ and $p_0\cdot\rho_0^{-1}$ are the corresponding state parameters.

4. Generalized Einstein's equations of motion II

I. Shapiro [17] studied the theory of gravity with torsion. He analyzed the four-dimensional space-time cosmological models. Into the second section of the paper [17], I. Shapiro considered the non-symmetric affine connection spaces whose affine connection coefficients are $\tilde{\Gamma}^i_{jk} = \Gamma^i_{\underline{jk}} + \mathcal{K}^i_{jk}$, for the Christoffel symbols $\Gamma^i_{\underline{jk}}$ (eq. 1.4, left) and the tensor $\hat{\mathcal{K}}$ of the type $(1,2)$ whose components satisfy the equality $\mathcal{K}^i_{jk} = -\mathcal{K}^i_{kj}$. The torsion tensor $\hat{\tilde{T}}$ is $\hat{\tilde{T}} = 2\hat{\mathcal{K}}$ in the Shaprio's article [17].

We will analyze a generalization of this concept below.

Fourteen years after Shapiro, S. Ivanov and M. Lj. Zlatanović published the paper [6] where they involved the model of the generalized Riemannian space that generalizes the Eisenhart's one.

We considered above the Einstein's equations of motion covered by the generalized Riemannian space with respect to Eisenhart's definition [5]. In this section, we will derive the equations of motion with respect to the generalized Riemannian space $\mathbb{G}\tilde{\mathbb{R}}_4$ defined by S. Ivanov and M. Zlatanović in [6] as the manifold $\mathcal{M}_4$ equipped with non-symmetric metric tensor $\hat{g}$.

The covariant affine connection coefficients $\tilde{\Gamma}_{ijk}$ for the space $\mathbb{G}\tilde{\mathbb{R}}_4$ are [6]

$$\tilde{\Gamma}_{ijk} = \Gamma_{i\underline{jk}} + \frac{1}{2}\Big[\tilde{T}_{jki} + \tilde{T}_{ijk} - \tilde{T}_{kij}\Big] - \frac{1}{2}\Big[g_{\underline{ki}\underset{1}{|}j} + g_{\underline{ij}\underset{1}{|}k} - g_{\underline{kj}\underset{1}{|}i}\Big], \tag{4.1}$$

for the Christoffel symbol of the first kind $\Gamma_{i\underline{jk}}$ obtained with respect to the symmetric metric tensor $\underline{\hat{g}}$ and the covariant derivative $a^i_{j\underset{1}{|}k} = a^i_{j,k} + \tilde{\Gamma}^i_{\alpha k} a^\alpha_j - \tilde{\Gamma}^\alpha_{jk} a^i_\alpha$.

With respect to the equation (4.1), we obtain

$$\tilde{\Gamma}_{i\underline{jk}} = \Gamma_{i\underline{jk}} - \frac{1}{2}\Big[\tilde{T}_{jik} + \tilde{T}_{kij}\Big] - \frac{1}{2}\Big[g_{\underline{ki}\underset{1}{|}j} + g_{\underline{ij}\underset{1}{|}k} - g_{\underline{kj}\underset{1}{|}i}\Big] \quad \text{and} \quad \tilde{\Gamma}_{i\underset{\vee}{jk}} = \frac{1}{2}\tilde{T}_{ijk}. \tag{4.2}$$

After rising the index i in the last equation, we get

$$\tilde{\Gamma}^i_{\underline{jk}} = \Gamma^i_{\underline{jk}} - \frac{1}{2}g^{\underline{i\alpha}}\Big[\tilde{T}_{j\alpha k} + \tilde{T}_{k\alpha j}\Big] - \frac{1}{2}g^{\underline{i\alpha}}\Big[g_{\underline{k\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}k} - g_{\underline{kj}\underset{1}{|}\alpha}\Big] \quad \text{and} \quad \tilde{\Gamma}^i_{\underset{\vee}{jk}} = \frac{1}{2}\tilde{T}^i_{jk}. \tag{4.3}$$

The covariant derivative $\tilde{|}$ with respect to the symmetric affine connection coefficients $\tilde{\Gamma}^i_{\underline{jk}}$ and the covariant derivative (1.5) satisfy the equation

$$a^i_{j\tilde{|}k} = a^i_{j|k} - \frac{1}{2}g^{\underline{i\alpha}}\Big(\tilde{T}_{\beta\alpha k} + \tilde{T}_{k\alpha\beta} + g_{\underline{k\alpha}\underset{1}{|}\beta} + g_{\underline{\alpha\beta}\underset{1}{|}k} - g_{\underline{k\beta}\underset{1}{|}\alpha}\Big)a^\beta_j + \frac{1}{2}g^{\underline{\alpha\beta}}\Big(\tilde{T}_{j\alpha k} + \tilde{T}_{k\alpha j} + g_{\underline{k\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}k} - g_{\underline{kj}\underset{1}{|}\alpha}\Big)a^i_\beta. \tag{4.4}$$

The components of the curvature tensor $\hat{\tilde{R}}$ for the associated space $\tilde{\mathbb{R}}_4$ are

$$\tilde{R}^i_{jmn} = \tilde{\Gamma}^i_{\underline{jm},n} - \tilde{\Gamma}^i_{\underline{jn},m} + \tilde{\Gamma}^\alpha_{\underline{jm}}\tilde{\Gamma}^i_{\underline{\alpha n}} - \tilde{\Gamma}^\alpha_{\underline{jn}}\tilde{\Gamma}^i_{\underline{\alpha m}}. \tag{4.5}$$

These components and the components (1.6) of the curvature tensor $\hat{R}$ for the space $\mathbb{R}_4$ satisfy the equation

$$\tilde{R}^i_{jmn} = R^i_{jmn} - \frac{1}{2}\eta^i_{jm,n} + \frac{1}{2}\eta^i_{jn,m} - \frac{1}{2}\Big(\eta^\alpha_{jm}\Gamma^i_{\underline{\alpha n}} + \eta^i_{\alpha n}\Gamma^\alpha_{\underline{jm}} - \eta^\alpha_{jn}\Gamma^i_{\underline{\alpha m}} - \eta^i_{\alpha m}\Gamma^\alpha_{\underline{jn}}\Big) + \frac{1}{4}\Big(\eta^\alpha_{jm}\eta^i_{\alpha n} - \eta^\alpha_{jn}\eta^i_{\alpha m}\Big), \tag{4.6}$$

for

$$\eta^i_{jk} = g^{\underline{i\alpha}}\Big(\tilde{T}_{j\alpha k} + \tilde{T}_{k\alpha j} + g_{\underline{k\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}k} - g_{\underline{kj}\underset{1}{|}\alpha}\Big). \tag{4.7}$$

With respect to the equations (1.5, 4.6), we proved the next proposition.

Proposition 4.1. *The components of the curvature tensors $\tilde{\hat{R}}$ and $\hat{R}$ respectively given by the equations* (4.5) *and* (1.6) *satisfy the equation*

$$\tilde{R}^i_{jmn} = R^i_{jmn} - \frac{1}{2}\eta^i_{jm|n} + \frac{1}{2}\eta^i_{jn|m} + \frac{1}{4}\Big(\eta^\alpha_{jm}\eta^i_{\alpha n} - \eta^\alpha_{jn}\eta^i_{\alpha m}\Big), \tag{4.8}$$

for the components η^i_{jk} of the tensor $\hat{\eta}$ of the type $(1,2)$*, defined by the equation* (4.7). □

The space $\mathbb{G}\tilde{\mathbb{R}}_4$ is a special affine connection space. For this reason, the components of the curvature tensors $\tilde{\hat{K}}$ of this space are elements of the family [13, 14]

$$\tilde{K}^i_{jmn} = \tilde{R}^i_{jmn} + u\tilde{T}^i_{jm\tilde{|}n} + u'\tilde{T}^i_{jn\tilde{|}m} + v\tilde{T}^\alpha_{jm}\tilde{T}^i_{\alpha n} + v'\tilde{T}^\alpha_{jn}\tilde{T}^i_{\alpha m} + w\tilde{T}^\alpha_{mn}\tilde{T}^i_{\alpha j}. \tag{4.9}$$

After applying the equations (4.4, 4.8) and the equality $\tilde{\Gamma}^i_{\underline{jk}} = \Gamma^i_{\underline{jk}} - \frac{1}{2}\eta^i_{jk}$ as well, one proves the next proposition.

Proposition 4.2. *The family of components of the curvature tensors $\tilde{\hat{K}}$ for the generalized Riemannian space $\mathbb{G}\tilde{\mathbb{R}}_4$ is*

$$\begin{aligned}
\tilde{K}^i_{jmn} = {} & R^i_{jmn} - \frac{1}{2}\eta^i_{jm|n} + \frac{1}{2}\eta^i_{jn|m} + \frac{1}{4}\Big(\eta^\alpha_{jm}\eta^i_{\alpha n} - \eta^\alpha_{jn}\eta^i_{\alpha m}\Big)\\
& + u\tilde{T}^i_{jm|n} + u'\tilde{T}^i_{jn|m} + v\tilde{T}^\alpha_{jm}\tilde{T}^i_{\alpha n} + v'\tilde{T}^\alpha_{jn}\tilde{T}^i_{\alpha m} + w\tilde{T}^\alpha_{mn}\tilde{T}^i_{\alpha j}\\
& - \frac{u}{2}g^{\underline{i\alpha}}\Big(\tilde{T}_{\beta\alpha n} + \tilde{T}_{n\alpha\beta} + g_{\underline{n\alpha}\underset{1}{|}\beta} + g_{\underline{\alpha\beta}\underset{1}{|}n} - g_{\underline{n\beta}\underset{1}{|}\alpha}\Big)\tilde{T}^\beta_{jm}\\
& - \frac{u}{2}g^{\underline{\alpha\beta}}\Big(\tilde{T}_{j\alpha n} + \tilde{T}_{n\alpha j} + g_{\underline{n\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}n} - g_{\underline{nj}\underset{1}{|}\alpha}\Big)\tilde{T}^i_{m\beta}\\
& - \frac{u'}{2}g^{\underline{i\alpha}}\Big(\tilde{T}_{\beta\alpha m} + \tilde{T}_{m\alpha\beta} + g_{\underline{m\alpha}\underset{1}{|}\beta} + g_{\underline{\alpha\beta}\underset{1}{|}m} - g_{\underline{m\beta}\underset{1}{|}\alpha}\Big)\tilde{T}^\beta_{jn}\\
& - \frac{u'}{2}g^{\underline{\alpha\beta}}\Big(\tilde{T}_{j\alpha m} + \tilde{T}_{m\alpha j} + g_{\underline{m\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}m} - g_{\underline{mj}\underset{1}{|}\alpha}\Big)\tilde{T}^i_{n\beta}\\
& + \frac{u+u'}{2}g^{\underline{\alpha\beta}}\Big(\tilde{T}_{m\alpha n} + \tilde{T}_{n\alpha m} + g_{\underline{n\alpha}\underset{1}{|}m} + g_{\underline{\alpha m}\underset{1}{|}n} - g_{\underline{nm}\underset{1}{|}\alpha}\Big)\tilde{T}^i_{j\beta}.
\end{aligned} \tag{4.10}$$

The corresponding family of components $\tilde{K}_{ij} = \tilde{K}^\alpha_{ij\alpha}$ of the Ricci curvatures is

$$\begin{aligned}
\tilde{K}_{ij} = {} & R_{ij} - \frac{1}{2}\eta^\alpha_{ij|\alpha} + \frac{1}{2}\eta^\alpha_{i\alpha|j} + \frac{1}{4}\Big(\eta^\alpha_{ij}\eta^\beta_{\alpha\beta} - \eta^\alpha_{i\beta}\eta^\beta_{j\alpha}\Big)\\
& + u\tilde{T}^\alpha_{ij|\alpha} + u'\tilde{T}^\alpha_{i\alpha|j} + v\tilde{T}^\alpha_{ij}\tilde{T}^\beta_{\alpha\beta} - (v'+w)\tilde{T}^\alpha_{i\beta}\tilde{T}^\beta_{j\alpha}\\
& - \frac{u}{2}g^{\underline{\alpha\gamma}}\Big(\tilde{T}_{\gamma\alpha\beta} + g_{\underline{\gamma\alpha}\underset{1}{|}\beta} + g_{\underline{\alpha\beta}\underset{1}{|}\gamma} - g_{\underline{\gamma\beta}\underset{1}{|}\alpha}\Big)\tilde{T}^\beta_{ij}\\
& - \frac{u}{2}g^{\underline{\alpha\beta}}\Big(\tilde{T}_{i\alpha\gamma} + \tilde{T}_{\gamma\alpha i} + g_{\underline{\gamma\alpha}\underset{1}{|}i} + g_{\underline{\alpha i}\underset{1}{|}\gamma} - g_{\underline{\gamma i}\underset{1}{|}\alpha}\Big)\tilde{T}^\gamma_{j\beta}\\
& + \frac{u}{2}g^{\underline{\alpha\beta}}\Big(\tilde{T}_{j\alpha\gamma} + \tilde{T}_{\gamma\alpha j} + g_{\underline{\gamma\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}\gamma} - g_{\underline{\gamma j}\underset{1}{|}\alpha}\Big)\tilde{T}^\gamma_{i\beta}\\
& + \frac{u'}{2}g^{\underline{\alpha\beta}}\Big(\tilde{T}_{i\alpha j} + \tilde{T}_{j\alpha i} + g_{\underline{j\alpha}\underset{1}{|}i} + g_{\underline{\alpha i}\underset{1}{|}j} - g_{\underline{ji}\underset{1}{|}\alpha}\Big)\tilde{T}^\gamma_{\beta\gamma}.
\end{aligned} \tag{4.11}$$

The family $\tilde{K} = g^{\underline{\gamma\delta}}\tilde{K}_{\gamma\delta}$ of scalar curvatures of the space $\mathbb{G}\tilde{\mathbb{R}}_4$ is

$$\begin{aligned}\tilde{K} = R &- \frac{1}{2}g^{\underline{\beta\gamma}}\eta^{\alpha}_{\beta\gamma|\alpha} + \frac{1}{2}g^{\underline{\beta\gamma}}\eta^{\alpha}_{\alpha\beta|\gamma} + \frac{1}{4}g^{\underline{\gamma\delta}}\Big(\eta^{\alpha}_{\gamma\delta}\eta^{\beta}_{\alpha\beta} - \eta^{\alpha}_{\beta\gamma}\eta^{\beta}_{\alpha\delta}\Big) - (v' + w)g^{\underline{\gamma\delta}}\tilde{T}^{\alpha}_{\gamma\beta}\tilde{T}^{\beta}_{\delta\alpha}\\ &+ u' g^{\underline{\beta\gamma}}\tilde{T}^{\alpha}_{\beta\alpha|\gamma} + \frac{u'}{2}g^{\underline{\alpha\beta}}g^{\underline{\delta\epsilon}}\Big(\tilde{T}_{\delta\alpha\epsilon} + \tilde{T}_{\epsilon\alpha\delta} + g_{\underline{\epsilon\alpha}\underset{1}{|}\delta} + g_{\underline{\alpha\delta}\underset{1}{|}\epsilon} - g_{\underline{\epsilon\delta}\underset{1}{|}\alpha}\Big)\tilde{T}^{\gamma}_{\beta\gamma}.\end{aligned} \tag{4.12}$$

In the equations (4.10, 4.11, 4.12)*, u, u', v, v', w are the corresponding coefficients.* □

As we may see, the part of the scalar curvature $\tilde{K}$ which corresponds to the matter is

$$\begin{aligned}\tilde{\mathcal{L}}_M = &-\frac{1}{2}g^{\underline{\beta\gamma}}\eta^{\alpha}_{\beta\gamma|\alpha} + \frac{1}{2}g^{\underline{\beta\gamma}}\eta^{\alpha}_{\alpha\beta|\gamma} + \frac{1}{4}g^{\underline{\gamma\delta}}\Big(\eta^{\alpha}_{\gamma\delta}\eta^{\beta}_{\alpha\beta} - \eta^{\alpha}_{\beta\gamma}\eta^{\beta}_{\alpha\delta}\Big) - (v' + w)g^{\underline{\gamma\delta}}\tilde{T}^{\alpha}_{\gamma\beta}\tilde{T}^{\beta}_{\delta\alpha}\\ &+ u' g^{\underline{\beta\gamma}}\tilde{T}^{\alpha}_{\beta\alpha|\gamma} + \frac{u'}{2}g^{\underline{\alpha\beta}}g^{\underline{\delta\epsilon}}\Big(\tilde{T}_{\delta\alpha\epsilon} + \tilde{T}_{\epsilon\alpha\delta} + g_{\underline{\epsilon\alpha}\underset{1}{|}\delta} + g_{\underline{\alpha\delta}\underset{1}{|}\epsilon} - g_{\underline{\epsilon\delta}\underset{1}{|}\alpha}\Big)\tilde{T}^{\gamma}_{\beta\gamma}.\end{aligned} \tag{4.13}$$

Let be $\tilde{\mathcal{L}}_M = \tilde{\mathcal{L}}[u', v', w]$, for the coefficients u', v', w. The meaning of the square brackets in the last equality is that the field $\tilde{\mathcal{L}}$ depends of the linear combination of necessary terms with respect to the coefficients u', v', w. In this case, the full lagrangian with torsion is $\mathcal{L} = \Big(R - 2\Lambda + \tilde{\mathcal{L}}[u', v', w]\Big)\sqrt{|g|}$.

The corresponding Einstein-Hilbert action with torsion is

$$\tilde{S} = \int d^4x\,\sqrt{|g|}\Big(R - 2\Lambda + \tilde{\mathcal{L}}[u', v', w]\Big). \tag{4.14}$$

We need to consider the variations of the functionals

$$\tilde{S}_1 = \int d^4x\,\sqrt{|g|}\Big(R - 2\Lambda\Big) \quad \text{and} \quad \tilde{S}_2 = \int d^4x\,\sqrt{|g|}\tilde{\mathcal{L}}[u', v', w].$$

The variation of the first of these functionals is given by the equation (3.5). The variation of the second functional is

$$\delta\tilde{S}_2 = \int d^4x\,\sqrt{|g|}\Big\{\frac{\delta\tilde{\mathcal{L}}[u', v, w]}{\delta g^{\underline{\alpha\beta}}} - \frac{1}{2}g_{\underline{\alpha\beta}}\tilde{\mathcal{L}}[u', v', w]\Big\}\delta g^{\underline{\alpha\beta}}. \tag{4.15}$$

With respect to the Quotient Rule, the variational derivatives $\delta\tilde{\mathcal{L}}[u', v', w]/\delta g^{\underline{ij}}$ are the components of the tensor $\hat{\tilde{V}}$ of the type $(0, 2)$, i.e.

$$\frac{\delta\tilde{\mathcal{L}}[u', v', w]}{\delta g^{\underline{ij}}} = \tilde{\mathcal{V}}_{ij}. \tag{4.16}$$

If sum the equations (3.5) and (4.15), we will obtain

$$\delta\tilde{S} = \int d^4x\,\sqrt{|g|}\Big\{R_{\alpha\beta} - \frac{1}{2}Rg_{\underline{\alpha\beta}} + \Lambda g_{\underline{\alpha\beta}} + \tilde{\mathcal{V}}_{\alpha\beta} - \frac{1}{2}g_{\underline{\alpha\beta}}\tilde{\mathcal{L}}[u', v', w]\Big\}\delta g^{\underline{\alpha\beta}}. \tag{4.17}$$

The right side of the last equation vanishes if and only if

$$R_{ij} - \frac{1}{2}Rg_{\underline{ij}} + \Lambda g_{\underline{ij}} = -\tilde{\mathcal{V}}_{ij} + \frac{1}{2}g_{\underline{ij}}\tilde{\mathcal{L}}[u', v', w], \tag{4.18}$$

which are the corresponding Einstein's equations of motion.

It holds the following theorem.

Theorem 4.3. *With respect to the family of the Einstein's equations of motion* (4.18)*, the family of energy-momentum tensors and the family of their traces are*

$$\tilde{T}_{ij} = -\tilde{\mathcal{V}}_{ij} + \frac{1}{2}g_{\underline{ij}}\tilde{\mathcal{L}}[u',v',w] \quad \text{and} \quad \tilde{T}^{\alpha}_{\alpha}[u',v',w] = -\tilde{\mathcal{V}}^{\alpha}_{\alpha} + 2\tilde{\mathcal{L}}[u',v',w]. \tag{4.19}$$

The pressure, energy-density and state-parameter may be obtained by the substituting the equation (4.19) *into the equations* (2.7, 2.8)*.*

The pEQM and ρEQM are

$$\frac{1}{3}R_{\alpha\beta}u^{\alpha}u^{\beta} + \frac{1}{6}R - \Lambda = \frac{1}{3}\tilde{\mathcal{V}}^{\alpha}_{\alpha} - \frac{1}{3}\tilde{\mathcal{V}}_{\alpha\beta}u^{\alpha}u^{\beta} - \frac{1}{2}\tilde{\mathcal{L}}[u',v',w], \tag{4.20}$$

$$R_{\alpha\beta}u^{\alpha}u^{\beta} - \frac{1}{2}R + \Lambda = -\tilde{\mathcal{V}}_{\alpha\beta}u^{\alpha}u^{\beta} + \frac{1}{2}\tilde{\mathcal{L}}[u',v',w], \tag{4.21}$$

in the reference frame u^i and

$$\frac{1}{3}R_{00} + \frac{1}{6}R - \Lambda = \frac{1}{3}\tilde{\mathcal{V}}^{\alpha}_{\alpha} - \frac{1}{3}\tilde{\mathcal{V}}_{00} - \frac{1}{2}\tilde{\mathcal{L}}[u',v',w], \tag{4.22}$$

$$R_{00} - \frac{1}{2}R + \Lambda = -\tilde{\mathcal{V}}_{00} + \frac{1}{2}g_{\underline{00}}\tilde{\mathcal{L}}[u',v',w], \tag{4.23}$$

in the comoving reference frame. □

The part $\tilde{\mathcal{L}}[u',v',w]$ which corresponds to the matter with respect to the space $\mathbb{G}\tilde{\mathbb{R}}_4$ and the corresponding part $\mathcal{L}_M$ which corresponds to the matter with respect to the space $\mathbb{G}\mathbb{R}_4$ satisfy the equality

$$\tilde{\mathcal{L}}[u',v',w] = \mathcal{L}_M + \Big(\tilde{\mathcal{L}}[u',v',w] - \mathcal{L}_M\Big).$$

Indirectly, we will find the difference between the energy-momentum tensors, pressures, energy-densities and state parameters obtained with respect to the families (3.4) and (4.13) in the following section.

5. Linearity

The question which arises is how much would we change the energy-momentum tensor, the pressure, the energy-density and the state-parameter obtained with respect to the part $\mathcal{L}_M$ if we get the part $l\mathcal{L}_M + f\mathcal{F}$ for some field $\mathcal{F}$ and the real or complex scalars l and f. We will answer this question more generally below.

Let us consider the field

$$\overset{\diamond}{\mathcal{L}}_M = \underset{1}{\alpha}\underset{1}{\mathcal{L}}_M + \ldots + \underset{s}{\alpha}\underset{s}{\mathcal{L}}_M, \tag{5.1}$$

for some $s \in \mathbb{N}$, fields $\underset{1}{\mathcal{L}}, \ldots, \underset{s}{\mathcal{L}}$ and real or complex coefficients $\underset{1}{\alpha}, \ldots, \underset{s}{\alpha}$.

The corresponding Einstein-Hilbert action is

$$\overset{\diamond}{S} = \int d^4x\,\sqrt{|g|}\Big(R - 2\Lambda + \overset{\diamond}{\mathcal{L}}_M\Big). \tag{5.2}$$

The last equation may be equivalently treated as the equation (4.14) after changing the field $\tilde{\mathcal{L}}_M[u',v',w]$ by the field $\overset{\diamond}{\mathcal{L}}_M$. Any of the fields $\underset{r}{\mathcal{L}}$, $r=1,\dots,s$, generates the corresponding tensor $\underset{r}{\hat{\mathcal{V}}}$ analogous to the tensor $\hat{\mathcal{V}}$ whose components are given by the equation (4.16). For this reason, the components of the tensor $\overset{\diamond}{\hat{\mathcal{V}}}$ obtained with respect to the field $\overset{\diamond}{\mathcal{L}}_M$ are

$$\overset{\diamond}{\mathcal{V}}_{ij} = \underset{1}{\alpha}\underset{1}{\mathcal{V}}_{ij} + \ldots + \underset{s}{\alpha}\underset{s}{\mathcal{V}}_{ij}. \tag{5.3}$$

Hence, the Einstein's equations of motion are

$$R_{ij} - \frac{1}{2}Rg_{\underline{ij}} + 2\Lambda g_{\underline{ij}} = -\sum_{r=1}^{s}\underset{r}{\alpha}\underset{r}{\mathcal{V}}_{ij} + \frac{1}{2}g_{\underline{ij}}\sum_{r=1}^{s}\underset{r}{\alpha}\underset{r}{\mathcal{L}}. \tag{5.4}$$

It is the set of the corresponding Einstein's equations of motion.
The equations of motion (5.4) may be rewritten as

$$R_{ij} - \frac{1}{2}Rg_{\underline{ij}} + 2\Lambda g_{\underline{ij}} = -\sum_{r=1}^{s}\underset{r}{\alpha}\Big(\underset{r}{\mathcal{V}}_{ij} - \frac{1}{2}g_{\underline{ij}}\underset{r}{\mathcal{L}}\Big). \tag{5.5}$$

Based on the equations (2.7, 2.8, 5.5), we obtain the following equalities

<table>
<tr><td>In reference system u^i</td><td>In comoving reference system $u^i=\delta^i_0$</td></tr>
<tr><td colspan="2">$\overset{\diamond}{T}_{ij} = -\sum_{r=1}^{s}\underset{r}{\alpha}\Big(\underset{r}{\mathcal{V}}_{ij} - \frac{1}{2}g_{\underline{ij}}\underset{r}{\mathcal{L}}\Big)$
$\overset{\diamond}{T}{}^{\alpha}_{\alpha} = -\sum_{r=1}^{s}\underset{r}{\alpha}\Big(\underset{r}{\mathcal{V}}{}^{\alpha}_{\alpha} - 2\underset{r}{\mathcal{L}}\Big)$</td></tr>
<tr><td>$\overset{\diamond}{p} = -\frac{1}{3}\sum_{r=1}^{s}\underset{r}{\alpha}\Big(\underset{r}{\mathcal{V}}_{\alpha\beta}u^{\alpha}u^{\beta} - \underset{r}{\mathcal{V}}{}^{\alpha}_{\alpha} - \underset{r}{\mathcal{L}}\Big)$
$\overset{\diamond}{\rho} = -\sum_{r=1}^{s}\underset{r}{\alpha}\Big(\underset{r}{\mathcal{V}}_{\alpha\beta}u^{\alpha}u^{\beta} - \frac{1}{2}\underset{r}{\mathcal{L}}\Big)$
$\overset{\diamond}{\omega} = \overset{\diamond}{p}\overset{\diamond}{\rho}{}^{-1}$</td><td>$\overset{\diamond}{p} = -\frac{1}{3}\sum_{r=1}^{s}\underset{r}{\alpha}\Big(\underset{r}{\mathcal{V}}_{00} - \underset{r}{\mathcal{V}}{}^{\alpha}_{\alpha} - \underset{r}{\mathcal{L}}\Big)$
$\overset{\diamond}{\rho}_0 = -\sum_{r=1}^{s}\underset{r}{\alpha}\Big(\underset{r}{\mathcal{V}}_{00} - \frac{1}{2}\underset{r}{\mathcal{L}}\Big)$
$\overset{\diamond}{\omega}_0 = \overset{\diamond}{p}_0\overset{\diamond}{\rho}{}_0^{-1}$</td></tr>
</table>

Table 2: Linear combinations of energy-momentums, pressures and energy-densities

With respect to the equalities (5.3) and the expressions in the Table 2, we proved the validity of the following theorem.

Theorem 5.1. *The energy-momentum tensor and its trace, the pressure and the energy-density are linear by the summands into the lagrangian which corresponds the matter. Their values are equal to the linear combinations of the corresponding values generated by the separate summands of the lagrangian.*

The corresponding pEQM and ρEQM of the system are the linear combinations of the pEQMs and ρEQMs of the separate subsystems generated by the summands in the lagrangian which corresponds to the whole system.

The state-parameter is not linear as the previous magnitudes. □

6. Conclusion

In the section 2, we geometrically interpreted the Madsen's formulae about energy-momentum tensors, pressures and energy-densities. With respect to these interpretations, we computed these magnitudes (see the Table 1).

In the section 3, we expressed the energy-momentum tensor with respect to the curvature tensors of a generalized Riemannian space in the sense of Eisenhart's definition. We also proved that the part $\mathcal{L}_M$ generates two generalized Riemannian spaces in the sense of Eisenhart's definition. In this section, it is concluded that the anti-symmetric part of a metric tensor corresponds to a matter.

In the section 4, we analyzed the differences and similarities between the results presented in the Shapiro's paper [17] and the model of the generalized Riemannian space involved by S. Ivanov and M. Lj. Zlatanović [6]. We explicitly obtained the corresponding energy-momentum tensor but the pressure, energy-density and state parameter may be obtained with respect to the corresponding formulae from the section 2.

In the Sections 3 and 4, we obtained the systems of equations for the equilibriums between symmetric affine connections and torsions $\big($the systems pEQM and ρEQM$\big)$.

In the section 5, we analyzed the linearity of the energy-momentum tensors, energy-densities, pressures and state parameters under summing of matter fields $\big($see the Table 2$\big)$. We also analyzed the linearity of pEQM and ρEQM in this section.

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