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The Fixed Point Property of the Infinite K-Sphere in the Set $Con^{\bigstar}((\mathbb{Z}^2)^*)$

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Abstract. In this paper the Alexandroff one point compactification of the 2-dimensional Khalimsky (K-, for brevity) plane (*resp.* the 1-dimensional Khalimsky line) is called the infinite K-sphere (*resp.* the infinite K-circle). The present paper initially proves that the infinite K-circle has the fixed point property (*FPP*, for short) in the set $Con(\mathbb{Z}^*)$, where $Con(\mathbb{Z}^*)$ means the set of all continuous self-maps f of the infinite K-circle. Next, we address the following query which remains open: Under what condition does the infinite K-sphere have the *FPP*? Regarding this issue, we prove that the infinite K-sphere has the *FPP* in the set $Con^{\bigstar}((\mathbb{Z}^2)^*)$ (see Definition 1.1). Finally, we compare the *FPP* of the infinite K-sphere and that of the infinite M-sphere, where the infinite M-sphere means the one point compactification of the Marcus-Wyse topological plane.

1. Introduction

The present paper focuses on studying the fixed point property (*FPP*, for short) of the Alexandroff one point compactifications of the Khalimsky topological plane and the Khalimsky topological line. Since we will often use the term "Khalimsky" in this paper, hereafter we will use the notation "K-" for short instead of the "Khalimsky" if there is no danger of ambiguity.

In this paper we denote by $(\mathbb{Z}, \kappa)$ (*resp.* $(\mathbb{Z}^2, \kappa^2)$) the Khalimsky line (*resp.* the Khalimsky plane) (see Section 2). It is obvious that the n-dimensional K-topological space $(\mathbb{Z}^n, \kappa^n)$ is neither a Hausdorff nor a compact space but a locally compact space [10], where $n \in \mathbb{N}$: the set of natural numbers. Hence we can establish the Alexandroff one point compactification of $(\mathbb{Z}^n, \kappa^n), n \in \{1, 2\}$ [10]. We recall that a topological space (X, T) is locally compact [21] if for each point $x \in X$ there is a compact neighborhood containing the point x. Then, it turns out that $(\mathbb{Z}^n, \kappa^n)$ is locally compact but neither compact nor Hausdorff (for more details see the property (2.1)). Thus, we can proceed with the Alexandroff one point compactification of $(\mathbb{Z}^n, \kappa^n), n \in \{1, 2\}$, instead of the Hausdorff compactification.

In order to proceed with this work and make the present paper self-contained, let us firstly recall the well-known one point compactification of a non-compact [2] and non-Hausdorff topological space. Let

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(X, T) be a non-compact and locally compact space. Take some object outside X, denote by the symbol $*$ (or ∞) $\notin X$, adjoin it to X, and finally form the set $X \cup \{*\} := X^*$. Topologize X^* by defining the collection of open sets in X^* to be all sets of the following types.
(c-1) $U \in T$
(c-2) $U(\ni *)$ where the complement of U in X^* is a compact and closed subset of (X, T).

This topological space is called the Alexandroff one-point compactification of (X, T) and denoted by (X^*, T^*). According to (X^*, T^*), it is obvious that the singleton $\{*\}$ is not an open but a closed subset of (X^*, T^*). Hereafter, we denote by $(\mathbb{Z}^*, \kappa^*)$ (*resp.* $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ the one point compactification of $(\mathbb{Z}, \kappa)$ (*resp.* $(\mathbb{Z}^2, \kappa^2)$) [10]. Besides, for convenience, we often call $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ (*resp.* $(\mathbb{Z}^*, \kappa^*)$) *the infinite K-sphere* (*resp. the infinite K-circle*). Hereafter, as usual, we often denote by $\aleph_0$ the first infinite cardinal number. Besides, $|\cdot|$ means the cardinality of the given set. To study the *FPP* problems of the infinite *K*-sphere and the infinite *K*-circle [7], we need to define the following sets.

Definition 1.1. (1) Let $Con(\mathbb{Z}^*)$ be the set of all continuous self-maps of the infinite *K*-circle.
(2) We denote by $Con((\mathbb{Z}^2)^*)$ the set of all continuous self-maps f of the infinite *K*-sphere.
(3) Let $Con^{\bigstar}((\mathbb{Z}^2)^*)$ be the set of all maps $f \in Con((\mathbb{Z}^2)^*)$ such that

$$\begin{cases} (a)\, |Im(f)| = \aleph_0 \text{ with } * \in Im(f) \text{ or} \\ (b) \text{ constant maps on the infinite } K\text{-sphere, or} \\ (c)\, (Im(f), (\kappa^2)^*_{Im(f)}) \text{ has a point } y \in Im(f) \\ \text{such that } f|_{Im(f)}(y) = y, \text{ where } Im(f) \subset \mathbb{Z}^2, \text{and} \\ f|_{Im(f)} \text{ means the restriction of } f \text{ to } Im(f). \end{cases}$$

In general, for a given category we say that an object X has the fixed point property (*FPP*, for brevity) in the category if for any self-morphism f of X there is some element $x \in X$ such that $f(x) = x$. Regarding the *FPP* for the infinite *K*-sphere, up to now there is one of the unsolved problems, as follows: Assume a map $f \in Con((\mathbb{Z}^2)^*)$ such that $Im(f)(\subset \mathbb{Z}^2)$. Then, does the subspace $(Im(f), \kappa^2_{Im(f)})$ have the *FPP*? Regarding this issue, up to now, only a few cases were proved. For instance, in the case $Im(f)$ is one of the smallest open sets [6] or it is equal to a *K*-retractable subspace of the *K*-square $(I^2, \kappa^2_{I^2})$ [13], it turns out that $(Im(f), \kappa^2_{Im(f)})$ has the *FPP* [8, 13]. Thus, we need to investigate a certain subset of $Con((\mathbb{Z}^2)^*)$ supporting the *FPP* for $(Im(f), (\kappa^2)^*_{Im(f)})$ or the infinite *K*-sphere. Hence we may raise the following queries.
($*$1) Does a continuous self-bijection of the infinite *K*-sphere (*resp.* the infinite *K*-circle) imply a self-homeomorphism of the infinite *K*-sphere (*resp.* the infinite *K*-circle)?
($*$2) Does the infinite *K*-circle have the *FPP* in the set $Con(\mathbb{Z}^*)$?
($*$3) Does the infinite *K*-sphere have the *FPP* in the set $Con^{\bigstar}((\mathbb{Z}^2)^*)$?
($*$4) What difference exists between the *FPP* of the infinite *K*-sphere and that of the infinite *M*-sphere.

The present paper suggests positive answers to the queries ($*$1), ($*$2), and ($*$3). Owing to these features, comparing with the non-*FPP* of the Hausdorff compactifications of the n-dimensional Euclidean topological spaces, $n \in \{1, 2\}$, we can recognize that the *FPP* problems of $(\mathbb{Z}^*, \kappa^*)$ and $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ have their own features.

The rest of the paper is organized as follows: Section 2 refers to some notions related to the *K*-topology and the Alexandroff one point compactification. Section 3 investigates various properties of continuous self-maps (or surjections or bijections) of the infinite *K*-sphere and the infinite *K*-circle. Section 4 proves that each map $f \in Con(\mathbb{Z}^*)$ has some point $x \in \mathbb{Z}^*$ such that $f(x) = x$. Concretely, we prove that $(\mathbb{Z}^*, \kappa^*)$ has the *FPP*. Section 5 proves that every map f of $Con^{\bigstar}((\mathbb{Z}^2)^*)$ has a certain point $x \in (\mathbb{Z}^2)^*$ such that $f(x) = x$. Section 6 compares the *FPP* of the infinite *K*-sphere and that of the infinite *M*-sphere. Section 7 concludes the paper with some remarks.

2. Preliminaries

We say that a topological space (X, T) is an Alexandroff space if every point $x \in X$ has the smallest (or minimal) open neighborhood in (X, T) [2]. Based on the Alexandroff topological structure [1, 2], the Khalimsky nD space was established and the study of its properties includes the papers [3, 10, 12, 14–18, 20, 23]. Let us now recall basic notions of the Khalimsky nD space. The *Khalimsky line topology* on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbb{Z}} \,|\, n \in \mathbf{Z}\}$ as a subbase [2] (see also [15]), where for $a, b \in \mathbb{Z}$, $[a, b]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, a \leq x \leq b\}$ [22]. Besides, we will use the notations $[a, +\infty)_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, a \leq x\}$ and $(-\infty, a]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, x \leq a\}$. In the present paper we call $([a, b]_{\mathbb{Z}}, \kappa_{[a,b]_{\mathbb{Z}}})$ (or $[a, b]_{\mathbb{Z}}$ for short, if there is no danger of ambiguity) a Khalimsky interval. Depending on the situation, we may use it with the K-topology or without topology. Furthermore, the product topology on $\mathbb{Z}^n$ induced by $(\mathbb{Z}, \kappa)$ is called the *Khalimsky product topology* on $\mathbb{Z}^n$ (or the *Khalimsky nD space*), denoted by $(\mathbb{Z}^n, \kappa^n)$. Hereafter, for a subset $X \subset \mathbb{Z}^n$ we will denote by (X, κ^n_X), $n \geq 1$, the subspace induced by $(\mathbb{Z}^n, \kappa^n)$, and we call it a K-topological space.

Let us now examine the structure of $(\mathbb{Z}^n, \kappa^n)$. A point $x = (x_i)_{i \in [1,n]_{\mathbb{Z}}} \in \mathbb{Z}^n$ is *pure open* if all coordinates are odd, *pure closed* if each of the coordinates is even and the other points in $\mathbf{Z}^n$ are called *mixed* [15]. We also denote by $(\mathbb{Z}^n)_e$ (*resp.* $(\mathbb{Z}^n)_o$) the set of pure closed (*resp.* pure open) points in $(\mathbb{Z}^n, \kappa^n)$. Besides, we denote by $(\mathbb{Z}^n)_m$ the set of mixed points in $(\mathbb{Z}^n, \kappa^n)$.

In relation to the further statement of a mixed point in $(\mathbb{Z}^2, \kappa^2)$, for the points $p = (2m, 2n+1)$ (*resp.* $p = (2m+1, 2n)$), we call the point p *closed-open* (*resp. open-closed*). With this perspective, we clearly observe that for the point $p = (p_1, p_2)$ of $\mathbb{Z}^2$ the *smallest (open) neighborhood* of the point, denoted by $SN_K(p) \subset \mathbb{Z}^2$, is the following [1, 2, 4]:

$$SN_K(p) = \left\{ \begin{array}{l} \{p\} \text{ if } p \text{ is pure open}, \\ \{(p_1 - 1, p_2), p, (p_1 + 1, p_2)\} \text{ if } p \text{ is closed-open}, \\ \{(p_1, p_2 - 1), p, (p_1, p_2 + 1)\} \text{ if } p \text{ is open-closed}, \\ N_8(p) \text{ if } p := (2m, 2n), m, n \in \mathbb{Z} \text{ is pure closed}, \textit{where} \\ N_8(p) := [2m-1, 2m+1]_{\mathbb{Z}} \times [2n-1, 2n+1]_{\mathbb{Z}}. \end{array} \right\} \tag{2.1}$$

Hereafter, in (X, κ^n_X), for a point $p \in X$ we use the notation $SN_K(p) \cap X := SN_X(p)$ or $O(p)$ for short.

Remark 2.1. In view of the property (2.1), any infinite subset of $(\mathbb{Z}^n, \kappa^n)$ is not compact in $(\mathbb{Z}^n, \kappa^n)$.

In $(\mathbb{Z}^n, \kappa^n)$, let us now recall the notion of K-continuity of a map between two K-topological spaces [14] as follows: For two K-topological spaces $(X, \kappa^{n_0}_X) := X$ and $(Y, \kappa^{n_1}_Y) := Y$, a function $f : X \to Y$ is said to be *K-continuous* at a point $x \in X$ if f is continuous at the point x from the viewpoint of Khalimsky product topology. Furthermore, we say that a map $f : X \to Y$ is K-continuous if it is K-continuous at every point $x \in X$. Indeed, since $(\mathbb{Z}^n, \kappa^n)$ is an Alexandroff space (see the property (2.1)), we can represent the K-continuity of f at a point $x \in X$ [4], as follows:

$$f(SN_K(x)) \subset SN_K(f(x)). \tag{2.2}$$

In addition, we recall the notion of a *K-homeomorphism* as follows: For two spaces $(X, \kappa^{n_0}_X)$ and $(Y, \kappa^{n_1}_Y)$, a map $h : X \to Y$ is called a K-homeomorphism if h is a K-continuous bijection and further, $h^{-1} : Y \to X$ is K-continuous. Besides, we say that two distinct points $x, y \in \mathbb{Z}^n$ is K-adjacent if $x \in SN_K(y)$ or $y \in SN_K(x)$ [15]. Using this notions, the following notions are defined [4].
(•) Consider two distinct points $x, y \in X := (X, \kappa^n_X)$ if there is the sequence (or a path) $(x_0, x_1, \cdots, x_l)$ on X with $\{x_0 = x, x_1, \cdots, x_l = y\}$ such that x_i and x_{i+1} are K-adjacent, $i \in [0, l-1]_{\mathbb{Z}}, l \geq 1$, then we say the sequence is a K-path connecting the two given points x and y.
Besides, for any two points $x, y \in X$, there is a K-path connecting the two points, then X is called K-path connected (or connected).
(•) A *simple K-path* in X is the K-path $(x_i)_{i \in [0,l]_{\mathbb{Z}}}$ in X such that x_i and x_j are K-adjacent if and only if $|i - j| = 1$.
Owing to the structure of (2.1), we have the following:

Proposition 2.2. *Each* $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ *and* $(\mathbb{Z}^*, \kappa^*)$ *are connected.*

Proof. Let us prove the connectedness of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. According to the definition of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ and owing to the connectedness of $(\mathbb{Z}^2, \kappa^2)$ [15] (or [10]), we obtain $\mathbb{Z}^2 \subset (\mathbb{Z}^2)^* = cl(\mathbb{Z}^2)$: the closure of $\mathbb{Z}^2$ in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$, which implies the connectedness of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$.
Using a method similar to the proof of the connectedness of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$, the connectedness of $(\mathbb{Z}^*, \kappa^*)$ was proved in [10]. □

3. Continuous Self-Maps of the Infinite *K*-Sphere and the Infinite *K*-Circle

Let us now investigate various properties of continuous self-maps of the infinite *K*-sphere and the infinite *K*-circle.

Remark 3.1. Each of the following is a continuous self-surjection of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$.
(1) The identity map.
(2) $f^*((x, y)) = (-y, -x), (x, y) \in \mathbb{Z}^2$ and $f^*(*) = *$.
(3) $f^*((x, y)) = (-x, y), (x, y) \in \mathbb{Z}^2$ and $f^*(*) = *$.
(4) $f^*((x, y)) = (x, -y), (x, y) \in \mathbb{Z}^2$ and $f^*(*) = *$.
(5) $f^*((x, y)) = (x + 2m, y + 2n), (x, y) \in \mathbb{Z}^2, m, n \in \mathbb{Z}$ and $f^*(*) = *$.

Proof. Each of the surjections of (1)-(4) is obviously continuous.
Let us now prove the continuity of the surjection f^* of (5). Given the self-surjection f^* of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$, for any point $p := (x, y) \in \mathbb{Z}^2$ we clearly have the following cases.

(Case 1) For any point $p \in \mathbb{Z}^2$ and the smallest open set $SN_K(f^*(p))$ in κ^2, we clearly have the corresponding smallest open neighborhood of p, denoted by $SN_K(p)$ in κ^2, such that $SN_K(p) = (f^*)^{-1}(SN_K(f^*(p)))$ according to the point $p \in \mathbb{Z}^2 = (\mathbb{Z}^2)_e \cup (\mathbb{Z}^2)_m \cup (\mathbb{Z}^2)_o$ (see the property (2.1)), which implies the continuity of the map f at the point p.

(Case 2) For the point $* \in (\mathbb{Z}^2)^*$ and any open neighborhood of $f^*(*)$, denoted by $O_K(f^*(*))$, we have an open neighborhood of $*$, denoted by $O_K(*)$ in $(\kappa^2)^*$, with

$$O_K(*) := (f^*)^{-1}(O_K(f^*(*)))$$

fulfilling the *K*-continuity of f at the point $*$ because we may take the open sets $O_K(*)$ and $O_K(f^*(*))$ in such a way

$$\begin{cases} (a)\, |\, O_K(*)\, | = |\, O_K(f^*(*))\, | \ = \aleph_0, \\ (b) \text{ both } (\mathbb{Z}^2)^* \setminus O_K(*) \text{ and } (\mathbb{Z}^2)^* \setminus O_K(f^*(*)) \\ \text{are compact and closed in } (\mathbb{Z}^2, \kappa^2), \text{ and} \\ (c)\, |(\mathbb{Z}^2)^* \setminus O_K(f^*(*))| = |(\mathbb{Z}^2)^* \setminus O_K(*)| \lneq \aleph_0. \end{cases}$$

Then, we need to point out that the open sets $O_K(f^*(*))$ and $O_K(*)$ are not minimal open sets of the corresponding points $f^*(*)$ and $*$ because both the points $*$ and $f(*)$ do not have their smallest open neighborhoods in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ (for more details, see the proof of Lemma 5.4 in the present paper). □

Using a method similar to the proof of Remark 3.1, we have the following:

Example 3.2. The followings are continuous self-surjections of $(\mathbb{Z}^*, \kappa^*)$.
(1) The identity map.
(2) $f^*(x) = -x, x \in \mathbb{Z}$ and $f^*(*) = *$.
(3) $f^*(x) = x + 2m, x \in \mathbb{Z}, m \in \mathbb{Z}$ and $f^*(*) = *$.

The Hausdorff one point compactification of the n-dimensional Euclidean topological space, $n \in \{1, 2\}$ [21] is often denoted by $((\mathbb{R}^n)^*, \mathbf{U}^*)$, $n \in \{1, 2\}$ [21]. Unlike the continuity of rotations of $((\mathbb{R}^2)^*, \mathbf{U}^*)$ and $((\mathbb{R})^*, \mathbf{U}^*)$, we have the following:

Lemma 3.3. *(1) A continuous self-map f of the infinite K-sphere such that $f(*) = p \in \mathbb{Z}^2$ is not injective.*
(2) A continuous self-map f of the infinite K-circle such that $f(*) = p \in \mathbb{Z}$ is not injective.

Proof. (1) Suppose a continuous self-map f of the infinite K-sphere such that $f(*) = p \in \mathbb{Z}^2$. Take the smallest open neighborhood of the point p, denoted by $SN_K(p)$. Since the set

$$f^{-1}(SN_K(p))(\ni *) \text{ is an open set in } (\kappa^2)^*, \tag{3.1}$$

it has an infinite cardinality because

$$(\mathbb{Z}^2)^* \setminus f^{-1}(SN_K(p)) = \mathbb{Z}^2 \setminus f^{-1}(SN_K(p)) \text{ is compact in } (\mathbb{Z}^2, \kappa^2), \tag{3.2}$$

which implies that the set $(\mathbb{Z}^2)^* \setminus f^{-1}(SN_K(p))$ of (3.2) should be finite (see Remark 2.1). Hence we have $|f^{-1}(SN_K(p))| = \aleph_0$. Thus, the map f should not be injective because $SN_K(p)$ is finite for any point $p \in (\mathbb{Z}^2, \kappa^2)$.
(2) Using a method similar to the proof of (1) above, the proof is completed. □

Corollary 3.4. *Assume that f is a rotation of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Then f should support $f(*) = *$.*

Proof. Any rotation of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ should be a self-homeomorphism of it, the proof is completed. □

Remark 3.5. (1) Not every continuous self-surjection of the infinite K-circle is an injection.
(2) Not every continuous self-surjection of the infinite K-sphere is an injection.

Proof. (1) Let us consider the self-map g of the infinite K-circle such that

$$\left\{\begin{array}{l} g(x) = x + 2,\ x \in [2, \infty)_{\mathbb{Z}}, \\ g(\{-1, 0, 1\}) = \{3\}, \\ g(x) = x + 4,\ x \in (-\infty, -2]_{\mathbb{Z}},\ \text{and} \\ g(*) = *. \end{array}\right\} \tag{3.3}$$

Then, the map g is a continuous self-surjection of the infinite K-circle which is not injective.
(2) Using a method similar to the approach of (1), the proof is completed. □

Let us now investigate a relation between a continuous self-bijection of the infinite K-sphere and a self-homeomorphism of the infinite K-sphere, which address the query of ($*$1) posed in Section 1.

Theorem 3.6. *(1) A continuous self-bijection of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ is a self-homeomorphism.*
(2) A continuous self-bijection of $(\mathbb{Z}^*, \kappa^*)$ is a self-homeomorphism.

Proof. (1) Owing to Lemma 3.3, every continuous self-bijection f of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ has the properties

$$\left\{\begin{array}{l} f(*) = * \text{ and} \\ f(SN_K(p)) = SN_K(f(p)) \text{ with } |SN_K(p)| = |SN_K(f(p))|, \end{array}\right\} \tag{3.4}$$

where $SN_K(p), SN_K(f(p)) \in \kappa^2$, and if $p \in (\mathbb{Z}^2)_e$, $p \in (\mathbb{Z}^2)_o$, or $p \in (\mathbb{Z}^2)_m$, then $f(p) \in (\mathbb{Z}^2)_e$, $f(p) \in (\mathbb{Z}^2)_o$, or $f(p) \in (\mathbb{Z}^2)_m$, respectively.
Next, by using a method similar to the approach of Remark 3.1 (Case 2), for the point $* \in (\mathbb{Z}^2)^*$ and any open neighborhood of $f(*)$, denoted by $O_K(f(*))$, we have an open neighborhood of $*$, denoted by $O_K(*)$, such that

$$O_K(*) = f^{-1}(O_K(f(*))).$$

Then, we need to point out that the open neighborhoods of $f(*)$ and $*$ such as $O_K(f(*))$ and $O_K(*)$ are not minimal open sets of the corresponding points because the points $*$ and $f(*)$ do not have their minimal open neighborhoods in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Thus, the inverse of the given continuous self-bijection f of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ also has the property

$$| SN_K(q) | = | SN_K(f^{-1}(q)) | \text{ and } f^{-1}(*) = *,$$

where $q \in \mathbb{Z}^2$. Hence the given map f should be a homeomorphism.

(2) Using a method similar to the proof of (1) above, the proof is completed. □

In view of Lemma 3.3 and Theorem 3.6, we obtain the following:

Corollary 3.7. *(1) A self-homeomorphism f of the infinite K-sphere has the property $f(*) = *$.*
(2) A self-homeomorphism f of the infinite K-circle has the property $f(*) = *$.

Regarding the *FPP* of the infinite *K*-sphere, the following properties with Lemma 3.8 and Proposition 3.9 will be used in Section 4.

Hereafter, given a topological space (X, T), we denote by $Con(X)$ the set of all self-continuous maps f of (X, T). Besides, we use the notation $Fix(f) := \{x \in X \,|\, f(x) = x, f \in Con(X)\}$ and $Im(f) := \{f(x) \,|\, x \in X\}$.

Lemma 3.8. *Consider a topological space (X, T) and a map $f \in Con(X)$. Let $f|_{Im(f)}$ be the restriction of f to $Im(f)$. If there is a point $y \in Im(f)$ such that $f(y) = y$, then there is a point $x \in X$ such that $f(x) = x$.*

Proof. Let $f|_{Im(f)} : (Im(f), T_{Im(f)}) \to (X, T)$ be the restriction of f to $Im(f)$. Owing to the hypothesis, there is some point $y \in Im(f)$ such that $f|_{Im(f)}(y) = y$, which implies $f(y) = y$. Hence $y \in Fix(f)$. Then, put $x := y$ so that $f(x) = x$. □

By Lemma 3.8, we obtain the following:

Proposition 3.9. *Consider a topological space (X, T). For any $f \in Con(X)$, if $(Im(f), T_{Im(f)})$ has a point $y \in Im(f)$ such that $f|_{Im(f)}(y) = y$, then (X, T) has the FPP.*

4. Fixed Point Property of the Infinite *K*-circle in the Set *Con*($\mathbb{Z}^*$)

This section addresses the query (*2) referred to in Section 1. Naively, we may ask if for any $f \in Con(\mathbb{Z}^*)$ there is some point $x \in \mathbb{Z}^*$ such that $f(x) = x$. Besides, a *K-topological invariant* [8] is said to be a property of a *K*-topological space that is invariant under *K*-homeomorphisms. In other words, we often call that property is a *K*-topological property [8]. This section is devoted to proving the *FPP* of $(\mathbb{Z}^*, \kappa^*)$ in the set $Con(\mathbb{Z}^*)$ (see Theorem 4.3). To address the query (*2) in Section 1, we need the following lemma.

Lemma 4.1. *For any map $f \in Con(\mathbb{Z}^*)$, $Im(f)$ is connected in $(\mathbb{Z}^*, \kappa^*)$.*

Proof. Owing to Proposition 2.2 and the continuity of f, the proof is completed. □

Lemma 4.2. *Let us consider a map $f \in Con(\mathbb{Z}^*)$.*
(1) In the case $* \notin Im(f)$, we obtain $|Im(f)| \leq \aleph_0$.
(2) In the case $* \in Im(f)$, we obtain $|Im(f)| = \aleph_0$ or $Im(f) = \{*\}$.

Proof. (1) It is sufficient to prove the following: There is no continuous self-map $f \in Con(\mathbb{Z}^*)$ such that

$$\begin{Bmatrix} f(\mathbb{Z}^*) \subset \mathbb{Z}, \text{ and} \\ | f(\mathbb{Z}^*) | = \aleph_0. \end{Bmatrix} \tag{4.1}$$

For the sake of a contradiction, suppose a continuous self-map f of $\mathbb{Z}^* := (\mathbb{Z}^*, \kappa^*)$ satisfying the property (4.1). Namely, since $Im(f) \subset \mathbb{Z}$, consider a point $p \in \mathbb{Z}$ such that $f(*) = p \in \mathbb{Z}$. For convenience, put

$f(\mathbb{Z}^*) := X \subset \mathbb{Z}$. Then, there is a finite open set $O(p) := SN_K(p) \cap X$ in $\kappa \subset \kappa^*$ such that (see the just above of Remark 2.1)

$$\begin{cases} * \in f^{-1}(O(p)) \text{ and} \\ |\mathbb{Z}^* \setminus f^{-1}(O(p))| \lneq \aleph_0. \end{cases}$$

Hence there is an infinite open set containing the point $*$, denoted $O(*) \in \kappa^*$, such that

$$O(*) = f^{-1}(O(p)).$$

Thus, we have the remaining set $\mathbb{Z}^* \setminus O(*)$ which is finite so that $f(\mathbb{Z}^* \setminus O(*))$ should be finite. Hence

$$Im(f) = f(\mathbb{Z}^* \setminus O(*)) \cup f(O(*))$$

should be finite, which leads to a contradiction to the hypothesis of (4.1).

(2) (Case 2-1) In the case $* \in Im(f)$ and $Im(f) \neq \{*\}$, let us prove $|Im(f)| = \aleph_0$. It suffices to prove that there is no continuous self-map f of $(\mathbb{Z}^*, \kappa^*)$ such that

$$\begin{Bmatrix} 2 \leq |f(\mathbb{Z}^*)| \lneq \aleph_0, \text{ and} \\ * \in f(\mathbb{Z}^*). \end{Bmatrix} \tag{4.2}$$

For the sake of a contradiction, suppose a continuous self-map f of the infinite K-sphere satisfying the property (4.2). Due to the hypothesis of (4.2), the complement of $f(\mathbb{Z}^*)$ in $\mathbb{Z}^*$ is denumerable.
Due to the hypothesis of (4.2), since $f(\mathbb{Z}^*)(\subset \mathbb{Z}^*)$ is finite, we obtain the two open sets

$$A(\text{finite}) \in \kappa \subset \kappa^* \text{ and } B(\text{infinite}) \in \kappa^* \tag{4.3}$$

such that

$$f(\mathbb{Z}^*) \setminus \{*\} \subset A, \ * \in B, \text{ and } A \subset B^c, \tag{4.4}$$

where $A \cup B$ need not be the set $\mathbb{Z}^*$. Thus, we obtain a separation set

$$\{A \cap f(\mathbb{Z}^*), B \cap f(\mathbb{Z}^*)\} \tag{4.5}$$

for disconnectedness of the subspace $(f(\mathbb{Z}^*), (\kappa^*)_{f(\mathbb{Z}^*)})$, which invokes a contradiction to the connectedness of $(f(\mathbb{Z}^*), (\kappa^*)_{f(\mathbb{Z}^*)})$ as stated in Lemma 4.1.

(Case 2-2) In the case $* \in Im(f)$ and $|Im(f)| \neq \aleph_0$, we prove that $Im(f) = \{*\}$. For the sake of a contradiction, with the hypothesis, suppose there is a finite and nonempty set $T \subset \mathbb{Z}$ such that

$$Im(f) = \{*\} \cup T.$$

Since $Im(f)$ is finite and connected (see Lemmas 4.1 and 4.2(1)), using methods similar to those of (4.3), (4.4), and (4.5), we have a contradiction to the connectedness of $(f(\mathbb{Z}^*), (\kappa^*)_{f(\mathbb{Z}^*)})$ as stated in Lemma 4.1. Thus, with the hypothesis we conclude that $Im(f) = \{*\}$. □

Using this result, we now prove the following:

Theorem 4.3. *The infinite K-circle has the FPP in the set Con($\mathbb{Z}^*$).*

Proof. To prove this assertion, for any $f \in Con(\mathbb{Z}^*)$ we follow the two cases.
(Case 1) In the case $f(*) = *$, the proof is completed.
(Case 2) In the case $f(*) \neq *$, we prove that f has some point $x \in \mathbb{Z}^*$ such that $f(x) = x$. To prove this assertion, we need the following two steps.
(Step 1) Since $f(*) \neq *$, for some point $p \in \mathbb{Z}$ we may assume $f(*) = p$. Then, by Lemma 4.2(1), $Im(f)$ is a

finite subset of $\mathbb{Z}$.
(Step 2) Owing to the property from (Step 1), since $Im(f)$ is a finite, by Lemma 4.1, we obtain that $(Im(f), \kappa_{Im(f)})$ is a finite and connected subspace in $(\mathbb{Z}, \kappa)$. Hence $(Im(f), \kappa_{Im(f)})$ should be a finite (simple) K-path on $(\mathbb{Z}, \kappa)$ which is a subspace of $(\mathbb{Z}, \kappa)$. Since any finite K-path in $(\mathbb{Z}, \kappa)$ has the *FPP* [8] and further, the *FPP* is the K-topological property [8], $(Im(f), \kappa_{Im(f)}) = (Im(f), \kappa^*_{Im(f)})$ has the *FPP* with respect to the map f. By Lemma 3.8 and Proposition 3.9, we conclude that the given map f has a point $x \in \mathbb{Z} \subset \mathbb{Z}^*$ such that $f(x) = x$. □

Example 4.4. As shown in Fig.1, let us consider the map $f \in Con(\mathbb{Z}^*)$ defined by

$$\begin{cases} f(\mathbb{Z}^* \setminus [-3, 1]_{\mathbb{Z}}) = \{0\}, \\ f(1) = -1, f(0) = -2, f(-1) = -3, f(-2) = -2, f(-3) = -1. \end{cases}$$

Then, f is a continuous map such that $f(-2) = -2$, which implies that the element -2 is a fixed point of f.

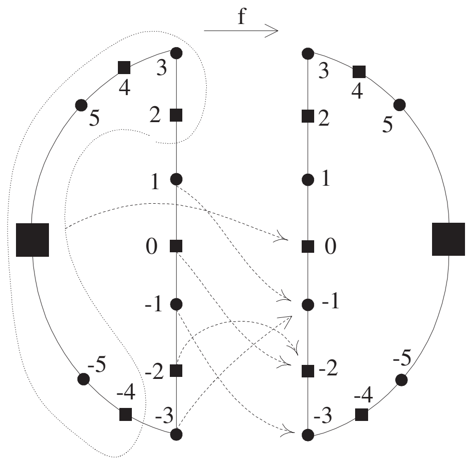

Figure 1: Configuration of $f \in Con(\mathbb{Z}^*)$ defined in Example 4.4. In this figure we observe that $f(\mathbb{Z}^*) = [-3, 0]_{\mathbb{Z}} \subset \mathbb{Z}$ and $f(*) = 0$. Then, it is clear that the point -2 has the property $f(-2) = -2$ (see Steps (1) and (2) of Theorem 4.3). In this figure we read each of the two jumbo black squares as $*$.

Owing to Theorems 3.6 and 4.3 and Corollary 3.7, we obtain the following results:

Corollary 4.5. *The infinite K-circle has the FPP in the set of self-homeomorphisms of the infinite K-circle.*

Corollary 4.6. *The infinite K-circle has the FPP in the set of continuous self-surjections of the infinite K-circle.*

Example 4.7. With $(\mathbb{Z}^*, \kappa^*)$, consider the self-map g of $\mathbb{Z}^*$ satisfying the properties

$$\begin{cases} g(x) = x, x \leq 0, x \neq *, \\ g(x) = -x, x \gneq 0, x \neq *, \text{ and} \\ g(*) = *. \end{cases}$$

Then, the map g satisfies the condition of Corollary 4.6, and further, the map g has some fixed points.

5. Fixed Point Property of the Infinite K-Sphere in the Set $Con^{\bigstar}((\mathbb{Z}^2)^*)$

First of all, we may ask if every continuous self-map f of the spaces $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ has some point $x \in (\mathbb{Z}^2)^*$ such that $f(x) = x$. The recent paper [6] proved that in $(\mathbb{Z}^n, \kappa^n), n \in \mathbb{N}$, each of the smallest open neighborhood of a point $p \in \mathbb{Z}^n$ has the *FPP* in the set of continuous self-maps of the space. In general, we say that a topological space X has the *FPP* if every continuous self-map f of X has a point $x \in X$ such that $f(x) = x$. With the perspective, it is clear that the K-topological plane $(\mathbb{Z}^2, \kappa^2)$ does not have the *FPP*.

This section now addresses the query (∗3) previously posed in Section 1. After firstly referring to an unsolved problem in Remark 5.1 below, we secondly explain some reason why we need a certain subset of the set of all continuous self-maps of the infinite K-sphere which supports the *FPP* in the subset (see Definition 1.1 and Theorem 5.7). As referred to in Section 1, this section mainly focuses on exploring a certain set in which the *FPP* of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ holds. It is obvious that the Hausdorff compactification of the 2-dimensional Euclidean topological space $(\mathbb{R}^2, \mathbf{U})$ is homeomorphic to the topological space $(S^2, \mathbf{U}_{S^2})$ as a subspace of the 3-dimensional Euclidean topological space $(\mathbb{R}^3, \mathbf{U})$, where $S^2 := \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Since the *FPP* is a topological property, we conclude that $((\mathbb{R}^2)^*, \mathbf{U}^*)$ do not have the *FPP*. For instance, some rotations of $((\mathbb{R}^2)^*, \mathbf{U}^*)$ does not have any fixed point in $(\mathbb{R}^2)^*$. Unlike this feature, this section proves that every map $f \in Con^{\bigstar}((\mathbb{Z}^2)^*)$ has a point $x \in (\mathbb{Z}^2)^*$ such that $f(x) = x$ (see Theorem 5.7). Let us now address the query (∗3) in Section 1, let us recall the following property.

Remark 5.1. Given a map $f \in Con((\mathbb{Z}^2)^*)$, in the case $Im(f) \subset \mathbb{Z}^2$, up to now there is still the following unsolved problem. Naively,

$$\text{does } (Im(f), (\kappa^2)^*_{Im(f)}) \text{ have a point } y \in Im(f) \text{ such that } f|_{Im(f)}(y) = y\,? \tag{5.1}$$

As referred to in Lemma 3.8 and Proposition 3.9, in the case the restriction map f to $Im(f)$, denoted by $f|_{Im(f)}$, has a point $y \in Im(f)$ such that $f|_{Im(f)}(y) = y$, it is clear that the given map f has a fixed point. For instance, consider a point $p \in \mathbb{Z}^2$ with the 2-dimensional K-topological structure. Let $SN_K(p)$ be the smallest open set of p according to the point p. Then $SN_K(p)$ has the *FPP* [6]. Thus, in the case a map $f \in Con((\mathbb{Z}^2)^*)$ has the property $Im(f) = SN_K(p) \subset \mathbb{Z}^2$, the given map f has the property $f(x) = x$, where $x \in (\mathbb{Z}^2)^*$ (see Example 2 with Fig.2).
Meanwhile, regarding (5.1), unlike the case above, in the case the restriction map $f|_{Im(f)}$ does not have any point $y \in Im(f)$ such that $f|_{Im(f)}(y) = y$, we have some difficulty in dealing with the *FPP* of $(\mathbb{Z}^2)^*$. Hence we need to establish the set $Con^{\bigstar}((\mathbb{Z}^2)^*)$ in Definition 1.1.

Example 5.2. Take the point $p := 0_2 := (0, 0)$ and further, $SN_K(p)$ (see (2.1)). Let us consider the self-map f of $(\mathbb{Z}^2)^*$ (see Fig.2) defined by

$$\left\{\begin{array}{l} f((\mathbb{Z}^2)^* \setminus SN_K(p)) = \{p\}, \text{ and} \\ f((p_1, p_2)) = (-p_1, p_2) \text{ for any } (p_1, p_2) \in SN_K(0_2). \end{array}\right\} \tag{5.2}$$

Then, we obtain $f \in Con((\mathbb{Z}^2)^*)$ because each $(\mathbb{Z}^2)^* \setminus SN_K(p)$ and $\{p\}$ are closed sets in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ and the other points also support the continuity of f. By Proposition 3.9, we obtain the point $p \in \{(0, \pm 1), 0_2\}$ such that $f(p) = p$.

As to the case which is different from those of Remark 5.1 and Example 5.2, we need to address the query (∗3) in Section 1 using the following lemmas.

Lemma 5.3. *For any map $f \in Con((\mathbb{Z}^2)^*)$, $Im(f)$ is connected.*

Proof. Owing to Proposition 2.2 and the continuity of f, the proof is completed. □

Lemma 5.4. *With $((\mathbb{Z}^2)^*, (\kappa^2)^*)$, the point ∗ does not have an open set $U(\ni *)$ that is homeomorphic to $SN_K(p)$, where $p \in \mathbb{Z}^2$.*

Proof. Suppose that there is an open set $U(\ni *)$ in $(\kappa^2)^*$ which is homeomorphic to $SN_K(p)(\in \kappa^2 \subset (\kappa^2)^*)$, where $p \in \mathbb{Z}^2$. Let us now consider the following cases depending on the point $p \in \mathbb{Z}^2 = (\mathbb{Z}^2)_e \cup (\mathbb{Z}^2)_m \cup (\mathbb{Z}^2)_o$.

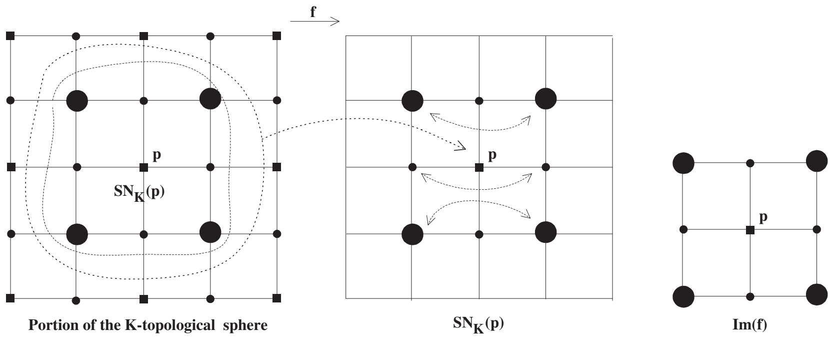

Figure 2: Configuration of $f \in Con^{\bigstar}((\mathbb{Z}^2)^*)$ defined in Example 5.2. We observe that the map f have certain fixed points.

(Case 1) Suppose there are an open set $U(\ni *)$ and a point $p := (p_1, p_2) \in (\mathbb{Z}^2)_e$ such that the open set U is homeomorphic to $SN_K(p)$ in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Then, the set $(\mathbb{Z}^2)^* \setminus U = \mathbb{Z}^2 \setminus U$ should be closed and compact in $(\mathbb{Z}^2, \kappa^2)$. Meanwhile, by the hypothesis, the space $SN_K(p)$ is equal to a certain space $\{(p_1 \pm 1, p_2), (p_1, p_2 \pm 1), p, (p_1 \pm 1, p_2 \pm 1)\}(\subset \mathbb{Z}^2)$ as a subspace of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Hence the set $(\mathbb{Z}^2)^* \setminus U = \mathbb{Z}^2 \setminus U$ should be denumerable so that $(\mathbb{Z}^2)^* \setminus U$ it is not compact in $(\mathbb{Z}^2, \kappa^2)$ (see Remark 2.1). Thus, $U(\ni *)$ cannot be an open set in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ which is homeomorphic to $SN_K(p)$ for $p \in (\mathbb{Z}^2)_e$, which contradicts the hypothesis.

(Case 2) Suppose there are an open set $U(\ni *)$ and a point $p \in (\mathbb{Z}^2)_m$ such that the open set $U(\ni *)$ is homeomorphic to $SN_K(p)$ in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Then, the set $(\mathbb{Z}^2)^* \setminus U = \mathbb{Z}^2 \setminus U$ should be closed and compact in $(\mathbb{Z}^2, \kappa^2)$. Depending on the point

$$p := (p_1, p_2) \in \{(2m, 2n+1), (2m+1, 2n) \,|\, m, n \in \mathbb{Z}\} := (\mathbb{Z}^2)_m, \tag{5.3}$$

$SN_K(p)$ is determined according to the point $p \in (\mathbb{Z}^2)_m$.
In the case $p := (p_1, p_2) = (2m, 2n+1)$ (*resp.* $p := (p_1, p_2) = (2m+1, 2n)$), the space $SN_K(p)$ is equal to the certain space $\{(2m-1, 2n+1), (2m, 2n+1), (2m+1, 2n+1)\}$ (*resp.* $\{(2m+1, 2n-1), (2m+1, 2n), (2m+1, 2n+1)\}$) as a subspace of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Thus, the set $(\mathbb{Z}^2)^* \setminus U = \mathbb{Z}^2 \setminus U$ is denumerable so that $(\mathbb{Z}^2)^* \setminus U$ it is not compact in $(\mathbb{Z}^2, \kappa^2)$ (see Remark 2.1)). Hence $U(\ni *)$ is not an open set in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ which is homeomorphic to $SN_K(p)$, which contradicts the hypothesis.

(Case 3) For the point $p \in (\mathbb{Z}^2)_o$, it is obvious that any open set $U(\ni *)$ is not homeomorphic to $SN_K(p)$ in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Indeed, the singleton $\{p\}$ is an open set in $\kappa^2 \subset (\kappa^2)^*$. Meanwhile, the singleton $\{*\}$ is not open but closed in $(\kappa^2)^*$. Hence it is clear that the point $*$ does not have an open set $U(\ni *)$ in $(\kappa^2)^*$ which is homeomorphic to $SN_K(p) = \{p\}$, where $p \in (\mathbb{Z}^2)_o$.
According to these three cases, the proof is completed. □

Owing to Proposition 2.2 and the definition of the infinite K-sphere, we have the following:

Lemma 5.5. *There is no $f \in Con((\mathbb{Z}^2)^*)$ such that*

$$\left\{\begin{array}{l} 2 \leq |\, f((\mathbb{Z}^2)^*)\,| \lneq \aleph_0, \text{ and} \\ * \in f((\mathbb{Z}^2)^*). \end{array}\right\} \tag{5.4}$$

Proof. For the sake of a contradiction, suppose a map $f \in Con((\mathbb{Z}^2)^*)$ satisfying the property (5.4). Then, we have at least the following properties:

$$\left\{\begin{array}{l} (a)\ f((\mathbb{Z}^2)^*) \text{ is connected (see Lemma 5.3) and} \\ (b)\ f((\mathbb{Z}^2)^*) \text{ is not an open set in } ((\mathbb{Z}^2)^*, (\kappa^2)^*). \end{array}\right\} \tag{5.5}$$

To be precise, with the hypothesis, while $* \in f((\mathbb{Z}^2)^*)$, it is clear there is no open set containing the point $*$, denoted by $O(*)$, in $(\kappa^2)^*$ such that

$$O(*) \subset f((\mathbb{Z}^2)^*),$$

because $O(*)$ should be denumerable. Hence $f((\mathbb{Z}^2)^*)$ is not an open set in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$, which implies the property of (b) in (5.5). Owing to Proposition 2.2 and the continuity of f, we clearly obtain the property (a). Due to the hypothesis of (5.4), the complement of $f((\mathbb{Z}^2)^*)$ in $(\mathbb{Z}^2)^*$ is denumerable. Furthermore, since $f((\mathbb{Z}^2)^*)$ is finite, we obtain the two open sets

$$A(\text{finite}) \in \kappa^2 \subset (\kappa^2)^* \text{ and } B(\text{infinite}) \in (\kappa^2)^*$$

such that

$$f((\mathbb{Z}^2)^*) \setminus \{*\} \subset A,\ * \in B,\ \text{and } A \subset B^c,$$

where $A \cup B$ need not be the set $(\mathbb{Z}^2)^*$. Thus, we obtain a separation set

$$\{A \cap f((\mathbb{Z}^2)^*), B \cap f((\mathbb{Z}^2)^*)\}$$

for disconnectedness of the subspace $(f((\mathbb{Z}^2)^*), (\kappa^2)_{f((\mathbb{Z}^2)^*)})$, which invokes a contradiction to the property (a) of (5.5). □

According to Lemma 5.5, it turns out that the map f satisfying the hypothesis of (5.4) is not a continuous self-map of the infinite K-sphere. Hence we need to consider the following:

Lemma 5.6. *There is no map $f \in Con((\mathbb{Z}^2)^*)$ such that*

$$\left\{\begin{array}{l} f((\mathbb{Z}^2)^*) \subset \mathbb{Z}^2, \text{ and} \\ |\, f((\mathbb{Z}^2)^*)\,| = \aleph_0. \end{array}\right\} \tag{5.6}$$

Proof. Suppose a continuous self-map satisfying the property (5.6). To be specific, consider a point $p \in \mathbb{Z}^2$ such that $f(*) = p \in \mathbb{Z}^2$. For convenience, put $f((\mathbb{Z}^2)^*) := X \subset \mathbb{Z}^2$. Then, there is an open set $SN_X(p) := SN_K(p) \cap X$ (see the just above of Remark 2.1) such that

$$\begin{cases} * \in f^{-1}(SN_X(p)) \text{ and} \\ |\,(\mathbb{Z}^2)^* \setminus f^{-1}(SN_X(p))\,| \lneq \aleph_0. \end{cases}$$

Meanwhile, we have $|\,X \setminus SN_X(p)\,| = \aleph_0$. Thus, the map f should map the finite set $(\mathbb{Z}^2)^* \setminus f^{-1}(SN_X(p))$ onto the infinite set $X \setminus SN_X(p)$, which invokes a contradiction of being a map of f. □

In view of Lemmas 5.5 and 5.6, and the map satisfying the property (5.6), we need to prove the following:

Theorem 5.7. *The infinite K-sphere has the FPP in the set $Con^{\bigstar}((\mathbb{Z}^2)^*)$.*

Proof. (Case 1) In the case $f \in Con^{\bigstar}((\mathbb{Z}^2)^*)$ is a constant map, the proof is completed.

(Case 2) Consider a map $f \in Con^{\bigstar}((\mathbb{Z}^2)^*)$ such that $* \in Im(f)$ and $|Im(f)| = \aleph_0$ (see the condition (a) of Definition 1.1(3)) and f is not a constant map. Then, we prove that there is a point $x \in (\mathbb{Z}^2)^*$ such that $f(x) = x$. With the hypothesis, without loss of generality, we may consider the following cases.
(Case 2-1) In the case $f(*) = *$, which completes the proof.
(Case 2-2) Let us now suppose that given a map f has the property $f(*) \neq *$. Namely, assume the case $f(*) = p \in \mathbb{Z}^2$ so that we may consider the following three cases.
(Case 2-2-1) Let us assume that the point p is a pure closed point, *i.e.* $p \in (\mathbb{Z}^2)_e$. Then, the singleton $\{p\}$ is closed and compact in $(\mathbb{Z}^2, \kappa^2)$. Thus, it is also closed in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Hereafter, for convenience, with the hypothesis, put $f((\mathbb{Z}^2)^*) = X$ which is denumerable. Then, the point p has $SN_X(p) := SN_K(p) \cap X$ as

a subset of $\{(p_1 \pm 1, p_2), (p_1, p_2 \pm 1), p, (p_1 \pm 1, p_2 \pm 1)\}(\subset \mathbb{Z}^2)$ (see the property (2.1)). Owing to the map f, the set $f^{-1}(SN_X(p))(\ni *)$ is a denumerable open set in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Hence $(\mathbb{Z}^2)^* \setminus f^{-1}(SN_X(p))$ should be a finite set in $(\mathbb{Z}^2, \kappa^2)$. Thus, the remaining finite set $(\mathbb{Z}^2)^* \setminus f^{-1}(SN_X(p))$ should be mapped by the map f onto $X \setminus SN_X(p)$. Since the set $X \setminus SN_X(p)$ is denumerable, we have a contradiction of being a map of f. Namely, the existence of $SN_X(p)$ such that $f(*) = p \in \mathbb{Z}^2$ invokes a contradiction of being a map of f.

(Case 2-2-2) Let us assume that the point p is a mixed point. Then, the singleton $\{p\}$ is also closed-open or open-closed point in $(\mathbb{Z}^2, \kappa^2)$. Thus, according to the point p, in the case p is closed-open, we obtain $SN_K(p) = \{(p_1-1, p_2), p, (p_1+1, p_2)\}$ and the case p is open-closed, we obtain $SN_K(p) = \{(p_1, p_2-1), p, (p_1, p_2+1)\}$ (see the property (2.1)). Then, after putting $f((\mathbb{Z}^2)^*) = X$ which is denumerable, the point p has $SN_X(p) := SN_K(p) \cap X$. Owing to the map f with the hypothesis, the set $f^{-1}(SN_X(p))(\ni *)$ is a denumerable open set in $((\mathbb{Z}^2)^*, (\kappa^2)^*)$. Hence $(\mathbb{Z}^2)^* \setminus f^{-1}(SN_X(p))$ should be a finite set in $(\mathbb{Z}^2, \kappa^2)$. Thus, the remaining finite set $(\mathbb{Z}^2)^* \setminus f^{-1}(SN_X(p))$ should be mapped by the map f onto $X \setminus SN_X(p)$ which is denumerable. Since the set $X \setminus SN_X(p)$ is infinite, we have a contradiction of being a map of f. Namely, the existence of $SN_X(p)$ such that $f(*) = p \in \mathbb{Z}^2$ invokes a contradiction of being a map of f.

(Case 2-2-3) Let us assume that the point p is a pure open point. Then, the singleton $\{p\} = SN_K(p)$ is open in $(\mathbb{Z}^2, \kappa^2)$. Using certain methods similar to the proofs of Cases (2-2-1) and (2-2-2), we have a contradiction. In view of these cases, we conclude that every map f in $Con^{\bigstar}((\mathbb{Z}^2)^*)$ such that $* \in Im(f)$ and $|Im(f)| = \aleph_0$ has the property $f(*) = *$, which implies that $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ has the *FPP* in the set $Con^{\bigstar}((\mathbb{Z}^2)^*)$.

(Case 3) In the case $(Im(f), (\kappa^2)^*_{Im(f)})$ has a point $y \in Im(f)$ such that $f|_{Im(f)}(y) = y$, by Lemma 3.8 and Proposition 3.9, the proof is completed. □

Example 5.8. With $((\mathbb{Z}^2)^*, (\kappa^2)^*)$, consider the self-map f of $(\mathbb{Z}^2)^*$ satisfying the properties

$$\begin{cases} f((x, y) = (x, y), x \leq 0, (x, y) \neq *, \\ f((x, y) = (-x, y), x \gtrsim 0, (x, y) \neq *, \text{ and} \\ f(*) = *. \end{cases}$$

Then, we observe that the map f is continuous so that $f((\mathbb{Z}^2)^*)$ is a connected subset of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ and $* \in f((\mathbb{Z}^2)^*)$. Thus, the map f has some fixed points such as $*, (x, y) \in \mathbb{Z}^2$, where $x \leq 0$.

In view of (Case 3) of Theorem 5.7, we observe the following:

Remark 5.9. Not every $f \in Con((\mathbb{Z}^2)^*)$ has the property $f(*) = *$ (see Fig.2).

Remark 5.10. In view of Remarks 5.1 and 5.9, and Lemmas 5.5 and 5.6, the condition $Con^{\bigstar}((\mathbb{Z}^2)^*)$ of Theorem 5.7 cannot be omitted.

Using methods similar to those of Theorem 5.7, we obtain the following results:

Corollary 5.11. *The infinite K-sphere has the FPP in the set of self-homeomorphisms of the infinite K-sphere.*

Corollary 5.12. *The infinite K-sphere has the FPP in the set of continuous self-surjections of the infinite K-sphere.*

In view of Theorem 5.7 and Corollaries 5.11 and 5.12, we obtain the following:

Remark 5.13. (1) Unlike the non-fixed point property of the Hausdorff compactification of the one dimensional Euclidean topological space, the infinite K-sphere (*the infinite K-circle*) has the *FPP* in the set of continuous self-bijections of the infinite K-sphere (*the infinite K-circle*), which implies that both the infinite K-sphere and the infinite K-circle have their own features from the viewpoint of fixed point theory.
(2) A self-homeomorphism and a continuous self-surjection of of the infinite K-sphere satisfy the condition of Definition 1.1(3)(a) so that they belong to the set $Con^{\bigstar}((\mathbb{Z}^2)^*)$.

6. Comparison Between the *FPP* of the Infinite *K*-Sphere and that of Infinite *M*-Sphere

To study the Marcus-Wyse (*M*-, for brevity) topological structure on $\mathbb{Z}^2$ [19], the notion of a digital k-neighborhood of a point $p \in \mathbb{Z}^2$ has been used. Naively, for a point $p := (x, y) \in \mathbb{Z}^2$ we follow the notation [22].

$$N_4^*(p) := \{(x \pm 1, y), p, (x, y \pm 1)\} \subset \mathbb{Z}^2,$$

Then, we use the set $U(p)$ for generating the M-topology on $\mathbb{Z}^2$, where

$$U(p) := \begin{cases} N_4^*(p) \text{ if } x + y \text{ is even, and} \\ \{p\} : \text{ otherwise.} \end{cases} \tag{6.1}$$

The M-topology "γ" on $\mathbb{Z}^2$ is derived from the set $\{U(p) \,|\, p = (x, y) \in \mathbb{Z}^2\}$ in (6.1) as a base [19]. Then, we use the notation $(\mathbb{Z}^2, \gamma)$ for the topological space. To further state a point in $\mathbb{Z}^2$, we call a point $p = (x_1, x_2)$ *double even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number [19].

In a subspace of $(\mathbb{Z}^2, \gamma)$, an *odd point* (*resp.* a *double even point or even point*) is denoted by a black large dot (*resp.* the symbol $\diamond$). According to (6.1), under $(\mathbb{Z}^2, \gamma)$, it appears that the singleton being composed of "$\diamond$" is a closed set and the singleton consisting of a black large dots is an open set. Besides, we will denote by $(\mathbb{Z}^2)_o$ (*resp.* $(\mathbb{Z}^2)_e$) the set of all odd points (*resp.* double even or even points) in $(\mathbb{Z}^2, \gamma)$. In addition, for a set $X \subset \mathbb{Z}^2$, we denote by (X, γ_X), called an M-topological space, the subspace induced by $(\mathbb{Z}^2, \gamma)$. Owing to (6.1), it is obvious that $(\mathbb{Z}^2, \gamma)$ is an Alexandroff space.

Under $(\mathbb{Z}^2, \gamma)$, the *smallest (open) neighborhood* of the point $p := (p_1, p_2)$ of $\mathbb{Z}^2$, denoted by $SN_M(p) \subset \mathbb{Z}^2$, is determined according to the given point p, as follows:

$$SN_M(p) := \begin{cases} \{p\} \text{ if } p \in (\mathbb{Z}^2)_o, \\ N_4^*(p) \text{ if } p \in (\mathbb{Z}^2)_e. \end{cases} \tag{6.2}$$

From now on, for a point p in (X, γ_X), we follow the notation $SN_X(p) := SN_M(p) \cap X$[4]. For two spaces $X := (X, \gamma_X)$ and $Y := (Y, \gamma_Y)$, a map $g : X \to Y$ is said to be *M-continuous* at a point $x \in X$ if g is continuous at the point x from the viewpoint of M-topology. Furthermore, we say that a map $g : X \to Y$ is M-continuous if it is M-continuous at every point $x \in X$. Indeed, since $(\mathbb{Z}^2, \gamma)$ is an Alexandroff space (see the property (6.1)), we can represent the M-continuity of g at a point $x \in X$ [4], as follows:

$$g(SN_M(x)) \subset SN_M(g(x)). \tag{6.3}$$

Since $(\mathbb{Z}^2, \gamma)$ is also a locally compact space and is neither a compact nor a Hausdorff space, we can establish the Alexandroff one point compactification denoted by $((\mathbb{Z}^2)^*, \gamma^*)$.

In addition, we recall the notion of *M-homeomorphism* as follows [4]: For two spaces $(X,_X)$ and (Y, γ_Y), a map $h : X \to Y$ is called an M-homeomorphism if h is an M-continuous bijection and further, $h^{-1} : Y \to X$ is M-continuous. Besides, we say that two distinct points $x, y \in \mathbb{Z}^2$ is M-adjacent if $x \in SN_M(y)$ or $y \in SN_M(x)$ [6]. Using this notions, the following notions are defined [4].

(•) Consider two distinct points $x, y \in (X, \gamma_X) := X$ if there is the sequence (or a path) $(x_0, x_1, \cdots, x_l)$ on X with $\{x_0 = x, x_1, \cdots, x_l = y\}$ such that x_i and x_{i+1} are M-adjacent, $i \in [0, l-1]_{\mathbb{Z}}, l \geq 1$, then we say that the sequence is an M-path connecting the two given points x and y.

Besides, for any two points $x, y \in X$, there is an M-path connecting the two points, then X is called M-path connected (or connected).

(•) A *simple M-path* in X means the M-path $(x_i)_{i \in [0,l]_{\mathbb{Z}}}$ in X such that x_i and x_j are M-adjacent if and only if $|i - j| = 1$.

(•) A *M-topological invariant* [4] is said to be a property of an M-topological space that is invariant under M-homeomorphisms. In other words, we often call that property is an M-topological property [4].

Definition 6.1. ([11]) (1) Let us denote by $Con((\mathbb{Z}^2)^*, \gamma^*)$ the set of all continuous self-maps g of $((\mathbb{Z}^2)^*, \gamma^*)$.
(2) We denote by $Mop(\gamma^*)$ the set of all continuous self-maps g of $((\mathbb{Z}^2)^*, \gamma^*)$ such that $|g((\mathbb{Z}^2)^*)| = \aleph_0$ with $* \in g((\mathbb{Z}^2)^*)$ or $g((\mathbb{Z}^2)^*)$ is a singleton.

The paper [11] proved the following (see Theorem 3 of [11]).

Theorem 6.2. ([11]) $((\mathbb{Z}^2)^*, \gamma^*)$ *has the FPP in* $Mop(\gamma^*)$.

Regarding this fact, the paper [11] asserted the following (see Theorem 2 of [11]).

Theorem 6.3. ([11]) $((\mathbb{Z}^2)^*, \gamma^*)$ *does not have the FPP in* $Con((\mathbb{Z}^2)^*, \gamma^*)$.

Regarding this theorem, this assertion is correct. However, the paper [11] used a misprinted counterexample when proving this theorem (indeed, the map g in (9) and Fig.1 of [11] are misprinted). Hence we now need to correct and improve them related to the proof of Theorem 2 of [11], as follows:

Proof. Using a counterexample, we prove this assertion. Let $p := (1,0), x_1 := (1,-1), x_2 := (2,0), x_3 := (1,1), x_4 := (0,0), x_5 := (0,1)$ (see the map g of Fig.3)
With $((\mathbb{Z}^2)^*, \gamma^*)$, consider the self-map g of $(\mathbb{Z}^2)^*$ defined by

$$\begin{Bmatrix} g((\mathbb{Z}^2)^* \setminus C_M(p)) = \{p\}, \text{ where } C_M(p) := \{p, x_1, x_2, x_3, x_4\}, \\ g(p) = x_5, g(x_3) = x_4, g(x_4) = x_3, g(x_1) = x_4, g(x_2) = x_3 \\ \text{(see the map } g \text{ of Fig.3).} \end{Bmatrix} \tag{6.4}$$

Then, we obtain

$$g((\mathbb{Z}^2)^*) = \{p := (1,0), x_3 := (1,1), x_4 := (0,0), x_5 := (0,1)\} \subset \mathbb{Z}^2.$$

More precisely, since the set $C_M(p)$ is the closure of the singleton $\{p\}$ in $((\mathbb{Z}^2)^*, \gamma^*)$, we have an open set $(\mathbb{Z}^2)^* \setminus C_M(p)$ containing the point $*$ and further, the singleton $\{p\}$ is an open set in $((\mathbb{Z}^2)^*, \gamma^*)$. Thus, the map g is continuous at any point $x \in (\mathbb{Z}^2)^* \setminus C_M(p)$. In addition, we observe that $g(SN_M(x_1)) \subset SN_M(x_4)$ and $g(SN_M(x_2)) \subset SN_M(x_3)$. While the map g is continuous, $(\mathbb{Z}^2)^*$ does not have any point such that $g(x) = x$, where $x \in (\mathbb{Z}^2)^*$. $\square$

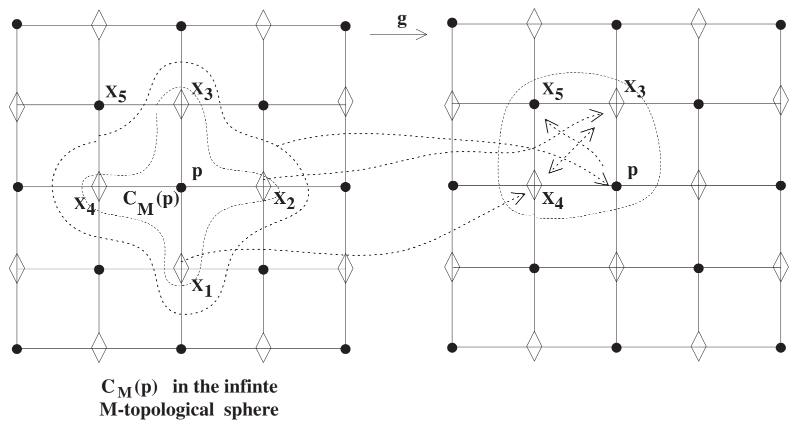

Figure 3: Configuration of $g \in Con((\mathbb{Z}^2)^*, \gamma^*)$ suggested in (6.4) of which g does not have any fixed point.

Motivated by Definition 1.1, to generalize Theorem 6.2, we establish the following:

Definition 6.4. Let $Con^\bigstar((\mathbb{Z}^2)^*, \gamma^*)$ be the set of all maps $f \in Con((\mathbb{Z}^2)^*, \gamma^*)$ such that

$$\begin{cases} (a) |\,Im(f)\,| = \aleph_0 \text{ with } * \in Im(f) \text{ or} \\ (b) \text{ constant maps on the infinite } M\text{-sphere or} \\ (c)\ (Im(f), \gamma^*_{Im(f)}) \text{ has a point } y \in Im(f) \\ \text{such that } f|_{Im(f)}(y) = y, \text{ where } Im(f) \subset \mathbb{Z}^2. \end{cases}$$

Comparing $Mop(\gamma^*)$ and $Con^\bigstar((\mathbb{Z}^2)^*, \gamma^*)$, we clearly obtain the following:

Lemma 6.5. *$Mop(\gamma^*) \subset Con^\bigstar((\mathbb{Z}^2)^*, \gamma^*)$.*

Let us now generalize Theorem 6.2, as follows:

Theorem 6.6. *$((\mathbb{Z}^2)^*, \gamma^*)$ has the FPP in $Con^\bigstar((\mathbb{Z}^2)^*, \gamma^*)$.*

Proof. By Theorem 6.2 and Lemma 6.5, for the cases of (a) and (b) of $Con^\bigstar((\mathbb{Z}^2)^*, \gamma^*)$, we prove the assertion. For the case of (c) of $Con^\bigstar((\mathbb{Z}^2)^*, \gamma^*)$, since $(Im(f), \gamma^*_{Im(f)})$ has a point $y \in Im(f)$ such that $f|_{Im(f)}(y) = y$, by Lemma 3.8 and Proposition 3.9, the proof is completed. □

Based on Remark 5.1, comparing Theorems 5.7 and 6.6, we obtain the following:

Remark 6.7. As to the case (c) of Definition 6.4, for instance, we can consider $(Im(f), \gamma_{Im(f)})$, where $Im(f)$ is a simple M-path in $(\mathbb{Z}^2, \gamma)$ because a (simple) M-path has the *FPP*.

7. Conclusions and a Further Work

We proved the *FPP* of the infinite K-circle. Besides, we also proved that the *FPP* of the infinite K-sphere holds in the set $Con^\bigstar((\mathbb{Z}^2)^*)$. In addition, it turns out that the infinite K-sphere has the *FPP* in the set of continuous self-surjections of the infinite K-sphere. Finally, we compared the *FPP* of the infinite K-sphere and that of the infinite M-sphere. Owing to this approach, we recognized some intrinsic features of $((\mathbb{Z}^2)^*, (\kappa^2)^*)$ and $(\mathbb{Z}^*, \kappa^*)$ compared to the non-*FPP* of the Hausdorff compactification of the n-dimensional Euclidean topological spaces, $n \in \{1, 2\}$.
As a further work, regarding the unsolved problem referred in (5.1), in the case $f \in Con((\mathbb{Z}^2)^*)$ such that $Im(f) \subset \mathbb{Z}^2$, we need to further investigate a certain condition that makes $(Im(f), (\kappa^2)^*_{Im(f)})$ have the *FPP*. Since the paper [9] established many types of topological structures which are different from $(\mathbb{Z}^2, \kappa^2)$, based on the obtained topological structures, we can establish the corresponding one point compactifications of them.

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