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Existence of the Solution to Second Order Differential Equation Through Fixed Point Results for Nonlinear F-Contractions Involving w_0-Distacne

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Abstract. In the present paper, the aim is to obtain some new fixed point theorems for nonlinear F-contractions involving generalized distance to prove the existence of solution to second order differential equation related to conversion of solar energy to electrical energy. Non-trivial examples are also presented, to illustrate the obtained results and to show that new results are proper generalization of recently appeared results in the literature.

1. Introduction and Preliminaries

A fundamental result in fixed point theory is the Banach Contraction Principle [2]. In the last few decades, many authors have been extended and generalized the Banach's contraction principle in several ways. There is vast amount of literature dealing with extensions of Banach contraction principle (see [1, 3, 4, 12, 16–19]). One of an attractive and important generalization is given by Wardowski in [23]. He introduced a new type of contraction called F-contraction and proved a new fixed point theorem concerning F-contraction.

Definition 1.1. *[23] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be F-contraction if there exist $\tau > 0$ such that*

$$d(Tx, Ty) > 0 \text{ implies } \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \text{ for all } x, y \in X, \tag{1}$$

where $F : (0, \infty) \to \mathbb{R}$ is a function satisfying:

(F1) F is strictly increasing;

(F2) for all sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \alpha_n = 0$, if and only if $\lim_{n\to\infty} F(t_n) = -\infty$;

(F3) there exist $0 < k < 1$ such that $\lim_{\alpha\to 0^+} t^k F(t) = 0$.

We denote by $\Delta(F)$, the collection of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying $(F1)$, $(F2)$ and $(F3)$.

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Theorem 1.2. *[23] Let (X, d) be a complete metric space and $T : X \to X$ be a F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Further, Turinici in [22], replaced $(F2)$ by the following condition:

$(F2^{'})$ $\lim_{t\to 0^+} F(t) = -\infty$.

Note that, in general, $F \in \Delta(F)$ is not continuous. However, by $(F1)$ and the properties of the monotone functions, we have the following proposition.

Proposition 1.3. *[22] Let $F : (0, \infty) \to \mathbb{R}$ be a function satisfying $(F1)$ and $(F2)$, then there exists a countable subset $\Lambda(F) \subseteq (0, 1)$ such that*

$$F(t-0) = F(t) = F(t+0) \text{ for each } t \in (0,1) \setminus \Lambda(F).$$

Lemma 1.4. *[22] Let $F : (0, \infty) \to \mathbb{R}$ be a function satisfying $(F1)$ and $(F2^{'})$. Then for each sequence $\{t_n\}$ in $(0, 1)$*

$$F(t_n) \to -\infty \Rightarrow t_n \to 0.$$

After this, many authors generalized the F-contraction in several ways (see [6, 9, 10]). In 2015, Klim and Wardowski [14] extended the concept of F-contractive mappings to the case of nonlinear F-contractions and proved a fixed point theorems via the dynamic processes. In 2017, Wardowski [24] omitted one of the conditions of F-contraction and introduced (φ, F)-contraction (or nonlinear F-contraction).

Definition 1.5. *[24] A mapping $T : X \to X$ is said to be a (φ, F)-contraction (or nonlinear F-contraction), if there exist $F \in \mathcal{F}$ and a function $\varphi : (0, \infty) \to (0, \infty)$ satisfying:*

(H1) $\liminf_{s\to t^+} \varphi(s) > 0$, *for all* $t \geq 0$.

(H2) $\varphi(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y))$, *for all* $x, y \in X$ *such that* $Tx \neq Ty$

Theorem 1.6. *[24] Let (X, d) be a complete metric space and let $T : X \to X$ be a (φ, F)-contraction. Then T has a unique fixed point in X.*

In 1996, Kada, Suzuki and Takahashi [15] introduced the generalized metric, which is known as the w-distance and improved Caristi's fixed point theorem, Ekeland's variational principle and nonconvex minimization theorem using the results of Takahashi [15], for more results on the w-distance, (see [5, 7, 11, 20, 21]).

Definition 1.7. *[15] Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X, if the following are satisfied:*

(a) $p(x, z) \leq p(x, y) + p(y, z)$, *for all* $x, y, z \in X$;

(b) for all $x \in X$, $p(x, .) : X \to [0, \infty)$ is lower semicontinuous (i.e., if $x \in X$ and $y_n \to y \in X$, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$);

(c) for any $\epsilon > 0$, $\exists\, \delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) \leq \epsilon$.

Example 1.8. *[15] Let (X, d) be a metric space. A mapping $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = k > 0 \quad \text{for all} \quad x, y \in X$$

is a w-distance on X. The mapping p is not a metric, since $p(x, x) \neq 0$ for any $x \in X$.

Example 1.9. *Let $(X, \|.\|)$ be a normed linear space. A mapping $p : X \times X \to [0, \infty)$ defined by $p(x, y) = \|x\| + \|y\|$ for all $x, y \in X$ is a w-distance on X.*

Lemma 1.10. *[15] Let X be a metric space with metric d and let p be a w-distance on X. Let $\{u_n\}$ and $\{v_n\}$ be sequences in X, let α_n and β_n be sequences in $[0, +\infty)$ converging to 0, and let $u, v, w \in X$. Then the following hold:*

(i) *if* $p(u_n, v) \leq \alpha_n$ *and* $p(u_n, w) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $v = w$. *In particular, if* $p(u, v) = 0$, *and* $p(u, w) = 0$, *then* $v = w$;

(ii) *if* $p(u_n, v_n) \leq \alpha_n$ *and* $p(u_n, w) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* v_n *converges to* w;

(iii) *if* $p(u_n, u_n) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{u_n\}$ *is a cauchy sequence;*

(iv) *if* $p(v, u_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{u_n\}$ *is a cauchy sequence.*

Recently, in [13], Kostić et al. introduced a special type of w-distance named as w_0- distance, to extend best proximity results of Tchier et al. [22] involving simulation functions. The w_0-distance is slightly different to the original w-distance, in regard that the lower semicontinuity with respect to both variables (when one of them is fixed) is supposed.

Definition 1.11. *[13] Let X be a metric space with metric d. Then a function* $p : X \times X \to [0.\infty)$ *is called a* w_0*-distance on X, if the following are satisfied:*

($P1$) $p(x, z) \leq p(x, y) + p(y, z)$, *for all* $x, y, z \in X$,

($P2$) *for any* $x \in X$, *functions* $p(x, .), \ p(., x) : X \to [0, \infty)$ *are lower semicontinuous,*

($P3$) *for any* $\epsilon > 0$, $\exists\, \delta > 0$ *such that* $p(z, x) \leq \delta$ *and* $p(z, y) \leq \delta$ *implies* $d(x, y) \leq \epsilon$.

Let (X, d) be a metric space, $p : X \times X \to [0, \infty)$ a w_0-distance on X, and, for every $x, y \in X$ let

$$\mu(x, y) := max\{p(x, y), p(y, x)\}.$$

Remark 1.12. *[13] The function* $\mu : X \times X \to [0, \infty)$ *has the following properties (for all* $x, y, z \in X$*)*

(1) $\mu(x, y) = 0 \Rightarrow x = y$;

(2) $\mu(x, y) = \mu(y, x)$, *i.e.* μ *is symmetric;*

(3) $\mu(x, y) \leq \mu(x, z) + \mu(z, y)$, *i.e.* μ *satisfies the triangle inequality.*

Example 1.13. *[13] Let* $X = [0, \infty)$ *be a metric space with metric* $d(x, y) =\mid x - y \mid$, *for all* $x, y \in X$, *then p defined by* $p(x, y) = x + y$ *for all* $x, y \in X$ *is a* w_0*-distance.*

Example 1.14. *[13] Let* $X = [0, \infty)$ *be a metric space with usual metric d. Let* $p : X \times X \to \mathbb{R}$ *be defined as*

$$p(x, y) = k \in (0, 1) \ for\ all\ x, y \in X$$

and let $\alpha : X \to [0, \infty)$ *be defined by*

$$\alpha(x) = \begin{cases} e^{-x} & if\ x > 0 \\ 2 & if\ x = 0 \end{cases}$$

A function $q : X \times X \to [0, \infty)$ *defined by*

$$q(x, y) = \max\{\alpha(x), k\}, \ for\ all\ x, y \in X$$

is then a w-distance on X. However, q is not a w_0*-distance on X, since for any sequence* $\{x_n\} \subset (0, \infty)$ *such that* $x_n \to 0$ *we have*

$$\liminf_{n\to\infty} q(x_n, y) = \liminf_{n\to\infty} \max\{e^{-x_n}, k\} = 1 < q(0, y) = \max\{\alpha(0), k\} = 2$$

In this paper, we prove fixed point theorems for generalized non-linear F-contraction involving w_0-distance in the setting of complete metric spaces. Our results generalize many results appearing recently in the literature including Wardowski [24].

2. Fixed Point Theorems for Generalized Non-Linear *F*-contractions

In this section, we obtain fixed point results for nonlinear F-contraction involving w_0-distance. Henceforth, we will denote by Φ, the collection of all functions $\varphi : (0,\infty) \to (0,\infty)$ satisfying

$$\liminf_{s\to t^+} \varphi(s) > 0, \quad \text{for all } t \geq 0. \tag{2}$$

Theorem 2.1. *Let (X, d) be a complete metric space with a w_0-distance p and $T : X \to X$. Assume that there exists $\varphi \in \Phi$, a non-decreasing real-valued function F_1 on $(0,\infty)$ and a continuous function $F_2 : (0,\infty) \to \mathbb{R}$ satisfying condition $(F2')$ such that following hold:*

(C1) *$F_1(a) \leq F_2(a)$ for all $a > 0$;*

(C2) *$\mu(Tx, Ty) > 0$ implies $\varphi(\mu(x, y)) + F_2(\mu(Tx, Ty)) \leq F_1(\mu(x, y))$ for all $x, y \in X$.*

Then T has a unique fixed point in X.

Proof. Take any $x_0 \in X$ and define the sequence $x_n = T^n x_0$ and $\gamma_n = \mu(x_{n-1}, x_n)$, $n \in \mathbb{N}$. If $\mu(x_{n-1}, x_n) = 0$, for some $n \in \mathbb{N}$ then $x_{n-1} = x_n$ and so x_{n-1}, is a fixed point of T. Assume that $\gamma_n > 0$ for all $n \in \mathbb{N}$, then from (C2), we have

$$\varphi(\gamma_n) + F_2(\gamma_{n+1}) \leq F_1(\gamma_n) \quad \text{for all} \quad n \in \mathbb{N}, \tag{3}$$

which implies

$$F_2(\gamma_{n+1}) \leq F_1(\gamma_n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{4}$$

From (C1) and using (4), we get

$$F_1(\gamma_{n+1}) \leq F_1(\gamma_n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{5}$$

Since F_1 is non-decreasing, so (5) implies $\{\gamma_n\}$ is a decreasing sequence of positive real numbers. Also, since $\varphi \in \Phi$, there exists $c > 0$ and $n_0 \in \mathbb{N}$ such that $\varphi(\gamma_n) > c$, for all $n \geq n_0$. From (C1) and (C2), we have

$$F_2(\gamma_n) \leq F_1(\gamma_{n-1}) - \varphi(\gamma_{n-1}) \leq F_2(\gamma_{n-1}) - \varphi(\gamma_{n-1}), \tag{6}$$

which further implies

$$\begin{aligned} F_2(\gamma_n) &\leq F_2(\gamma_{n-2}) - \varphi(\gamma_{n-1}) - \varphi(\gamma_{n-1}) \\ &\vdots \\ &\leq F_2(\gamma_1) - \sum_{i=1}^{n-1} \varphi(\gamma_i) \\ &= F_2(\gamma_1) - \sum_{i=1}^{n_0-1} \varphi(\gamma_i) - \sum_{i=n_0}^{n-1} \varphi(\gamma_i) \\ &\leq F_2(\gamma_1) - (n - n_0)c, \quad n \geq n_0. \end{aligned} \tag{7}$$

Tending with $n \to \infty$ in (7), we get $F_2(\gamma_n) \to -\infty$ and, by $(F2')$, we have

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \mu(x_{n-1}, x_n) = 0. \tag{8}$$

Next, we claim that

$$\lim_{n,m\to\infty} \mu(x_n, x_m) = 0. \tag{9}$$

If (9) is not true then there exist $\eta > 0$ such that for every $q \geq 0$ with $m_k > n_k \geq q$

$$\mu(x_m, x_n) > \eta.$$

Also there exists $q_0 \in \mathbb{N}$ such that

$$\gamma_{q_0} = \mu(x_{n-1}, x_n) < \eta \quad \text{for all } n \geq q_0. \tag{10}$$

Consider two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ satisfying

$$q_0 \leq n_k < m_k < m_k + 1 \text{ and } \mu(x_{m_k}, x_{n_k}) > \eta \text{ for all } k. \tag{11}$$

Observe that

$$\mu(x_{m_k-1}, x_{n_k}) \leq \eta \text{ for all } k, \tag{12}$$

where m_k is chosen as minimal index for which (11) is satisfied. Also note that because of (10) and (11), the case $n_{k+1} \leq m_k$ is impossible. Thus, $n_{k+2} \leq m_k$ for all k. It implies that

$$n_k + 1 < m_k < m_k + 1 \text{ for all } k.$$

Using the triangle inequality for μ, by (11) and (12) we get

$$\eta < \mu(x_{m_k}, x_{n_k}) \leq \mu(x_{m_k}, x_{m_k-1}) + \mu(x_{m_k-1}, x_{n_k}) \leq \gamma_{m_k} + \eta. \tag{13}$$

Tending to the limit $k \to \infty$ in (13) and using (8), we get

$$\lim_{k\to\infty} \mu(x_{m_k}, x_{n_k}) = \eta. \tag{14}$$

Now tending limit $k \to \infty$ in the inequalities

$$\mu(x_{m_k+1}, x_{n_k+1}) \leq \mu(x_{m_k+1}, x_{m_k}) + \mu(x_{m_k}, x_{n_k+1}) + \mu(x_{n_k}, x_{n_k+1}) \tag{15}$$

and

$$\mu(x_{m_k}, x_{n_k}) - \mu(x_{m_k}, x_{m_k+1}) - \mu(x_{n_k}, x_{n_k+1}) \leq \mu(x_{m_k+1}, x_{n_k+1}), \tag{16}$$

by using (8) and (14), we obtain

$$\lim_{n\to\infty} \mu(x_{m_k+1}, x_{n_k+1}) = \eta. \tag{17}$$

From (C2) and (C2), we get

$$\begin{aligned} \varphi(\mu(x_{m_k}, x_{n_k})) &\leq F_1(\mu(x_{m_k}, x_{n_k})) - F_2(\mu(x_{m_k+1}, x_{n_k+1})) \\ &\leq F_2(\mu(x_{m_k}, x_{n_k})) - F_2(\mu(x_{m_k+1}, x_{n_k+1})). \end{aligned} \tag{18}$$

By passing limit $k \to \infty$, using (14), (17) and using the fact that F_2 is continuous, we get

$$\liminf_{s\to\eta^+} \varphi(s) \leq 0,$$

which is contradiction to (2) and hence (9) holds.
Thus, by Lemma 1.10(*iii*), $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete metric space, so

$$\lim_{n\to\infty} x_n = x^* \in X. \tag{19}$$

This means that for all $\epsilon > 0$ there exist $q \in \mathbb{N}$ such that $\mu(x_n, x_m) < \epsilon$ for all $m > n \geq q$. Now for a fixed $n \in \mathbb{N}$ with $n \geq q$ the function $p(x_n, .)$ is lower semi-continuous; hence we obtain

$$p(x_n, u) \leq \liminf_{m\to\infty} p(x_n, x_m) < \epsilon.$$

Thus,

$$\lim_{n\to\infty} p(x_n, u) = 0. \tag{20}$$

Similarly $\lim_{n\to\infty} p(u, x_n) = 0$, which together with (20) yields

$$\lim_{n\to\infty} \mu(x_n, u) = 0. \tag{21}$$

Now from (C2), we have

$$F_2(\mu(Tx_n, Tu)) \le F_1(\mu(x_n, u)) - \varphi(\mu(x_n, u)) \le F_1(\mu(x_n, u)). \tag{22}$$

By using (C1), (22) gives

$$F_1(\mu(Tx_n, Tu)) \le F_1(\mu(x_n, u)). \tag{23}$$

Since (F_1) is non-decreasing, therefore we obtain

$$\mu(Tx_n, Tu) \le \mu(x_n, u), \tag{24}$$

letting $n \to \infty$ and using (21), we have

$$\lim_{n\to\infty} \mu(Tx_n, Tu) = 0. \tag{25}$$

From triangular inequality for μ, we have

$$\mu(x_n, Tu) \le \mu(x_n, Tx_n) + \mu(Tx_n, Tu).$$

Letting $n \to \infty$ and using (8) and (25), we have

$$\lim_{n\to\infty} \mu(x_n, Tu) = 0,$$

which further implies

$$\lim_{n\to\infty} p(x_n, Tu) = 0. \tag{26}$$

Hence, by using Lemma 1.10, (20) and (26) gives $Tu = u$.
For uniqueness of fixed point, let $x^*, y^* \in X$ be such that $Tx^* = x^*$ and $Ty^* = y^*$. Assume that $x^* \neq y^*$. If $\mu(Tx^*, Ty^*) = 0$, then $Tx^* = Ty^*$, so, $\mu(Tx^*, Ty^*) > 0$. Thus, from (C2), we obtain

$$\begin{aligned}\varphi(\mu(x^*, y^*)) &\le F_1(\mu(x^*, y^*)) - F_2(\mu(Tx^*, Ty^*))\\ &\le F_1(\mu(x^*, y^*)) - F_1(\mu(Tx^*, Ty^*))\\ &= 0,\end{aligned}$$

a contradiction as $\varphi \in \Phi$. Hence $x^* = y^*$. □

Theorem 2.2. *Let (X, d) be a complete metric space with a w_0-distance p and $T : X \to X$. Assume that there exists $\varphi \in \Phi$, a non-decreasing real-valued function F_1 on $(0, \infty)$ and a function $F_2 : (0, \infty) \to \mathbb{R}$ satisfying condition $(F2')$ and $(F3)$ such that (C1) and (C2) hold. Then T has a unique fixed point in X.*

Proof. Take any $x_0 \in X$ and define the sequence $x_n = T^n x_0$ and $\gamma_n = \mu(x_{n-1}, x_n)$, $n \in \mathbb{N}$. If $\mu(x_{n-1}, x_n) = 0$, for some $n \in \mathbb{N}$ then $x_{n-1} = x_n$ and so x_{n-1}, is a fixed point of T. Assume that $\gamma_n > 0$ for all $n \in \mathbb{N}$, then as in proof of Theorem 2.1, we obtain

$$F_2(\gamma_n) \le F_2(\gamma_1) - (n - n_0)c, \quad n \ge n_0. \tag{27}$$

Tending with $n \to \infty$ in (27), we get $F(\gamma_n) \to -\infty$ and, by $(F2')$, we have

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \mu(x_{n-1}, x_n) = 0. \tag{28}$$

Now from $(F3)$, there exist $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F_2(\gamma_n) = 0. \tag{29}$$

Then from (27), for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\gamma_n^k F_2(\gamma_n) - \gamma_n^k F_2(\gamma_1) &\le \gamma_n^k (F_2(\gamma_1) - (n - n_0)c) - \gamma_n^k F_2(\gamma_1)\\ &= -\gamma_n^k (n - n_0)c \le 0.\end{aligned} \tag{30}$$

Letting $n \to \infty$ in (30), and using (28) and (29), we obtain

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{31}$$

Observe that from (31) there exists $q_0 \in \mathbb{N}$ such that $n\gamma_n^k \leq 1$ for all $n \geq q_0$. Consequently, we have

$$\gamma_n \leq \frac{1}{n^{\frac{1}{k}}} \quad \text{for all} \quad n \geq q_0. \tag{32}$$

In order to show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq q_0$. From the definition of μ and (32), we get

$$\mu(x_m, x_n) \leq \gamma_{m-1} + \gamma_{m-2} + \cdots + \gamma_n < \sum_{i=n}^{\infty} \gamma_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, so from above and using Lemma 1.10(*iii*) , $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete metric space, so

$$\lim_{n\to\infty} x_n = x^* \in X. \tag{33}$$

This means that for all $\epsilon > 0$ there exist $q \in \mathbb{N}$ such that $\mu(x_n, x_m) < \epsilon$ for all $m > n \geq q$. Now for a fixed $n \in \mathbb{N}$ with $n \geq q$ the function $p(x_n, .)$ is lower semi-continuous; hence we obtain

$$p(x_n, u) \leq \liminf_{m\to\infty} p(x_n, x_m) < \epsilon.$$

Thus,

$$\lim_{n\to\infty} p(x_n, u) = 0. \tag{34}$$

Similarly $\lim_{n\to\infty} p(u, x_n) = 0$, which together with (34) yields

$$\lim_{n\to\infty} \mu(x_n, u) = 0. \tag{35}$$

Now from (C2), (C1) and by using the fact that (F_1) is non-decreasing (see proof of Theorem 2.1), we get

$$\lim_{n\to\infty} p(x_n, Tu) = 0. \tag{36}$$

Now equations (34) and (36) by Lemma 1.10(*i*) imply that $Tu = u$.
For uniqueness of fixed point, let $x^*, y^* \in X$ be such that $Tx^* = x^*$ and $Ty^* = y^*$. Assume that $x^* \neq y^*$. If $\mu(Tx^*, Ty^*) = 0$, then $Tx^* = Ty^*$, so, $\mu(Tx^*, Ty^*) > 0$. Thus, from (C2), we obtain

$$\varphi(\mu(x^*, y^*)) \leq 0,$$

a contradiction as $\varphi \in \Phi$. Hence $x^* = y^*$. □

Theorem 2.3. *Let (X, d) be a complete metric space with a w_0-distance p. Assume that there exist $\varphi \in \Phi$, a non decreasing function $F : (0, \infty) \to \mathbb{R}$ satisfying $(F2')$ and a function $G : [0, \infty)^3 \to [0, \infty)$ satisfying*

(G1) $\max\{a, b\} \leq G(a, b, c), \quad$ *for all* $\quad a, b, c \geq 0;$

(G2) $G(a, b, c) = 0$ *if and only if* $a = b = c = 0$.

Then for a given function $\psi : X \to [0, \infty)$, the operator $T : X \to X$ satisfying

(G3) $G(\mu(Tx, Ty), \psi(Tx), \psi(Ty)) > 0 \Rightarrow \varphi(\mu(x, y)) + F(G(\mu(Tx, Ty), \psi(Tx), \psi(Ty))) \leq F(\mu(x, y))$

for all $x, y \in X$ has a unique fixed point in X.

Proof. Take any $x_0 \in X$ and define the sequence $x_n = T^n x_0$ and $\gamma_n = \mu(x_{n-1}, x_n), n \in \mathbb{N}$. If $G(\mu(x_{n-1}, x_n), \psi(x_{n-1}), \psi(x_n)) = 0$, for some $n \in \mathbb{N}$, then $x_{n-1} = x_n$ and so x_{n-1} is a fixed point of T. Assume that $G(\mu(Tx, Ty), \psi(Tx), \psi(Ty)) > 0$, then from $(G3)$ we have

$$F(G(\gamma_{n+1}, \psi(\gamma_n), \psi(\gamma_{n+1}))) \leq F(\gamma_n) - \varphi(\gamma_n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{37}$$

Since F is non decreasing, so by using $(G1)$, from (37), we get

$$\begin{aligned} F(\gamma_{n+1}) &\leq F(max\{\gamma_{n+1}, \psi(\gamma_n)\}) \\ &\leq F(G(\gamma_{n+1}, \psi(\gamma_n), \psi(\gamma_{n+1}))) \\ &\leq F(\gamma_n) - \varphi(\gamma_n) \\ &< F(\gamma_n). \end{aligned} \tag{38}$$

Thus, we get that $\{\gamma_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive real numbers. Since $\varphi \in \Phi$, so as in the proof of Theorem 2.1, we obtain

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \mu(x_{n-1}, x_n) = 0. \tag{39}$$

Next, we claim that

$$\lim_{n,m\to\infty} \mu(x_n, x_m) = 0. \tag{40}$$

If (40) is not true, then from Proposition 1.3, there exist $\eta \in (0, \infty) \setminus \Lambda(F)$ such that F is continuous at η and for every for every $q \geq 0$ with $m_k > n_k \geq q$ and $\mu(x_m, x_n) > \eta$. Also there exists $q_0 \in \mathbb{N}$ such that

$$\gamma_{q_0} = \mu(x_{n-1}, x_n) < \eta \quad \text{for all } n \geq q_0 \tag{41}$$

Consider two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ satisfying

$$q_0 \leq n_k < m_k < m_k + 1 \;\text{ and }\; \mu(x_{m_k}, x_{n_k}) > \eta \;\text{ for all } k. \tag{42}$$

Then, as in proof of Theorem 2.1, we get

$$\lim_{k\to\infty} \mu(x_{m_k}, x_{n_k}) = \eta \tag{43}$$

and

$$\lim_{n\to\infty} \mu(x_{m_k+1}, x_{n_k+1}) = \eta. \tag{44}$$

By using $(G1)$ and $(G3)$, we obtain

$$\begin{aligned} \varphi(\mu(x_{m_k}, x_{n_k})) &\leq F(\mu(x_{m_k}, x_{n_k})) - F(G(\mu(x_{m_k+1}, x_{n_k+1}), \psi(x_{m_k+1}), \psi(x_{n_k+1}))) \\ &\leq F(\mu(x_{m_k}, x_{n_k})) - F(\max\{\mu(x_{m_k+1}, x_{n_k+1}), \psi(x_{m_k+1})\}) \\ &\leq F(\mu(x_{m_k}, x_{n_k})) - F(\mu(x_{m_k+1}, x_{n_k+1})). \end{aligned} \tag{45}$$

By passing limit $k \to \infty$, using (43), (44) and the fact that F is continuous at η, we get

$$\liminf_{s\to\eta^+} \varphi(s) \leq 0,$$

which is contradiction to (2) and hence (40) holds.
Thus, by Lemma 1.10(iii), $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete metric space, so

$$\lim_{n\to\infty} x_n = x^* \in X. \tag{46}$$

This implies that for all $\epsilon > 0$ there exist $q \in \mathbb{N}$ such that $\mu(x_n, x_m) < \epsilon$ for all $m > n \geq q$. Now for a fixed $n \in \mathbb{N}$ with $n \geq q$ the function $p(x_n, .)$ is lower semi-continuous; hence we obtain

$$p(x_n, u) \leq \liminf_{m\to\infty} p(x_n, x_m) < \epsilon$$

Thus

$$\lim_{n\to\infty} p(x_n, u) = 0. \tag{47}$$

Similarly $\lim_{n\to\infty} p(u, x_n) = 0$ which together with (47) yields

$$\lim_{n\to\infty} \mu(x_n, u) = 0. \tag{48}$$

From $(G3)$, we obtain

$$F(G(\mu(Tx_n, Tu), \psi(Tx_n), \psi(Tu))) \leq F(\mu(x_n, u)) - \varphi(\mu(x_n, u)) \leq F(\mu(x_n, u)), \tag{49}$$

and by using $(G1)$ and $(F1)$, we get

$$\mu(Tx_n, Tu) \leq \mu(x_n, u). \tag{50}$$

Letting $n \to \infty$ and using (48), we have

$$\lim_{n\to\infty} \mu(Tx_n, Tu) = 0. \tag{51}$$

By using triangular inequality for μ, we obtain

$$\mu(x_n, Tu) \leq \mu(x_n, Tx_n) + \mu(Tx_n, Tu)$$

Letting $n \to \infty$ and using (39) and (14), we have

$$\lim_{n\to\infty} \mu(x_n, Tu) = 0,$$

which further implies

$$\lim_{n\to\infty} p(x_n, Tu) = 0. \tag{52}$$

Hence, by using Lemma 1.10, (47) and (52) gives $Tu = u$.
For uniqueness, let $x^*, y^* \in X$ be such that $Tx^* = x^*$ and $Ty^* = y^*$. Assume that $x^* \neq y^*$. If $G(\mu(Tx^*, Ty^*), \psi(Tx^*), \psi(Ty^*)) = 0$ then $\mu(x^*, y^*) = 0$ implies $x^* = y^*$, so, let $G(\mu(Tx^*, Ty^*), \psi(Tx^*), \psi(Ty^*)) > 0$. Thus, from $(G3)$, we obtain

$$\varphi(\mu(x^*, y^*)) \leq 0,$$

a contradiction as $\varphi \in \Phi$. Hence $x^* = y^*$.
□

3. Consequences and Examples

In this section, we derive special cases of the Theorems obtained in Section 2.

Corollary 3.1. *Let (X, d) be a complete metric space with a w_0-distance p and $T : X \to X$. Assume that there exists $\varphi \in \Phi$ and a continuous function $F : (0, \infty) \to \mathbb{R}$ satisfying $(F1)$ and $(F2')$ such that T satisfies*

$$\mu(Tx, Ty) > 0 \text{ implies } \varphi(\mu(x, y)) + F(\mu(Tx, Ty)) \leq F(\mu(x, y)), \tag{53}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Define $F_1 = F_2 = F$, then (C1) and (C2) hold true and result follows from Theorem 2.1. □

Corollary 3.2. *Let (X, d) be a complete metric space with a w_0-distance p and $T : X \to X$. Assume that there exists a non-decreasing real-valued function F_1 on $(0, \infty)$ and a continuous function $F_2 : (0, \infty) \to \mathbb{R}$ satisfying condition $(F2')$. If there exists $\tau > 0$ such that T satisfies* (C1) *and*

$$\mu(Tx, Ty) > 0 \text{ implies } \tau + F_2(\mu(Tx, Ty)) \leq F_1(\mu(x, y)), \tag{54}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Define $\varphi : (0,\infty) \to (0,\infty)$ by

$$\varphi(t) = \tau,$$

where $\tau > 0$. Then (C2) holds true and result follows from Theorem 2.1. □

Corollary 3.3. *Let (X, d) be a complete metric space with a w_0-distance p and $T : X \to X$. Assume that there exists $\varphi \in \Phi$ and a function $F : (0,\infty) \to \mathbb{R}$ satisfying $(F1)$, $(F2^{'})$ and $(F3)$ such that T satisfies*

$$\mu(Tx, Ty) > 0 \ \textit{ implies } \ \varphi(\mu(x, y)) + F(\mu(Tx, Ty)) \leq F(\mu(x, y)), \tag{55}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Define $F_1 = F_2 = F$, then (C1) and (C2) holds true and result follows from Theorem 2.2. □

Corollary 3.4. *Let (X, d) be a complete metric space with a w_0-distance p and $T : X \to X$. Assume that there exists a non-decreasing real-valued function F_1 on $(0,\infty)$ and a function $F_2 : (0,\infty) \to \mathbb{R}$ satisfying condition $(F2^{'})$ and $(F3)$. If there exists $\tau > 0$ such that T satisfies (C1) and*

$$\mu(Tx, Ty) > 0 \ \textit{ implies } \ \tau + F_2(\mu(Tx, Ty)) \leq F_1(\mu(x, y)), \tag{56}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Define $\varphi : (0,\infty) \to (0,\infty)$ by

$$\varphi(t) = \tau,$$

where $\tau > 0$. Then (56) holds true and result follows from Theorem 2.2. □

Corollary 3.5. *Let (X, d) be a complete metric space with a w_0-distance p. Assume that there is a non decreasing function $F : (0,\infty) \to \mathbb{R}$ satisfying $(F2^{'})$ and a function $G : [0,\infty)^3 \to [0,\infty)$ satisfying $(G1)$ and $(G2)$. If there exists $\tau > 0$ then for a given function $\psi : X \to [0,\infty)$, the operator $T : X \to X$ satisfying*

$$\begin{aligned} G(\mu(Tx, Ty), \psi(Tx), \psi(Ty)) > 0 \ \textit{ implies } \ &\tau + F(G(\mu(Tx, Ty), \psi(Tx), \psi(Ty))) \\ &\leq F(\mu(x, y)) \end{aligned} \tag{57}$$

for all $x, y \in X$ has a unique fixed point in X.

Proof. Define $\varphi : (0,\infty) \to (0,\infty)$ by

$$\varphi(t) = \tau,$$

where $\tau > 0$. Then (57) holds true and result follows from Theorem 2.3. □

Corollary 3.6. *Let (X, d) be a complete metric space with a w_0-distance p and $T : X \to X$. Assume that there exists $\varphi \in \Phi$, a non decreasing function $F : (0,\infty) \to \mathbb{R}$ satisfying $(F2^{'})$. If T satisfies*

$$\mu(Tx, Ty) > 0 \ \textit{ implies } \ \varphi(\mu(Tx, Ty)) + F(\mu(Tx, Ty)) \leq F(\mu(x, y))$$

for all $x, y \in X$, then T has a unique fixed point in X.

Proof. Define $G : [0,\infty)^3 \to [0,\infty)$ by

$$G(a, b, c) = a + b + c,$$

for all $a, b, c \in [0,\infty)$. Then G satisfies $(G1)$ and $(G2)$. Also, there exists a function $\psi : X \to [0,\infty)$, defined by $\psi(t) = 0$ for all $t \in [0,\infty)$ such that T satisfies $(G3)$. Thus, the result follows from Theorem 2.3. □

Remark 3.7. *By taking $p = d$ in Corollary 3.6, we get Theorem 1.6.*

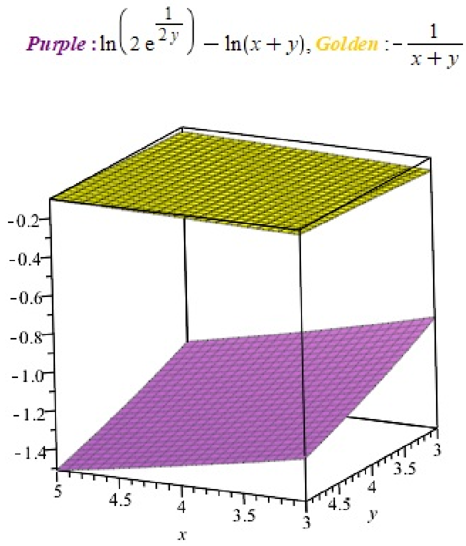

Figure 1: Graph of Inequality 58

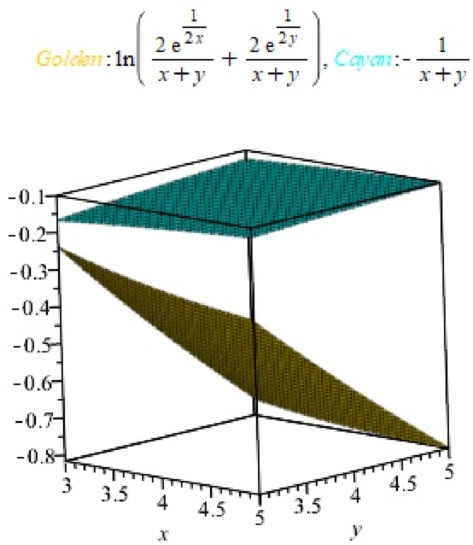

Figure 2: Graph of Inequality 59

Example 3.8. *Let $X = [0,\infty)$ be a metric space with usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and a w_0-distance p defined by $p(x,y) = x+y$ for all $x,y \in X$. Then*

$$\mu(x,y) = x+y.$$

Define the functions $F_1,\ F_2 : (0,\infty) \to \mathbb{R}$ by $F_1(t) = \ln t$ and $F_2(t) = \ln(2t)$ for all $t > 0$ respectively. Then the function F_1 is non-decreasing and the function F_2 satisfy the condition $(F2')$ and $(F3)$. Also $F_2(t) \geq F_1(t)$ for all $t > 0$. Define $\varphi : (0,\infty) \to (0,\infty)$ by $\varphi(t) = \frac{1}{t}$ for all $t > 0$. Then $\varphi \in \Phi$. Now define $T : X \to X$ by

$$T(x) = \begin{cases} 0 & 0 \leq x, < 3 \\ e^{\frac{1}{2x}} & x \geq 3 \end{cases}.$$

Assume that $\mu(Tx,Ty) > 0$, then there arises the following cases:
Case I *If $x \in [0,3)$ and $y \geq 3$, then we have*

$$\begin{aligned} F_2(\mu(Tx,Ty)) - F_1(\mu(x,y)) &= \ln(2e^{\frac{1}{2y}}) - \ln(x+y) \\ &= \ln\Big(\frac{2e^{\frac{1}{2y}}}{x+y}\Big) \\ &\leq -\frac{1}{x+y} \\ &= -\varphi(\mu(x,y)) \end{aligned} \tag{58}$$

Case II *If $x,y \geq 3$, then we have*

$$\begin{aligned} F_2(\mu(Tx,Ty)) - F_1(\mu(x,y)) &= \ln(2(e^{\frac{1}{2x}} + e^{\frac{1}{2y}})) - \ln(x+y) \\ &= \ln\Big(\frac{2(e^{\frac{1}{2x}} + e^{\frac{1}{2y}})}{x+y}\Big) \\ &\leq -\frac{1}{x+y} \\ &= -\varphi(\mu(x,y)). \end{aligned} \tag{59}$$

The inequalities 58 and 59 are shown in Figure 1 and Figure 2 respectively. Thus all conditions of Theorem 2.2 hold true and note that 0 is the unique fixed point of T.

Example 3.9. *Let $X = [0,\infty)$ be a metric space with usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and a w_0-distance p defined by $p(x,y) = x+y$ for all $x,y \in X$. Then*

$$\mu(x,y) = x+y.$$

Define $G : [0,\infty)^3 \to [0,\infty)$, $F : (0,\infty) \to \mathbb{R}$, $\varphi : (0,\infty) \to (0,\infty)$ and $\psi : X \to [0,\infty)$ by $G(a,b,c) = a+b+c$ for all $a,b,c \geq 0$, $F(t) = \ln t$, for all $t > 0$,

$$\varphi(t) = \begin{cases} t+1 & 1 > t > 0 \\ \ln 2 & t \geq 1 \end{cases}$$

and $\psi(x) = 2x$ for all $x \in X$, respectively then G satisfy $(G1)$ and $(G2)$ and $\varphi \in \Phi$. Now define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{3e^2} & x > 0 \\ 0 & x = 0 \end{cases}$$

Assume that $G(a,b,c) > 0$, then

$$\begin{aligned} F(G(\mu(Tx,Ty),\psi(Tx),\psi(Ty))) - F(\mu(x,y)) &= F(\frac{x+y}{e^2}) - F(x+y) \\ &= -2. \end{aligned} \tag{60}$$

Here arises the following cases:
Case I *If* $0 < x + y < 1$, *then from* (60), *we have*

$$\begin{aligned} F(G(\mu(Tx,Ty),\psi(Tx),\psi(Ty))) - F(\mu(x,y)) &= -2 \\ &\leq -(x+y+1) \\ &= -\varphi(\mu(x,y)). \end{aligned}$$

Case II *If* $x + y \geq 1$, *then from* (60), *we have*

$$\begin{aligned} F(G(\mu(Tx,Ty),\psi(Tx),\psi(Ty))) - F(\mu(x,y)) &= -2 \\ &\leq -\ln 2 \\ &= -\varphi(\mu(x,y)). \end{aligned}$$

In all cases contractive condition ($G3$) *is satisfied. Hence all the hypothesis of Theorem* 2.3 *are satisfied and note that 0 is the unique fixed point of T in X.*

Remark 3.10. *In Examples 3.8 and 3.9,* $\mu(2,2) = p(2,2) = 4 \neq 0$, *so* ($d1$) *does not hold and* $p \neq d$. *Therefore, Theorems 1.2 and 1.6 can not be applied for this example.*

4. Solution to differential equation of RLC circuit's current

For decades, solar panels has been praised as promising alternative energy source and a great way to offset energy costs. They absorb sunlight as a source of energy to generate direct current electricity. A solar panel works by allowing particles of light or photons, to knock electrons free from atoms, thus, in turn, generating a flow of electricity. With a basic understanding of how light is transformed into electricity, a mathematical model can be presented of the electric current in an RLC parallel circuit, also known as a tuning circuit (see [8]). In Figure 3, V is the voltage of the power source, I is the current in the circuit, R is

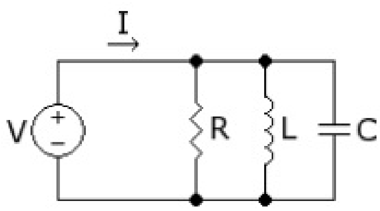

Figure 3: RLC parallel circuit

the resistance of the resistor, L is the inductance of the inductor and C is the capacitance of the capacitor. Such problems are mathematically modeled as initial value problem for second order ordinary differential equation of the form :

$$\begin{cases} \frac{d^2u}{dt^2} + \frac{R}{L}\frac{du}{dt} = f(t,u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \end{cases} \tag{61}$$

where $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.
In this section, we prove the existence of the solution to the RLC differential equation (61). The problem (61) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s) f(s,u(s))ds, \quad t \in [0,1], \tag{62}$$

where G is the Green's function defined by

$$G(t,s) = \begin{cases} (t-s)e^{\tau(t-s)} & if\ 0 \le s \le t \le 1 \\ 0 & if\ 0 \le t \le s \le 1 \end{cases}$$

Here $\tau > 0$ is a constant, calculated in terms of R and L, mentioned in (61).
Now, u is a solution of problem (61) if and only if u is a solution of the integral equation (62).

Theorem 4.1. *Let $X = C([0,1], \mathbb{R}^+)$ be the space of all continuous functions defined on $[0,1]$ with norm defined by*

$$\|u\| = \sup_{t\in[0,1]} e^{-2\tau t}|u(t)|$$

for all $u \in X$. Consider the non linear integral equation (62) and suppose that the following conditions hold:

(1) *there exist a continuous function $w : [0,1] \to (0,\infty)$ and $\tau > 0$ such that*

$$|f(s,u(s))| \le \frac{1}{2}\tau^2 w(s)|u(s)|,$$

for all $s \in [0,1]$, $u(s) \in \mathbb{R}$;

(2) $\max_{s\in[0,1]} w(s) = e^{-\alpha}$, *where $\alpha > e$.*

Then, the integral equation (62) has a solution.

Proof. Let $X = C([0,1], \mathbb{R}^+)$ and $\|u\| = \sup_{t\in[0,1]} e^{-2\tau t}|u(t)|$, then $(X, \|.\|)$ is a complete metric space.
Define $T : X \to X$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds, \tag{63}$$

for all $x \in X$ and $t \in [0,1]$.
Note that the existence of a solution to the equation (62) is equivalent to the existence of a fixed point of the mapping T.
Define $\mu : X \times X \to [0,\infty)$ and $F_1,\ F_2 : (0,\infty) \to \mathbb{R}$ by

$$\mu(x,y) = \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$, $F_1(t) = \ln t$ and $F_2(t) = \ln 2t$ for all $t \in (0,\infty)$ respectively. Then for $u, v \in X$, we obtain

$$\begin{aligned}
|Tu(t)| &= \left|\int_0^1 G(t,s)f(u,u(s))ds\right| \\
&\le \int_0^1 G(t,s)|f(s,u(s))|ds \\
&\le \int_0^1 \frac{1}{2}G(t,s)\tau^2 e^{-\alpha}|u(s)|ds \\
&= \int_0^1 \tau^2 e^{-\alpha}(t-s)e^{\tau(t-s)}e^{2\tau s}\|u\|ds \\
&= \frac{1}{2}\tau^2 e^{-\alpha+\tau t}\|u\| \int_0^1 (t-s)e^{\tau s}ds \\
&= \frac{1}{2}\tau^2 e^{-\alpha+\tau t}\|u\|\Big[\frac{-t}{\tau} - \frac{1}{\tau^2} + \frac{e^{\tau t}}{\tau^2}\Big] \\
&= \frac{1}{2}e^{-\alpha}\|u\|e^{2\tau t}[1 - \tau t e^{-\tau t} - e^{-\tau t}]
\end{aligned}$$

Since $[1 - \tau t e^{-\tau t} - e^{-\tau t}] \leq 1$, then

$$\|Tu\| \leq \frac{1}{2} e^{-\alpha} \|u\|$$

Similarly, we have that

$$\|Tv\| \leq \frac{1}{2} e^{-\alpha} \|v\|$$

This implies that

$$\begin{aligned} \mu(Tu, Tv) &= \max\{\|Tu\|, \|Tv\|\} \\ &\leq \frac{1}{2} e^{-\alpha} \max\{\|u\|, \|v\|\} \\ &\leq \frac{1}{2} e^{-e} \max\{\|u\|, \|v\|\}. \end{aligned} \tag{64}$$

Define $\varphi : (0, \infty) \to (0, \infty)$ such that

$$\min_{t \in (0,\infty)} \varphi(t) \geq e, \tag{65}$$

then $\varphi \in \Phi$. By combing (64) and (65), we get

$$\mu(Tu, Tv) \leq \frac{1}{2} e^{-\varphi(\mu(u,v))} \max\{\|u\|, \|v\|\},$$

which further implies that

$$\ln(2\mu(Tu, Tv)) \leq \ln(\mu(u, v)) - \varphi(\mu(u, v)).$$

Thus,

$$\varphi(\mu(u, v)) + F_2(\mu(Tu, Tv)) \leq F_1(\mu(u, v)).$$

Hence all the conditions of Theorem 2.1 are satisfied and so, the integral equation (62) has a solution. □

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