Filomat 34:12 (2020), 4043–4060
https://doi.org/10.2298/FIL2012043L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

General Family of Exponential Operators

Km. Lipi[a], Naokant Deo[a]

[a]*Delhi Technological University, Department of Applied Mathematics, Bawana Road, Delhi-110042, India*

Abstract. In this article, we deal with the approximation properties of Ismail-May operators [16] based on a non-negative real parameter λ. We provide some graphs and error estimation table for a numerical example depicting the convergence of our proposed operators. We further define the Bézier variant of these operators and give a direct approximation theorem using Ditizan-Totik modulus of smoothness and a Voronovoskaya type asymptotic theorem. We also study the error in approximation of functions having derivatives of bounded variation. Lastly, we introduce the bivariate generalization of Ismail May operators and estimate its rate of convergence for functions of Lipschitz class.

1. Introduction

In the year 1976, May [21] defined a positive exponential operator L_λ on $C(-\infty,\infty)$ into C^∞ as

$$(L_\lambda f(s))(t) = \int_{-\infty}^{\infty} W(\lambda,t,s) f(s)ds,$$

where $W(\lambda,t,s) \geq 0$ is a kernel of distribution and satisfy the following conditions:

1. $(L_\lambda e_0(s))(t) = \int_{-\infty}^{\infty} W(\lambda,t,s)ds = 1$ normalisation condition.
2. $\frac{\partial}{\partial t} W(\lambda,t,s) = \frac{\lambda}{p(t)} W(\lambda,t,s)(s-t)$, where $p(t)$ is analytic and positive for $t \in (-\infty,\infty)$.

The partial differential equation and the normalisation condition together define at most one kernel $W(\lambda,t,s)$ of an exponential operator for a given polynomial $p(t)$. Also, the normalization condition yields,

$$\exp\left(\lambda \int_{c}^{g(x)} \frac{\theta d\theta}{p(\theta)}\right) = \int_{-\infty}^{\infty} C(\lambda,s)e^{\lambda sx}ds, \quad x \in \text{Range of } q(t), \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 41A25 ; Secondary 41A36

Keywords. Rate of convergence, Voronovoskaya theorem, Peetre's-K functional, Bounded variation, Partial and mixed modulus of smoothness

Received: 18 December 2019; Revised: 21 March 2020; Accepted: 06 April 2020

Communicated by Hari M. Srivastava

The present work is supported by the Council of Scientific and Industrial Research (CSIR) with reference no.: 08/133(0029)/2019-EMR-I for the first author.

Email addresses: chaudhary.lipi123@gmail.com (Km. Lipi), naokantdeo@dce.ac.in (Naokant Deo)

where $q(t) = \int\limits_c^t \frac{dv}{p(v)}$ and $g(q(t)) = q(g(t))$.

Ismail and May [16] showed that for a linear or quadratic $p(t)$, there already exists various known operators such as Bernstein, Szász, Baskakov, Gauss-Weierstrass, Post-Widder etc. that satisfy the above conditions and thus can be referred to as exponential operators. For instance if $p(t) = t$ and $c = 1$ in (1), the corresponding approximation operators is Szász operators,

$$(S_\lambda f(s))(t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f\left(\frac{k}{\lambda}\right), \quad t \in (0, \infty).$$

For $p(t) = 1$ and $c = 0$, equation (1) becomes The Gauss-Weierstrass operators,

$$(W_\lambda f(s))(t) = \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} \exp\left\{\frac{-\lambda(s-t)^2}{2}\right\} f(s)ds, \quad t \in (-\infty, \infty).$$

Similarly, for a quadratic $p(t) = t(1-t)$ with $t \in (0, 1)$ and $c = \frac{1}{2}$, formula (1) transforms into the very known Bernstein operator,

$$(B_\lambda f(s))(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right), \quad n = 1, 2, .., \quad t \in [0, 1].$$

Ismail and May [16] also constructed some new approximation operators for cubic polynomials by determining a unique generalized function $C(\lambda, s)$ for which corresponding kernel is given by:

$$W(\lambda, t, s) = \exp\left\{\lambda \int_c^t \frac{s-\theta}{p(\theta)} d\theta\right\} C(\lambda, s). \tag{2}$$

Now if $p(t) = t(1+t)^2$ and $c = 1$ and consider the identity (see[24])

$$e^{\lambda x} = \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} (xe^{-x})^k \tag{3}$$

we get

$$C(\lambda, s) = 2^{-\lambda s} e^{-\frac{\lambda(1+s)}{2}} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} \delta(k - \lambda s).$$

For this value of $C(\lambda, s)$, the corresponding exponential operator is

$$(\tilde{R_\lambda} f(s))(t) = e^{-\frac{\lambda t}{(1+t)}} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} \left(\frac{t}{1+t}\right)^k e^{-\frac{\lambda t}{(1+t)}} f\left(\frac{k}{\lambda}\right).$$

With change of variables $x = \frac{t}{1+t}$, Ismail and May [16] defined a linear positive operator for a continuous function $f \in [0, 1]$, as follows:

$$(R_\lambda f(s))(x) = e^{-\lambda x} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} (xe^{-x})^k f\left(\frac{k}{\lambda+k}\right). \tag{4}$$

In [22], the Kantorovich form of the operator (4) and its bivariate is defined along with some important approximation results.

The aim of this paper is to discuss some approximation properties of operator (4) along with some graphs and an error estimation table for a numerical example. Further we define its Bézier variant and estimate the rate of convergence for functions with derivatives of bounded variation, a direct approximation theorem using Ditizan-Totik modulus of smoothness, and a Voronovoskaya type result . The last section is dedicated to bivariate generalization of operator (4) and its convergence properties.

2. Basic Results

In this section, we discuss some auxiliary results to prove our main results for the operators (4).

Lemma 2.1. *For $e_i(s) = s^i$, $i = 0, 1, 2, 3, 4$ and $\lambda > 0$, we have*

(i) $(R_\lambda e_0(s))(x) = 1$;

(ii) $(R_\lambda e_1(s))(x) = \frac{\lambda}{\lambda+1}x$;

(iii) $(R_\lambda e_2(s))(x) = \frac{\lambda^2}{(\lambda+1)(\lambda+2)}x^2 + \frac{\lambda}{(\lambda+1)^2}x$;

(iv) $(R_\lambda e_3(s))(x) = \frac{\lambda^3}{(\lambda+1)(\lambda+2)(\lambda+3)}x^3 + \frac{\lambda^2(3\lambda+4)}{(\lambda+1)^2(\lambda+2)^2}x^2 + \frac{\lambda}{(\lambda+1)^3}x$;

(v) $(R_\lambda e_4(s))(x) = \frac{\lambda^4}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}x^4 + \frac{2\lambda^3(3\lambda^2+11\lambda+9)}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)^2}x^3 + \frac{\lambda^2(7\lambda^2+18\lambda+12)}{(\lambda+1)^3(\lambda+2)^3}x^2 + \frac{\lambda}{(\lambda+1)^4}x$.

Lemma 2.2. *If $\mu_{\lambda,m} = (R_\lambda(s-x)^m)(x)$ denote the central moments of the operators (4), then for $m = 1, 2, 4$, we have*

(i) $\mu_{\lambda,1}(x) = -\frac{x}{(\lambda+1)}$;

(ii) $\mu_{\lambda,2}(x) = -\frac{(\lambda-2)}{(\lambda+1)(\lambda+2)}x^2 + \frac{\lambda}{(\lambda+2)}x$;

(iii) $\mu_{\lambda,4}(x) = \frac{(3\lambda^2-46\lambda+24)x^4}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} + \frac{\lambda(-6\lambda^3+36\lambda^2+216\lambda+216)x^3}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)^2} + \frac{\lambda(3\lambda^3-6\lambda^2-36\lambda-32)x^2}{(\lambda+1)^3(\lambda+2)^3} + \frac{\lambda x}{(\lambda+1)^4}$.

Remark 2.3. *For sufficiently large λ, we have*

(i) $\lim_{\lambda\to\infty} \lambda\mu_{\lambda,1}(x) = -x$;

(ii) $\lim_{\lambda\to\infty} \lambda\mu_{\lambda,2}(x) = x(1-x)$;

(iii) $\lim_{\lambda\to\infty} \lambda^2\mu_{\lambda,4}(x) = 3x^2(x-1)^2$.

Lemma 2.4. *Let f be a continuous function in $[0, 1]$, then we have*

$$\left|(R_\lambda f(s))(x)\right| \le \left\|f\right\|.$$

3. Main results

Theorem 3.1. *Suppose f be a continuous function in $[0, 1]$, then $(R_\lambda f(s))(x) \to f(x)$ uniformly in $[0, 1]$.*

Proof. From Lemma 2.1, $(R_\lambda e_0(s))(x) = 1$, $(R_\lambda e_1(s))(x) \to x$ and $(R_\lambda e_2(s))(x) \to x^2$ as $\lambda \to \infty$. Then by Bohman-Korovkin theorem, $(R_\lambda f(s))(x) \to f(x)$ uniformly in $[0, 1]$. □

For $C_B[0, 1]$, the space of all continuous and bounded functions on $[0, 1]$, let us consider the following K-functional:

$$K_2(f, \delta) = \inf_{g\in C_B^2[0,1]} \left\{\left\|f-g\right\| + \delta\left\|g''\right\|\right\} (\delta > 0),$$

where $C_B^2[0, 1] = \{g \in C_B[0, 1] : g', g'' \in C_B[0, 1]\}$. By [7], for $f \in C[0, 1]$, there exists a constant L such that

$$K_2(f; \delta) \le L\omega_2(f; \sqrt{\delta}). \tag{5}$$

Now we estimate the rate of convergence of operators (4), with the help of first and second order modulus of continuity (see also [27]).

Theorem 3.2. *For $f \in C[0,1]$, $\lambda > 0$, we have*

$$\left|(R_\lambda f(s))(x) - f(x)\right| \leq C\omega_2\left(f; \frac{\sqrt{\mu_{\lambda,2}(x) + \psi(x)}}{2}\right) + \omega(f; \psi^2(x)),$$

where $\psi(x) = \sqrt{\frac{x}{\lambda+1}}$ and $\mu_{\lambda,2}(x)$ is stated in Lemma 2.2.

Proof. We consider auxiliary operators

$$(\tilde{R_\lambda} f(s))(x) = (R_\lambda f(s))(x) - f\left(\frac{\lambda x}{\lambda+1}\right) + f(x). \tag{6}$$

Suppose $g \in C^2[0,1]$, by Taylor's expansion, we have

$$g(s) = g(x) + g'(x)(s-x) + \int_x^s (s-u)g''(u)du.$$

From (6), $(\tilde{R_\lambda} e_0(s))(x) = 1$, $(\tilde{R_\lambda} e_1(s))(x) = x$ and $(\tilde{R_\lambda}(s-x))(x) = 0$, we have

$$\begin{aligned}\left|(\tilde{R_\lambda}(g(s) - g(x))(x)\right| &\leq \left|\int_x^{\frac{\lambda x}{\lambda+1}} \left(\frac{\lambda x}{\lambda+1} - u\right) g''(u)du\right| + \left|\left(R_\lambda\Big(\int_x^t (s-u)g''(u)du\Big)\right)(x)\right| \\ &\leq \left\|g''\right\| \left(\int_x^{\frac{\lambda x}{\lambda+1}} \left|\frac{\lambda x}{\lambda+1} - u\right| du + \left(R_\lambda\left(\int_x^t |t-u| du\right)\right)(x)\right) \\ &\leq \left\|g''\right\| \left(\mu_{\lambda,2}(x) + \psi(x)\right).\end{aligned} \tag{7}$$

From (6) and using Lemma 2.4, we get

$$\left|(\tilde{R_\lambda} f(s)(x)\right| \leq 3\left\|f\right\|. \tag{8}$$

Using equations (6) -(8), we have

$$\begin{aligned}\left|(R_\lambda f(s))(x) - f(x)\right| &\leq \left|(\tilde{R_\lambda}(f(s) - g(s))(x) - (f-g)(x)\right| + \left|(\tilde{R_\lambda}(g(s) - g(x))(x)\right| + \left|f\left(\frac{\lambda x}{\lambda+1}\right) - f(x)\right| \\ &\leq 4\left[\left\|f-g\right\| + \frac{(\mu_{\lambda,2}(x) + \psi(x))}{4}\left\|g''\right\|\right] + \omega(f; \psi^2(x)).\end{aligned} \tag{9}$$

Taking infimum on the right hand side of (9), we have

$$\left|(R_\lambda f(s))(x) - f(x)\right| \leq 4K_2\left(f; \frac{\mu_{\lambda,2}(x) + \psi(x)}{4}\right) + \omega(f; \psi^2(x)).$$

By using relation (5), we get the desired result . □

In our next theorem, we discuss the rate of convergence for the operators $(R_\lambda f(s))(x)$ using the functions of Lipschitz class $Lip_M(\beta)$ defined by Lenze [20], where $M > 0$ and $0 < \beta \leq 1$. The function $f \in Lip_M(\beta)$ if

$$\left|f(s) - f(x)\right| \leq M|s-x|^\beta \quad s, x \in \mathbb{R}. \tag{10}$$

Theorem 3.3. *Let $f \in Lip_M(\beta)$, $x \in [0,1]$ and $\lambda > 0$, we have*

$$\left|(R_\lambda f(s))(x) - f(x)\right| \leq M(\mu_{\lambda,2}(x))^{\frac{\beta}{2}}$$

Proof. *For the linear positive operators* $(R_\lambda f(s))(x)$ *and* $f \in Lip_M(\beta)$ *, we have*

$$\begin{aligned}\left|(R_\lambda f(s))(x) - f(x)\right| \leq & (R_\lambda(\left|f(t) - f(x)\right|)(x) \\ = & e^{-\lambda x}\sum_{k=0}^{\infty}\frac{\lambda(\lambda+k)^{k-1}}{k!}(xe^{-x})^k\left|f\left(\frac{k}{\lambda+k}\right) - f(x)\right| \\ \leq & Me^{-\lambda x}\sum_{k=0}^{\infty}\frac{\lambda(\lambda+k)^{k-1}}{k!}(xe^{-x})^k\left|\frac{k}{\lambda+k} - x\right|^\beta,\end{aligned}$$

using Hölder's inequality,

$$\left|(R_\lambda f(s))(x) - f(x)\right| \leq M[\mu_{\lambda,2}(x)]^{\frac{\beta}{2}}.$$

Hence the proof. □

Theorem 3.4. *If the function* $f(x)$ *is bounded on* $[0,1]$ *,* $x \in (0,1)$ *and for which* $f'(x), f''(x)$ *exist then, we have*

$$\lim_{\lambda\to\infty}\lambda\left[(R_\lambda f(s))(x) - f(x)\right] = -xf'(x) + \frac{x(1-x)}{2}f''(x).$$

Proof. *Suppose* $x \in [0,1]$ *be a fixed point, by Taylor's formula, we can say*

$$f(s) = f(x) + (s-x)f'(x) + \frac{(s-x)^2}{2}f''(x) + r(s,x)(s-x)^2, \tag{11}$$

$r(s,x) \in C[0,1]$ *be the Peano form of the remainder. By using L'hospital rule we can easily say that* $r(s,x)$ *converges to* 0 *when* s *approaches to* x.
In (11), applying $(R_\lambda(.))(x)$*, we obtain*

$$\begin{aligned}\lim_{\lambda\to\infty}\lambda\left[(R_\lambda f(s))(x) - f(x)\right] = & f'(x)\lim_{\lambda\to\infty}\lambda(R_\lambda(s-x))(x) + \frac{f''(x)}{2}\lim_{\lambda\to\infty}\lambda(R_\lambda(s-x)^2)(x) \\ & + \lim_{\lambda\to\infty}\lambda(R_\lambda r(s,x)(s-x)^2)(x).\end{aligned} \tag{12}$$

Applying Cauchy-Schwarz inequality in the last term of (12), we get

$$\lambda(R_\lambda r(s,x)(s-x)^2)(x) \leq \sqrt{(R_\lambda r^2(s,x))(x)}.\sqrt{\lambda^2(R_\lambda(s-x)^4)(x)}$$

Since $r^2(x,x) = 0$ *and from Remark 2.3, we obtain*

$$\lim_{\lambda\to\infty}\lambda(R_\lambda r(s,x)(s-x)^2)(x) = 0. \tag{13}$$

From (12), (13) and using Remark 2.3, we get the required result. □

From [10] the unified Ditzian-Totik modulus of smoothness is given as follows:

$$\omega_{\phi^\tau}(f;t) = \underset{0<h\leq t}{Sup}\left\{\left|f\left(x + \frac{h\phi^\tau(x)}{2}\right) - f\left(x - \frac{h\phi^\tau(x)}{2}\right)\right|, x + \frac{h\phi^\tau(x)}{2} \in [0,1]\right\}.$$

Further, the appropriate K-functional is defined by

$$K_{\phi^\tau}(f,t) = \inf_{g\in W_{\phi^\tau[0,1]}}\left\{\left\|f-g\right\| + t\left\|\phi^\tau g'\right\|\right\} \qquad (t>0),$$

where $0 \leq \tau \leq 1$, $W_{\phi^\tau}[0,1] = \left\{g : g \in AC_{loc}[0,1], \left\|\phi^\tau g'\right\| < \infty\right\}$, $g \in AC_{loc}[0,1]$ denotes the class of all locally absolutely continuous function and $\|.\|$ is the sup norm on $C[0,1]$. It is well known [10] that there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\phi^\tau}(f;t) \leq K_{\phi^\tau}(f;t) \leq C\omega_{\phi^\tau}(f;t). \tag{14}$$

Theorem 3.5. *Let $f \in C[0,1]$. Then for $\phi(x) = \sqrt{x(1-x)}$ and for every $x \in (0,1)$, we have*

$$\left|(R_\lambda g(s))(x) - g(x)\right| \leq C\omega_{\phi^\tau}\left(h, \frac{\phi^{1-\tau}(x)}{\sqrt{\lambda}}\right),$$

where C is a constant.

Proof. Since $g \in W_{\phi^\tau}$, we obtain

$$g(s) = g(x) + \int_x^s g'(u)du,$$

Therefore, we can write

$$\left|(R_\lambda g(s))(x) - g(x)\right| \leq \left(R_\lambda\left(\left|\int_x^s g'(u)du\right|\right)\right)(x) \tag{15}$$

Applying Hölder's inequality, we get

$$\left|\int_x^s g'(u)du\right| \leq \left\|\phi^\tau g'\right\| \left|\int_x^s \frac{du}{\phi^\tau}\right| \leq \left\|\phi^\tau g'\right\| |s-x|^{1-\tau} \left|\int_x^s \frac{du}{\phi(u)}\right|^\tau \tag{16}$$

and

$$\begin{aligned}
\left|\int_x^s \frac{du}{\phi(u)}\right| &= \left|\int_x^s \frac{du}{\sqrt{u(1-u)}}\right| \\
&\leq \left|\int_x^s \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\
&\leq 2[\left|\sqrt{s} - \sqrt{x}\right| + \left|\sqrt{1-s} - \sqrt{1-x}\right|] \\
&= 2\,|s-x|\left[\frac{1}{\sqrt{s}+\sqrt{x}} + \frac{1}{\sqrt{1-s}+\sqrt{1-x}}\right] \\
&\leq 2\,|s-x|\left[\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right] \\
&\leq \frac{2\sqrt{2}\,|s-x|}{\phi(x)}.
\end{aligned}$$

From (16), we have

$$\int_x^s g'(u)du \leq \left\|\phi^\tau g'\right\| |s-x|^{1-\tau}\left(\frac{2\sqrt{2}\,|s-x|}{\phi(x)}\right)^\tau = \frac{\left\|\phi^\tau g'\right\| |s-x|\, 2^\tau 2^{\frac{\tau}{2}}}{\phi^\tau(x)}.$$

Applying Cauchy-Schwarz inequality,

$$\begin{aligned}\left|(R_\lambda g(s))(x)-g(x)\right| &\leq \frac{2^{\tau}2^{\frac{\tau}{2}}\left\|\phi^\tau g'\right\|(R_\lambda|s-x|)(x)}{\phi^\tau(x)}\\ &\leq \frac{2^{\frac{3\tau}{2}}\left\|\phi^\tau g'\right\|\sqrt{(R_\lambda(s-x)^2)(x)}}{\phi^\tau(x)}\\ &\leq \frac{2^{\frac{3\tau}{2}}\left\|\phi^\tau g'\right\|}{\phi^\tau(x)}\frac{C_1\phi(x)}{\sqrt{\lambda}}=\frac{C_1 2^{\frac{3\tau}{2}}\left\|\phi^\tau g'\right\|\phi^{1-\tau}(x)}{\sqrt{\lambda}}.\end{aligned}$$

Therefore

$$\begin{aligned}\left|(R_\lambda f(s))(x)-f(x)\right| &\leq \left|(R_\lambda(f-g))(x)\right|+\left|(R_\lambda g(s))(x)-g(x)\right|+\left|g(x)-f(x)\right|\\ &\leq 2\left\|f-g\right\|+\left|(R_\lambda g(s))(x)-g(x)\right|\\ &\leq 2\left\|f-g\right\|+\frac{C_1 2^{\frac{3\tau}{2}}\left\|\phi^\tau g'\right\|\phi^{1-\tau}(x)}{\sqrt{\lambda}}\\ &\leq C\left\{\left\|f-g\right\|+\frac{\phi^{1-\tau}(x)}{\sqrt{\lambda}}\left\|\phi^\tau g'\right\|\right\}\leq CK_{\phi^\tau}\left(f,\frac{\phi^{1-\tau}(x)}{\sqrt{\lambda}}\right),\end{aligned}$$

where $C = max\left\{2, C_1 2^{\frac{3\tau}{2}}\right\}$.
Using relation (14), we get required result. □

Example 3.6. *For $\lambda = 10, 20, 100$ the rate of convergence of the operators $(R_\lambda f(s))(x)$ to the function $f(x) = 9x^2 - 6x + 6/5$ is illustrated in Fig 1. Further, in Table 1, we estimated the absolute error $E_\lambda = \left|(R_\lambda f(s))(x) - f(x)\right|$ for different values of λ and given the corresponding graph for error depicting the convergence in Fig 2 . It can be clearly seen from Fig 1, Fig 2 and from the Table 1 that for larger values of λ the proposed operator (4) converges to $f(x)$.*

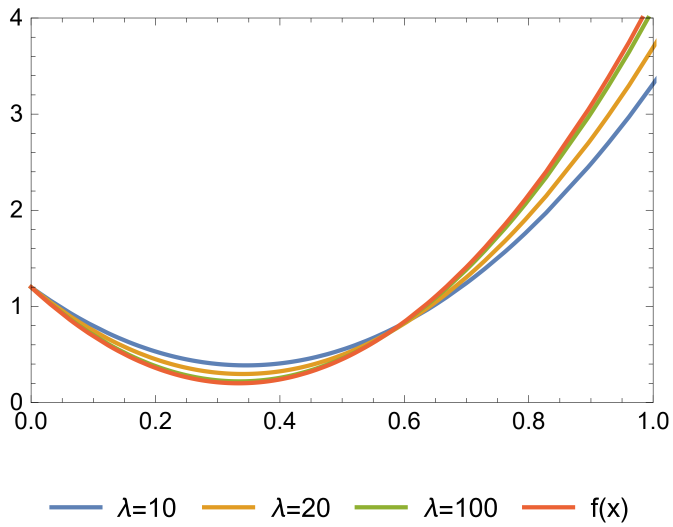

Figure 1: The Convergence of operators $(R_\lambda f(s))(x)$ to the function $f(x) = 9x^2 - 6x + 6/5$ for $\lambda = 10, 20, 100$

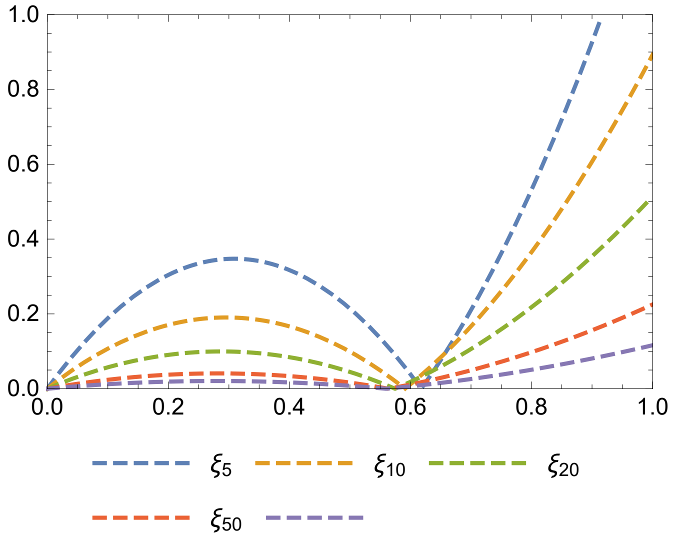

Figure 2: Graphical representation of absolute error of operators $(R_\lambda f(s))(x)$ to the function $f(x) = 9x^2 - 6x + 6/5$ for $\lambda = 5, 10, 20, 50, 100$. The error clearly is converging to zero for the given function.

Table 1: Error of approximation process for $f(x) = 9x^2 - 6x + 6/5$

x	E_5	E_{10}	E_{20}	E_{50}	E_{100}
0.08	0.156686	0.089180	0.047780	0.019951	0.010122
0.16	0.266743	0.150426	0.080101	0.033300	0.016867
0.24	0.330171	0.183749	0.096962	0.040046	0.020235
0.32	0.346971	0.189144	0.098363	0.040189	0.020226
0.40	0.317143	0.166612	0.084304	0.033729	0.016834
0.48	0.240686	0.116152	0.054786	0.020667	0.010077
0.56	0.117600	0.037765	0.009808	0.001002	0.000064
0.64	0.052114	0.068549	0.050630	0.025266	0.013581
0.72	0.268457	0.202790	0.126528	0.058137	0.030475
0.80	0.531429	0.364959	0.217885	0.097610	0.050747
0.88	0.841029	0.555055	0.324702	0.143686	0.074395
0.96	1.197260	0.773078	0.446979	0.196364	0.101421

4. Bézier Variant of Ismail-May Operators

Zeng and Piriou [34] in the year 1997, constructed the Bernstein-Bézier type operators and studied its rate of convergence for bounded variation functions. Gupta et.al.[15] proposed the Bézier variant of the Szász-Kantorovich operator and investigated a convergence theorem for locally bounded functions subsuming the approximation of functions of bounded variation as a special case. These works have been continued ever since by several authors with construction of Bézier variants of different operators and analyzed their approximation properties.For further references, one can refer to articles ([3], [6], [13], [14], [15], [18], [25], [29], [32] , [33]).

Motivated by the above stated work, in this section, we define the Bézier variant of the operators (4) as follows:

$$\left(R_\lambda^\alpha f(s)\right)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{\lambda + k}\right) \vartheta_{\lambda,k}^{\alpha}(x), \quad \alpha \geq 1,\ x \in [0,1], \tag{17}$$

where $\vartheta^{\alpha}_{\lambda,k}(x) = (P_{\lambda,k}(x))^{\alpha} - (P_{\lambda,k+1}(x))^{\alpha}$, and $P_{\lambda,k}(x) = \sum_{j=k}^{\infty} r_{\lambda,j}(x)$ for $k = 0, 1, 2, ..n$ are the Bézier basis functions and

$$r_{\lambda,k}(x) = e^{-\lambda x}\frac{\lambda(\lambda+k)^{k-1}}{k!}(xe^{-x})^{k}, \quad x \in [0,1].$$

Lemma 4.1. *Let $f \in C[0,1]$. Then the Bézier variant of the operators based on the Ismail-May operators verify:*

(i) $\|R^{\alpha}_{\lambda}\| \le \|f\|$;
(ii) $(R^{\alpha}_{\lambda} f(s))(x) \le \alpha(R_{\lambda} f(s))(x)$.

Proof. (i) $(R^{\alpha}_{\lambda}(e_0))(x) = \sum_{k=0}^{\infty} 1.\vartheta^{\alpha}_{\lambda,k}(x) = 1$,
it follows that

$$\left|(R^{\alpha}_{\lambda} f(s))(x)\right| \le \|f\| \sum_{k=0}^{\infty} \vartheta^{\alpha}_{\lambda,k}(x) = \|f\|.$$

(ii) using the inequality $|a^{\alpha} - b^{\alpha}| \le \alpha\,|a-b|$ where $0 \le a, b \le 1$, and $\alpha \ge 1$, we get

$$\begin{aligned} 0 &\le (P_{\lambda,k}(x))^{\alpha} - (P_{\lambda,k+1}(x))^{\alpha} \\ &\le \alpha\,(P_{\lambda,k}(x) - P_{\lambda,k+1}(x)) \\ &= \alpha r_{\lambda,k}(x), \end{aligned}$$

Hence in view of definition of R^{α}_{λ} and the positivity of f, we get the result.

□

4.1. Global approximation theorem

Now we establish a global approximation theorem for the operator (4) using the first order Ditizan Totik modulus of smoothness defined in [10] as follows:

$$\omega_{\phi}(f;t) = \sup_{0<h\le t}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right) - f\left(x-\frac{h\phi(x)}{2}\right)\right|, x+\frac{h\phi(x)}{2} \in [0,1]\right\}.$$

The corresponding K-functional is given as:

$$K_{\phi}(f;t) = \inf_{g\in W_{\phi}[0,1]}\left\{\|f-g\| + t\,\|\phi g'\| + t^2\,\|g'\|\right\}$$

where $W_{\phi}[0,1] = \left\{g : g \in AC_{loc}[0,1], \|\phi g'\| < \infty\right\}$, $g \in AC_{loc}[0,1]$ denotes the class of all locally absolutely continuous function and $\|.\|$ is the sup norm on $C[0,1]$.

Theorem 4.2. *Let $f \in C[0,1]$ and $\phi(x) = \sqrt{x(1-x)}$. For every $x \in [0,1)$ and sufficiently large λ, we have*

$$\left|\left(R^{\alpha}_{\lambda} f(s)\right)(x) - f(x)\right| \le C\omega_{\phi}\left(f;\frac{1}{\sqrt{\lambda}}\right).$$

Proof. By definition of $K_{\phi}(f,t)$, for fixed λ, x we can choose $g = g_{\lambda,x} \in W_{\phi}[0,1)$ such that

$$\|f-g\| + \frac{1}{\sqrt{\lambda}}\|\phi g'\| + \frac{1}{\lambda}\|g'\| \le \omega_{\phi}\left(f;\frac{1}{\sqrt{\lambda}}\right). \tag{18}$$

Then

$$\left|\left(R_\lambda^\alpha f(s)\right)(x) - f(x)\right| \leq \left|\left(R_\lambda^\alpha (f-g)(s)\right)(x)\right| + \left|f-g\right| + \left|\left(R_\lambda^\alpha g(s)\right)(x) - g(x)\right|$$
$$\leq C\left\|f-g\right\| + \left|\left(R_\lambda^\alpha g(s)\right)(x) - g(x)\right|.$$

Now to estimate the second relation above, we split the domain into two subintervals, $x \in I_\lambda = [0, \frac{1}{\lambda}]$ and $x \in I_\lambda^c = (\frac{1}{\lambda}, 1)$. Using the representation

$$g(s) = g(x) + \int_x^s g'(z)dz,$$

we can write

$$\left|\left(R_\lambda^\alpha g(s)\right)(x) - g(x)\right| \leq \left|\left(R_\lambda^\alpha\left(\int_x^s g'(z)dz\right)\right)(x)\right|. \tag{19}$$

Let $x \in I_\lambda^c = (\frac{1}{\lambda}, 1)$, we have

$$\left|\int_x^s g'(z)dz\right| \leq \left\|\phi g'\right\| \left|\int_x^s \frac{1}{\phi(z)}dz\right|$$
$$\leq \left\|\phi g'\right\| \left|\int_x^s \frac{1}{\sqrt{z(1-z)}}dz\right|$$
$$\leq \left\|\phi g'\right\| \frac{2\sqrt{2}\,|s-x|}{\phi(x)}. \tag{20}$$

By combining (19) and (20), we have

$$\left|\left(R_\lambda^\alpha g(s)\right)(x) - g(x)\right| \leq \frac{2\sqrt{2}\left\|\phi g'\right\|}{\phi(x)} \left|\left(R_\lambda^\alpha |s-x|\right)(x)\right|$$
$$\leq \frac{2\sqrt{2}\left\|\phi g'\right\|}{\phi(x)} \left|\left(\left(R_\lambda^\alpha (s-x)^2\right)(x)\right)^{1/2}\right|$$
$$\leq \frac{2\sqrt{2\alpha}\left\|\phi g'\right\|}{\phi(x)} \sqrt{\frac{cx(1-x)}{\lambda}}$$
$$\leq C\frac{\left\|\phi g'\right\|}{\sqrt{\lambda}}.$$

Again for $x \in I_\lambda = [0, \frac{1}{\lambda}]$, using Lemma 4.1 and Remark 2.3

$$\left|\left(R_\lambda^\alpha g(s)\right)(x) - g(x)\right| \leq \left\|g''\right\| \left|\left(R_\lambda^\alpha |s-x|\right)(x)\right|$$
$$\leq \sqrt{\alpha}\left\|g''\right\| \left(R_\lambda (s-x)^2(x)\right)^{\frac{1}{2}}$$
$$\leq \sqrt{\alpha}\left\|g''\right\| \sqrt{\frac{cx(1-x)}{\lambda}}$$
$$\leq \frac{C\left\|g''\right\|}{\lambda}.$$

Therefore,

$$\left|\left(R_{\lambda}^{\alpha}g(s)\right)(x)-g(x)\right|\leq C\left(\frac{\left\|\phi g''\right\|}{\sqrt{\lambda}}+\frac{\left\|g'\right\|}{\lambda}\right). \tag{21}$$

Collecting equations (18)-(21), we obtain the required result. □

4.2. *Voronovoskaya theorem*

In this section, we present a quantitative Voronovoskaya theorem for operators (4) involving Ditizan Totik modulus of smoothness (see also [30]).

Theorem 4.3. *Let $f\in C^2[0,1], x\in[0,1)$. For sufficiently large λ the following inequality hold*

$$\begin{aligned}&\left|\lambda\left[(R_{\lambda}^{\alpha}f(s))(x)-f(x)-\mu_{\lambda,\alpha}^{(1)}(x)f'(x)-\frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x)f''(x)\right]\right|\\&\leq\begin{cases}C\omega_{\phi}(f'';\lambda^{-1/2}\phi(x))\\C\phi(x)\omega_{\phi}(f'';\lambda^{-1/2}),\end{cases}\end{aligned}$$

where $\mu_{\lambda,\alpha}^{(n)}(x)=(R_{\lambda}^{\alpha}(s-x)^n)(x)$.

Proof. Let $f\in C^2[0,1]$ be given and $s,x\in[0,1]$. Using Taylor's expansion

$$f(s)-f(x)=(s-x)f'(x)+\int_x^s(s-u)f''(u)du,$$

we get

$$\begin{aligned}f(s)-f(x)-(s-x)f'(x)-\frac{1}{2}(s-x)^2f''(x)&=\int_x^s(s-u)f''(u)du-\int_x^s(s-u)f''(x)du\\&=\int_x^s(s-u)[f''(u)-f''(x)]du.\end{aligned}$$

Applying R_{λ}^{α} to both sides of the above relation, we obtain

$$\begin{aligned}&\left|(R_{\lambda}^{\alpha}f(s))(x)-f(x)-\mu_{\lambda,\alpha}^{(1)}(x)f'(x)-\frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x)f''(x)\right|\\&\leq R_{\lambda}^{\alpha}\left(\left|\int_x^s|s-u|\left|f''(u)-f''(x)\right|du\right|;x\right).\end{aligned} \tag{22}$$

From ([11]), for $g\in W_{\phi}$ the following estimates can be obtained,

$$\left|\int_x^s|s-u|\left|f''(u)-f''(x)\right|du\right|\leq 2\left\|f''-g\right\|(s-x)^2+2\left\|\phi g'\right\|\phi^{-1}(x)|s-x|^3. \tag{23}$$

Using relations (22)-(23), Lemma 4.1, Remark 2.3 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|(R_\lambda^\alpha f(s))(x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x) f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x) f''(x)\right| \\
&\le 2\left\|f'' - g\right\| R_\lambda^\alpha\left((s-x)^2; x\right) + 2\left\|\phi g'\right\| \phi^{-1}(x) R_\lambda^\alpha\left(|s-x|^3; x\right) \\
&\le 2\left\|f'' - g\right\| \alpha R_\lambda\left((s-x)^2; x\right) + 2\alpha\left\|\phi g'\right\| \phi^{-1}(x)\left\{R_\lambda(s-x)^2; x\right\}^{1/2}\left\{R_\lambda(s-x)^4; x\right\}^{1/2} \\
&\le 2\left\|f'' - g\right\| \alpha R_\lambda\left((s-x)^2; x\right) + \frac{2\alpha C}{\lambda}\left\|\phi g'\right\|\left\{R_\lambda(s-x)^2; x\right\}^{1/2} \\
&\le 2\left\|f'' - g\right\| \alpha \frac{Cx(1-x)}{\lambda} + \frac{2\alpha C}{\lambda}\left\|\phi g'\right\| \sqrt{\frac{Cx(1-x)}{\lambda}} \\
&\le \frac{C}{\lambda}\left\{\phi^2(x)\left\|f'' - g\right\| + \lambda^{-1/2}\phi(x)\left\|\phi g'\right\|\right\},
\end{aligned}$$

where the constant $C > 0$ is not the same at each occurrence.
Since $\phi^2(x) \le \phi(x) \le 1$ and $x \in [0, 1)$, we obtain

$$\begin{aligned}
&\left|\left(R_\lambda^\alpha f(s)\right)(x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x) f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x) f''(x)\right| \\
&\le \frac{C}{\lambda}\left\{\left\|f'' - g\right\| + \lambda^{-1/2}\phi(x)\left\|\phi g'\right\|\right\}. \qquad (24)
\end{aligned}$$

The above inequality can be rewritten as

$$\begin{aligned}
&\left|(R_\lambda^\alpha f(s))(x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x) f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x) f''(x)\right| \\
&\le \frac{C\phi(x)}{\lambda}\left\{\left\|f'' - g\right\| + \lambda^{-1/2}\left\|\phi g'\right\|\right\}. \qquad (25)
\end{aligned}$$

Taking infimum on RHS of (24) and (25) and for $g \in W_\phi$, we have

$$\left|\lambda\left[(R_\lambda^\alpha f(s))(x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x) f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x) f''(x)\right]\right| \le \begin{cases} CK_\phi\left(f''; \lambda^{-1/2}\phi(x)\right) \\ C\phi(x) K_\phi\left(f''; \lambda^{-1/2}\right). \end{cases}$$

□

4.3. Functions of bounded variation

Let $DBV[0,1]$ be the space of all absolutely continuous functions f defined on [0,1] and having a derivative f' equivalent with a function of bounded variation on $[0,1]$. For $f \in DBV[0,1]$ we may write,

$$f(x) = \int_0^x g(t)\,dt + f(0)$$

we can rewrite the operators given by (4) as

$$\left(R_\lambda^\alpha f(s)\right)(x) = \int_0^1 f(s) \frac{\partial}{\partial s}\left\{M_\lambda^\alpha(x,s)\right\} dw,$$

where

$$M_\lambda^\alpha(x,s) = \left\{\begin{array}{ll} \sum\limits_{\frac{k}{k+\lambda} \le s} Q_{\lambda,k}^\alpha(x) & , 0 < s < 1 \\ 0 & , s = 0. \end{array}\right\}$$

We mention the following Lemma which is necessary to prove the main theorem (see also [28]).

Lemma 4.4. *Let $x \in [0,1)$, then for sufficiently large n, we have*

(i) $\vartheta_{\lambda}^{\alpha}(x,s) \leq \frac{C\alpha x(1-x)}{\lambda(x-s)^2}$;

(ii) $1-\vartheta_{\lambda}^{\alpha}(x,z) \leq \frac{C\alpha x(1-x)}{\lambda(x-z)^2}$.

Proof. (i) $\vartheta_{\lambda}^{\alpha}(x,s) = \int_0^s \frac{\partial}{\partial v}\{\mathcal{M}_{\lambda}^{\alpha}(x,v)\}dv \leq \int_0^s \left(\frac{x-v}{x-s}\right)^2 \frac{\partial}{\partial v}\{\mathcal{M}_{\lambda}^{\alpha}(x,v)\}dv = \frac{1}{(x-s)^2}(R_{\lambda}^{\alpha}(x-v)^2)(x,v) \leq \frac{C\alpha x(1-x)}{\lambda(x-s)^2}$.

(ii) The proof of this part is left to the readers which is similar to part (i).

□

In the theorem given below, we study the rate of convergence for functions with derivative of bounded variation (see, e.g. [8])

Theorem 4.5. *Let $f \in DBV[0,1]$. Then, we have*

$$\left|(R_{\lambda}^{\alpha}f(s))(x)-f(x)\right| \leq \left|\frac{(f'(x+)+\alpha f'(x-))}{\alpha+1}\right|\sqrt{\frac{C\alpha x(1-x)}{\lambda}} + \left|(f'(x+)+\alpha f'(x-))\right|\sqrt{\frac{C\alpha x(1-x)}{\lambda}}$$
$$+\frac{C\alpha(1-x)}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\bigvee_{x-x/k}^{x} f_x' + \frac{x}{\sqrt{\lambda}}\bigvee_{x-x/\sqrt{\lambda}}^{x} f_x' + \frac{1-x}{\sqrt{\lambda}}\left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{\lambda}}} f_x'\right) + \frac{C\alpha x}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\left(\bigvee_{x}^{x+\frac{1-x}{k}} f_x'\right),$$

where

$$f_x'(s) = \begin{cases} f'(s)-f'(x-), & 0 \leq s < x \\ 0 & s = x \\ f'(s)-f'(x+), & x < s < \infty. \end{cases} \tag{26}$$

Proof. For $f \in DBV[0,1]$, we may write

$$f'(s) = f_x'(s) + \frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-)) + \frac{1}{2}(f'(x+)-f'(x-))\left(\text{sgn}(s-x)+\frac{\alpha-1}{\alpha+1}\right)$$
$$+\delta_x(s)\left[f'(s)-\frac{1}{2}(f'(x+)+f'(x-))\right], \tag{27}$$

where

$$\delta_x(s) = \begin{cases} 1 & s = x \\ 0, & s \neq x. \end{cases}$$

Again, we have

$$\begin{aligned}
&(R_{\lambda}^{\alpha}f(s))(x)-f(x) \\
&= \int_0^1 (f(s)-f(x))\frac{\partial}{\partial s}\left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\}ds \\
&= \int_0^x (f(s)-f(x))\frac{\partial}{\partial s}\left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\}ds + \int_x^1 (f(s)-f(x))\frac{\partial}{\partial s}\left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\}ds \\
&= -\int_0^x \left(\int_s^x f'(u)du\right)\frac{\partial}{\partial s}\left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\}ds + \int_x^1 \left(\int_s^x f'(u)du\right)\frac{\partial}{\partial s}\left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\}ds \\
&= -A_{\lambda}^{\alpha}(x)+B_{\lambda}^{\alpha}(x).
\end{aligned} \tag{28}$$

Now, from equation (27), we have

$$A_{\lambda}^{\alpha}(x) = \frac{(f'(x+) + \alpha f'(x-))}{\alpha+1} \int_0^x (x-s) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds + \int_0^x \left(\int_s^x f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds$$
$$- \frac{2}{\alpha+1} \frac{f'(x+) - f'(x-)}{2} \int_0^x (x-s) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds. \tag{29}$$

Similarly,

$$B_{\lambda}^{\alpha}(x) = \frac{(f'(x+) + \alpha f'(x-))}{\alpha+1} \int_x^1 (s-x) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds + \int_x^1 \left(\int_x^s f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds$$
$$+ \frac{2\alpha}{\alpha+1} \frac{f'(x+) - f'(x-)}{2} \int_x^1 (s-x) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds. \tag{30}$$

Using (29)-(30)and from (27), we get

$$(R_{\lambda}^{\alpha} f(s))(x) - f(x)$$
$$= \frac{(f'(x+) + \alpha f'(x-))}{\alpha+1} \int_0^1 (s-x) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds + \frac{2}{\alpha+1} \frac{f'(x+) - f'(x-)}{2}$$
$$\times \int_0^x (x-s) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds + \frac{2\alpha}{\alpha+1} \frac{f'(x+) - f'(x-)}{2} \int_x^1 (s-x) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds$$
$$- \int_0^x \left(\int_s^x f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds + \int_x^1 \left(\int_x^s f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds.$$

Hence

$$\left|(R_{\lambda}^{\alpha} f(s))(x) - f(x)\right| \leq \left|\frac{(f'(x+) + \alpha f'(x-))}{\alpha+1}\right| (R_{\lambda}^{\alpha} |s-x|)(x) + \left|f'(x+) - f'(x-)\right| \left|R_{\lambda}^{\alpha} |s-x|)(x)\right|$$
$$+ \left| \int_0^x \left(\int_s^x f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds \right| + \left| \int_x^1 \left(\int_x^s f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds \right|.$$

Applying Cauchy-Schwarz inequality, we get

$$\left|(R_{\lambda}^{\alpha} f(s))(x) - f(x)\right| \leq \left|\frac{(f'(x+) + \alpha f'(x-))}{\alpha+1}\right| \sqrt{(R_{\lambda,\alpha}(s-x)^2)(x)} + \left|f'(x+) - f'(x-)\right| \sqrt{(R_{\lambda,\alpha}(s-x)^2)(x)}$$
$$+ \left| \int_0^x \left(\int_s^x f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds \right| + \left| \int_x^1 \left(\int_x^s f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds \right|. \tag{31}$$

Now, using Lemma 2.1 and integration by parts, we get

$$\int_0^x \left(\int_s^x f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\mathcal{M}_{\lambda}^{\alpha}(x,s)\right\} ds = \int_0^x \left(\int_s^x f_x'(u) du \right) \frac{\partial}{\partial s} \left\{\vartheta_{\lambda}^{\alpha}(x,s)\right\} ds = -\int_0^x f_x'(s) \vartheta_{\lambda}^{\alpha}(x,s) ds$$

Therefore

$$\left|\int_0^x\left(\int_s^x {f_x}'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_n^\alpha(x,s)\}ds\right| \le \int_0^x \left|{f_x}'(s)\right| \vartheta_n^\alpha(x,s)ds$$
$$\le \int_0^{x-\frac{x}{\sqrt{\lambda}}} \left|{f_x}'(s)\right| \vartheta_n^\alpha(x,s)ds + \int_{x-\frac{x}{\sqrt{\lambda}}}^x \left|{f_x}'(s)\right| \vartheta_n^\alpha(x,s)ds.$$

Using ${f_x}'(x)=0$ and $\vartheta_\lambda^\alpha(x,s)\le 1$, we get

$$\int_{x-\frac{x}{\sqrt{\lambda}}}^x \left|{f_x}'(s)\right| \vartheta_\lambda^\alpha(x,s)ds = \int_{x-\frac{x}{\sqrt{\lambda}}}^x \left|{f_x}'(s)-{f_x}'(x)\right| \vartheta_\lambda^\alpha(x,s)ds$$
$$\le \int_{x-\frac{x}{\sqrt{\lambda}}}^x \bigvee_s^x ({f_x}')ds \le \bigvee_{x-x/\sqrt{\lambda}}^x {f_x}' \int_{x-\frac{x}{\sqrt{\lambda}}}^x \bigvee_s^x {f_x}'ds = \frac{x}{\sqrt{\lambda}}\bigvee_{x-x/\sqrt{\lambda}}^x {f_x}'.$$

Again, using $\vartheta_\lambda^\alpha(x,s)\le \frac{C\alpha x(1-x)}{\lambda(x-s)^2}$ and putting $s=x-\frac{x}{u}$, we get

$$\int_0^{x-\frac{x}{\sqrt{\lambda}}} \left|{f_x}'(s)\right| \vartheta_\lambda^\alpha(x,s)ds \le \frac{C\alpha(1-x)}{\lambda}\int_1^{\sqrt{\lambda}}\bigvee_{x-x/\sqrt{\lambda}}^x {f_x}'du \le \frac{C\alpha(1-x)}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\bigvee_{x-x/k}^x {f_x}'$$

Hence

$$\left|\int_0^x\left(\int_s^x {f_x}'(u)du\right)\frac{\partial}{\partial s}\left\{\mathcal{M}_\lambda^\alpha(x,s)\right\}ds\right| \le \frac{C\alpha(1-x)}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\bigvee_{x-x/k}^x {f_x}' + \frac{x}{\sqrt{\lambda}}\bigvee_{x-x/\sqrt{\lambda}}^x {f_x}' \tag{32}$$

Now

$$\left|\int_x^1\left(\int_x^s {f_x}'(u)du\right)\frac{\partial}{\partial s}\left\{\mathcal{M}_\lambda^\alpha(x,s)\right\}ds\right|$$
$$= \left|\int_x^z\left(\int_x^s {f_x}'(u)du\right)\frac{\partial}{\partial s}\left\{1-\vartheta_\lambda^\alpha(x,s)\right\}ds + \int_z^1\left(\int_x^s {f_x}'(u)du\right)\frac{\partial}{\partial s}\left\{1-\vartheta_\lambda^\alpha(x,s)\right\}ds\right|$$
$$= \left|\left(\int_x^z {f_x}'(u)du\right)\left\{1-\vartheta_\lambda^\alpha(x,z)\right\} - \int_x^z {f_x}'(s)\left\{1-\vartheta_\lambda^\alpha(x,s)\right\}ds\right|$$
$$- \left|\left(\int_x^z {f_x}'(u)du\right)\left\{1-\vartheta_\lambda^\alpha(x,z)\right\} - \int_z^1 {f_x}'(s)\left\{1-\vartheta_\lambda^\alpha(x,s)\right\}ds\right|$$
$$= \left|\int_x^z {f_x}'(s)\left\{1-\vartheta_\lambda^\alpha(x,s)\right\}ds + \int_z^1 {f_x}'(s)\left\{1-\vartheta_\lambda^\alpha(x,s)\right\}ds\right|$$
$$\le \int_x^z \bigvee_x^s {f_x}'ds + \frac{C\alpha x(1-x)}{\lambda}\int_z^1\left(\bigvee_x^s {f_x}'\right)(s-x)^{-2}ds$$

Now, let $z = x + \frac{1-x}{\sqrt{\lambda}}$ and then putting $u = \frac{1-x}{s-x}$, we get

$$\left|\int_x^1 \left(\int_x^s f_x'(u)du\right) \frac{\partial}{\partial s}\left\{\mathcal{M}_\lambda^\alpha(x,s)\right\} ds\right| \leq \frac{1-x}{\sqrt{\lambda}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{\lambda}}} f_x'\right) + \frac{C\alpha x(1-x)}{\lambda}\int_{x+\frac{1-x}{\sqrt{\lambda}}}^1 \left(\bigvee_x^s f_x'\right)(s-x)^{-2}ds$$

$$\leq \frac{1-x}{\sqrt{\lambda}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{\lambda}}} f_x'\right) + \frac{C\alpha x(1-x)}{\lambda}\int_1^{\sqrt{\lambda}}\left(\bigvee_x^{x+\frac{1-x}{u}} f_x'\right)(1-x)^{-1}du$$

$$\leq \frac{1-x}{\sqrt{\lambda}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{\lambda}}} f_x'\right) + \frac{C\alpha x}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\left(\bigvee_x^{x+\frac{1-x}{k}} f_x'\right). \tag{33}$$

Collecting estimates from (31-33), we get required result. □

5. Bivariate Generalisation of Ismail-May Operators

In this section, we introduce the bivariate generalisation of the operators (4). A lot of work has already been done on construction of bivariate form of various linear positive operators and analysis of their convergence results. We refer to the readers some interesting articles (see [1],[2],[4],[9],[5],[12],[17],[19],[23],[26],[31]) for more information. The bivariate extension of the operators (4) for $(x, y) \in I^2 = [0, 1] \times [0, 1]$ and $\lambda_1 > 0, \lambda_2 > 0$ is defined as follows:

$$(R_{\lambda_1\lambda_2}^{k_1k_2} f(s_1,s_2))(x,y) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \hbar_{\lambda_1\lambda_2}^{k_1k_2}(x,y) f\left(\frac{k_1}{\lambda_1+k_1}, \frac{k_2}{\lambda_2+k_2}\right), \tag{34}$$

where the basis function is considered as:

$$\hbar_{\lambda_1\lambda_2}^{k_1k_2}(x,y) = e^{-(\lambda_1 x+\lambda_2 y)}\frac{\lambda_1\lambda_2(\lambda_1+k_1)^{k_1}(\lambda_2+k_2)^{k_2}}{k_1!k_2!}(xe^{-x})^{k_1}(ye^{-y})^{k_2}$$

Lemma 5.1. *Let $e_{n_1n_2}(s_1,s_2) = s_1^{n_1}s_2^{n_2}, 0 \leq n_1+n_2 \leq 2$. For $(x,y) \in I^2 = [0,1]\times[0,1]$ and $\lambda_1, \lambda_2 \in [0,\infty)$, we have*

$$(R_\lambda e_{00}(s_1,s_2))(x,y) = 1;$$

$$(R_\lambda e_{10}(s_1,s_2))(x,y) = \frac{\lambda_1}{\lambda_1+1}x;$$

$$(R_\lambda e_{01}(s_1,s_2))(x,y) = \frac{\lambda_2}{\lambda_2+1}y;$$

$$(R_\lambda e_{20}(s_1,s_2))(x,y) = \frac{{\lambda_1}^2}{(\lambda_1+1)(\lambda_1+2)}x^2 + \frac{\lambda_1}{(\lambda_1+1)^2}x;$$

$$(R_\lambda e_{02}(s_1,s_2))(x,y) = \frac{{\lambda_2}^2}{(\lambda_2+1)(\lambda_2+2)}y^2 + \frac{\lambda_2}{(\lambda_2+1)^2}y.$$

Remark 5.2. *Using Lemma 5.1, we have*

$$(R_{k_1k_2}^{\lambda_1\lambda_2}(e_{10}-x))(x,y) = -\frac{x}{(\lambda_1+1)};$$

$$(R_{k_1k_2}^{\lambda_1\lambda_2}(e_{01}-y))(x,y) = -\frac{y}{(\lambda_2+1)};$$

$$(R_{\lambda_1\lambda_2}^{k_1k_2}(e_{20}-x)^2)(x,y) = -\frac{(\lambda_1-2)}{(\lambda_1+1)(\lambda_1+2)}x^2 + \frac{\lambda_1}{(\lambda_1+2)}x;$$

$$(R_{\lambda_1\lambda_2}^{k_1k_2}(e_{02}-y)^2)(x,y) = -\frac{(\lambda_2-2)}{(\lambda_2+1)(\lambda_2+2)}y^2 + \frac{\lambda_2}{(\lambda_2+2)}y.$$

Now we estimate the degree of approximation of bivariate operators (34) with the help of Lipschitz class functions. We define Lipschitz class $Lip_M(\zeta_1, \zeta_2)$ for bivariate functions for $0 < \zeta_1 \leq 1$ and $0 < \zeta_2 \leq 1$ as follows:

$$\left|f(s_1, s_2) - f(x, y)\right| \leq M|s_1 - x|^{\zeta_1}\left|s_2 - y\right|^{\zeta_2}.$$

Theorem 5.3. *If $f \in Lip_M(\zeta_1, \zeta_2)$, then for $\zeta_1, \zeta_2 \in (0, 1]$*

$$\left|\left(R^{k_1 k_2}_{\lambda_1 \lambda_2} f(s_1, s_2)\right)(x, y) - f(x, y)\right| \leq M\delta_n^{\frac{\zeta_1}{2}} \delta_m^{\frac{\zeta_2}{2}}.$$

Proof. If $f \in Lip_M(\zeta_1, \zeta_2)$, we can write

$$\begin{aligned}
&\left|\left(R^{k_1 k_2}_{\lambda_1 \lambda_2} f(s_1, s_2)\right)(x, y) - f(x, y)\right| \\
&\leq \left(R^{k_1 k_2}_{\lambda_1 \lambda_2} \left|f(s_1, s_2) - f(x, y)\right|\right)(x, y) \\
&\leq \left(R^{k_1 k_2}_{\lambda_1 \lambda_2} \left(M|s_1 - x|^{\zeta_1}\left|s_2 - y\right|^{\zeta_2}\right)\right)(x, y) \\
&\leq M\left(R_{\lambda_1}|s_1 - x|^{\zeta_1}\right)(x)\left(R_{\lambda_2}\left|s_2 - y\right|^{\zeta_2}\right)(y).
\end{aligned}$$

Using Hölder's inequality

$$\begin{aligned}
&\left|\left(R^{k_1 k_2}_{\lambda_1 \lambda_2} f(s_1, s_2)\right)(x, y) - f(x, y)\right| \\
&\leq M\left(R_{\lambda_1}(e_{10} - x)^2\right)^{\frac{\zeta_1}{2}}(x, y)\left(R_{\lambda_1}(e_{00})\right)^{\frac{2-\zeta_1}{2}}(x, y) \\
&\times \left(R_{\lambda_2}(e_{01} - y)^2\right)^{\frac{\zeta_2}{2}}(x, y)\left(R_{\lambda_2}(e_{00})\right)^{\frac{2-\zeta_2}{2}}(x, y) \\
&\leq M\delta_n^{\frac{\zeta_1}{2}} \delta_m^{\frac{\zeta_2}{2}}.
\end{aligned}$$

□

Acknowledgement

The authors are thankful to the referee for his valuable comments as well as suggestions, leading to a better presentation of the paper and the first author expresses her sincere thanks to Mr. Ram Pratap for his continued support and helpful discussions during the preparation of this paper.

References

[1] O. Agratini, On a class of linear positive bivariate operators of King type, Studia Universitatis Babeş-Bolyai Mathematica 51(4) (2006) 13–22.
[2] P. N. Agrawal, N. Ispir, Degree of approximation for bivariate Chlodowsky-Szász-Charlier type operators, Results in Mathematics 69 (2016) 369–385.
[3] P. N. Agrawal, N. Ispir, A. Kajla, Approximation properties of Bézier-summation-integral type operators based on Polya-Bernstein functions, Applied Mathematics and Computation 259 (2015) 533–539.
[4] N. Deo, N. Bhardwaj, Some approximation theorems for multivariate Bernstein operators, Southeast Asian Bulletin of Mathematics 34 (2010) 1023–1034.
[5] N. Deo, M. Dhamija, D. Miclăuş, New modified Baskakov operators based on the inverse Pólya-Eggenberger distribution, Filomat 33(11) (2019) 3537–3550.
[6] S. Deshwal, A. M. Acu, P. N. Agrawal, Pointwise approximation by Bézier Variant of an operator based on Laguerre polynomials, Journal of Mathematical Inequalities 12(3) (2018) 693–707
[7] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.
[8] M. Dhamija, N. Deo, Approximation by generalized positive linear Kantorovich operators, Filomat 31(14) (2017), 4353–4368.
[9] M. Dhamija, Durrmeyer modification of Lupas type Baskakov operators based on IPED, Mathematical Analysis I: Approximation Theory, Springer Proceedings in Mathematics & Statistics (eds. Deo et al.) 306 (2020) 111–120.

[10] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer, New York, 1987.
[11] Z. Finta, Remark on Voronovskaja theorem for q-Bernstein operators, Studia Universitatis Babeş-Bolyai Mathematica 56 (2011) 335–339.
[12] V. Gupta, N. Deo, On the rate of convergence for bivariate Beta operators, General Mathematics 13(3) (2005) 107–114.
[13] V. Gupta, N. Ispir, On the Bézier variant of generalized Kantorovich type Balázs operators, Applied Mathematics Letters 18 (2005) 1052–1061.
[14] V. Gupta, H. M. Srivastava, A general family of the Srivastava-Gupta operators preserving linear functions, European Journal of Pure and Applied Mathematics 11(3) (2018) 575–579.
[15] V. Gupta, X. M. Zeng, Approximation by Bézier variant of the Szász−Kantorovich operator in case $0 < \alpha < 1$, Georgian Mathematical Journal 17 (2010) 253–260.
[16] E. H. Ismail Mourad, C. P. May, On a family of approximation operators, Journal of Mathematical Analysis and Applications 63 (1978) 446–462.
[17] N. Ispir, I. B*ü*y*ü*kyazici, Quantitative estimates for a certain bivariate Chlodowsky-Szász-Kantorovich type operators, Mathematical Communications 21 (2016) 31–44.
[18] N. Ispir, I. Yuksel, On the Bézier variant of Srivastava-Gupta operators, Applied Mathematics E-Notes 5 (2005) 129–137.
[19] H. Karsli, M. A. Özarslan, Direct local and global approximation results for the operators of gamma type, Hacettepe Journal of Mathematics and Statistics 39(2) (2010) 241–253.
[20] B. Lenze, On Lipschitz-type maximal functions and their smoothness spaces, Nederl Akad Wetensch Indag Math 50(1) (1988) 53–63.
[21] C. P. May, Saturation and inverse theorems for combinations of a class of exponential type operators, Canadian Journal of Mathematics 28 (1976) 1224–1250.
[22] N. S. Mishra, N. Deo, Kantorovich variant of Ismail-May operators, Iranian Journal of Science and Technology. Transaction A. Science (2020) In-press.
[23] E. Özergin, M. A. Özarslan, H. M. Srivastava, Some families of generating functions for a class of bivariate polynomials, Mathematical and Computer Modelling (50) (2009) 1113–1120.
[24] G. Pólya, G. Szegö, Problems and Theorems in Analysis, Vol.I (English translation), Springer-Verlag, New York, 1972.
[25] R. Pratap, N. Deo, Rate of convergence of Gupta-Srivastava operators based on certain parameters, Journal of Classical Analysis 14(2) (2019) 137–153.
[26] P. Sharma, V. N. Mishra, Bivariate generalization of q-Bernstein-Kantorovich type operator, Cogent Mathematics (3) (2016) Article 1160587.
[27] H. M. Srivastava, Z. Finta, V. Gupta, Direct results for a certain family of summation-integral type operators, Applied Mathematics and Computation 190 (2007) 449–457.
[28] H. M. Srivastava, V. Gupta., A certain family of summation-integral type operators, Mathematical and Computer Modelling 37 (2003) 1307–1315.
[29] H. M. Srivastava, V. Gupta, Rate of convergence for Bézier variant of the Bleimann-Butzer-Hahn operators, Applied Mathematics Letters 18(8) (2005) 849–857.
[30] H. M. Srivastava, G. Içöz, B. Çekim, Approximation properties of an extended family of the Szász-Mirakjan Beta-type operators, Axioms 8(4) (2019) Article 111.
[31] H. M. Srivastava, M. Mursaleen, A. M. Alotaibi, M. Nasiruzzaman, A. A. H. Al-Abied, Some approximation results involving the q-Szász-Mirakjan-Kantorovich type operators via Dunkl's generalization, Mathematical Methods in the Applied Sciences 40(15) (2017) 5437–5452.
[32] H. M. Srivastava, F. Özger, S. A. Mohiuddine, Construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter λ, Symmetry 11(3) (2019) Article 316.
[33] H. M. Srivastava, X. M. Zeng, Approximation by means of the Szász-Bézier integral operators, International Journal of Pure and Applied Mathematics 14 (2004) 283–294.
[34] X. M. Zeng, A. Piriou, On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions, Journal of Approximation Theory 95(3) (1998) 369–387.