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Equivalent Conditions for Digital Covering Maps

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Abstract. It is known that every digital covering map $p : (E, \kappa) \to (B, \lambda)$ has the unique path lifting property. In this paper, we show that its inverse is true when the continuous surjective map p has no conciliator point. Also, we prove that a digital (κ, λ)−continuous surjection $p : (E, \kappa) \to (B, \lambda)$ is a digital covering map if and only if it is a local isomorphism, when all digital spaces are connected. Moreover, we find out a loop criterion for a digital covering map to be a radius n covering map.

1. Introduction and Motivation

In image processing, computer graphics and modeling topology in medical image processing algorithms, an object in the plane or 3-space is often approximated digitally by a set of pixels or voxels. Digital topology deals with topological properties of this set of pixels or voxels that correspond to topological properties of the original object. It provides theoretical foundations for important operations such as digitization, connected component labeling and counting, boundary extraction, contour filling, and thinning. Digitization is replacing an object by a discrete set of its points[16, 17].

In recent years, computing topological invariants has been of great importance in understanding the shape of an arbitrary 2-dimensional (2D) or 3-dimensional (3D) object [13]. The most powerful invariant of these objects is the fundamental group [20], which is unfortunately difficult to work with, although for 3D objects, this problem is decidable but no practical algorithm has been found yet.

The digital fundamental group of a discrete object was introduced in Digital Topology by Kong and Stout [15, 21]. Boxer [3] has shown how classical methods of Algebraic Topology may be used to construct the digital fundamental group which is a useful tool for Image Analysis. Digital covering spaces are important tools for computing fundamental groups of digital images. A digital covering space has been introduced by Han [7]. Boxer [4] has developed further the topic of the digital covering space by deriving digital analogs of classical results of Algebraic Topology concerning the existence and properties of digital universal covering spaces. Boxer and Karaca [5, 6] have classified digital covering spaces by conjugacy classes of image subgroups corresponding to a digital covering space.

Lots of researches in the digital covering theory are digitization of concepts in Topology and Algebraic Topology. In Algebraic Topology, covering maps are local isomorphism (local homeomorphism) and also satisfy all lifting problems. But the converse is not true. In fact, every local isomorphism is not necessarily a

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covering map and there are some maps in which enjoy various concepts of the lifting, but are not a covering map.

Despite the locally complicated behavior of spaces in classical topology, digital images are locally simple and this leads us to investigate conditions such that makes some concepts to be equivalent. For this, after some reminders and preliminary results about the digital topology and the digital covering map, we introduce notions "digital path lifting property", "uniqueness of digital path lifts" and "unique digital path lifting property" for a digitally continuous map and will compare them by some examples.

The digital path lifting property means that every digital path has a lifting started at a given point in the appropriate fiber. By uniqueness of digital path lifts we mean that if a digital path has a lifting at a given point, it must be unique. Eventually, a map has the unique digital path lifting property if it has both of the digital path lifting property and the uniqueness of digital path lifts. Every digital covering map has the unique digital path lifting property [7]. By proving some basic results about maps equipped with such properties, we show that every continuous surjection with the unique digital path lifting property is a covering map when it has no conciliator point and by some example will emphasis that these hypotheses are essential. This shows that the digital covering theory is not a special case of the well known concept of a graph covering projection because in covering graph theory a graph map is a covering graph if and only if it has the unique path lifting property [1].

In classical topology, a covering map is a local isomorphism, but the inverse is not necessarily true. Inspired by this, in all of researches it is claimed by a misplaced example that a digital local isomorphism is not necessarily a covering map (for example see[9]). Here, we show that for connected digital images, a digital continuous surjection is a digital covering map if and only if it is a digital local isomorphism. Also, since digital versions of some fundamental theorems in Algebraic Topology are satisfied for radius 2 local isomorphisms, this motivates us to find a loop criterion for a digital covering map to be a radius n local isomorphism.

2. Notations and Preliminaries

Let $\mathbb{Z}$ be the set of integers. Then $\mathbb{Z}^n$ is the set of lattice points in the n-dimensional Euclidean space. Let $X \subseteq \mathbb{Z}^n$ and let κ be some adjacency relation for the members of X. Then the pair (X, κ) is said to be a (binary) digital image. For a positive integer u with $1 \leq u \leq n$, an adjacency relation of a digital image in $\mathbb{Z}^n$ is defined as follows:
Two distinct points $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n)$ in $\mathbb{Z}^n$ are l_u-adjacent [18] if there are at most u distinct indices i such that $|p_i - q_i| = 1$ and for all indices j, $p_j = q_j$ if $|p_j - q_j| \neq 1$. An l_u-adjacency relation on $\mathbb{Z}^n$ can be denoted by the number of points that are l_u-adjacent to a given point $p \in \mathbb{Z}^n$. For example,

- The l_1-adjacent points of $\mathbb{Z}$ are called 2-adjacent.
- The l_1-adjacent points of $\mathbb{Z}^2$ are called 4-adjacent and the l_2-adjacent points in $\mathbb{Z}_2$ are called 8-adjacent.
- The l_1-adjacent, l_2-adjacent and l_3-adjacent points of $\mathbb{Z}^3$ are called 6-adjacent, 18-adjacent, and 26-adjacent, respectively.

More general adjacency relations are studied in [12].

Let κ be an adjacency relation defined on $\mathbb{Z}^n$. A digital image $X \subseteq \mathbb{Z}^n$ is κ-**connected** [19] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_r\}$ of points of X such that $x = x_0, y = x_r$ and x_i and x_{i+1} are κ-adjacent where $i = 0, 1, ..., r - 1$. A κ-component of a digital image X is a maximal κ-connected subset of X.

Definition 2.1. Let $X \subseteq \mathbb{Z}^n$ and $Y \subseteq \mathbb{Z}^m$ be digital images with κ-adjacency and λ-adjacency, respectively. A function $f : X \longrightarrow Y$ is said to be (κ, λ)-**continuous** ([3, 19]) if for every κ-connected subset U of X, $f(U)$ is a λ-connected subset of Y. We say that such a function is digitally continuous.

The following proposition let us to interpret the digital continuity by the adjacency relations.

Proposition 2.2. ([3, 18]) *Let (X,κ) in $\mathbb{Z}^n$ and (Y,λ) in $\mathbb{Z}^m$ be digital images. A function $f : X \longrightarrow Y$ is (κ,λ)-continuous if and only if for every κ-adjacent points $x_0, x_1 \in X$, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are λ-adjacent in Y.*

For $a, b \in \mathbb{Z}$ with $a < b$, a **digital interval** [2] is the set of the form

$$[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\}.$$

Definition 2.3. By a **digital κ-path** from x to y in digital image (X,κ), we mean a $(2,\kappa)$-continuous function $f : [0,m]_{\mathbb{Z}} \longrightarrow X$ such that $f(0) = x$ and $f(m) = y$. If $f(0) = f(m)$ then the κ-path is said to be closed, and f is called a κ-loop.

Let $f : [0,m-1]_{\mathbb{Z}} \longrightarrow X \subseteq \mathbb{Z}^n$ be a $(2,\kappa)$-continuous function such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm 1 \bmod m$. Then f is called simple κ-loop and the set $f([0,m-1]_{\mathbb{Z}})$ is a simple closed κ-curve containing m point which is denoted by $SC_\kappa^{n,m}$. If f is a constant function, it is called a trivial loop.

If $f : [0,m_1]_{\mathbb{Z}} \longrightarrow X$ and $g : [0,m_2]_{\mathbb{Z}} \longrightarrow X$ are digital κ-paths with $f(m_1) = g(0)$, then define the product [14] $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \longrightarrow X$ by

$$(f*g)(t) = \begin{cases} f(t) \;\; if \;\; t \in [0,m_1]_{\mathbb{Z}}; \\ g(t-m_1) \;\; if \;\; t \in [m_1, m_1+m_2]_{\mathbb{Z}}. \end{cases}$$

Let (E,κ) be a digital image and let $\varepsilon \in N$. The κ-neighborhood [8] of $e_0 \in E$ with radius ε is the set $N(e_0,\varepsilon) = \{e \in E | \, l_\kappa(e_0,e) \leq \varepsilon\} \cup \{e_0\}$, where $l_\kappa(e_0,e)$ is the length of the shortest κ-path in E from e_0 to e.

By the above notations, a function $f : X \longrightarrow Y$ is a (κ,λ)**-isomorphism** [4], denoted by $X \overset{(\kappa,\lambda)}{\approx} Y$, if f is a (κ,λ)-continuous bijection and further $f^{-1} : Y \longrightarrow X$ is (λ,κ)-continuous. If $n = m$ and $\kappa = \lambda$, then f is called κ-isomorphism.

Definition 2.4. ([3]) Let (X,κ) and (Y,λ) be digital images and let $f, g : X \longrightarrow Y$ be (κ,λ)-continuous functions. Suppose that there is a positive integer m and a function $F : X \times [0,m]_{\mathbb{Z}} \longrightarrow Y$ such that

- For all $x \in X$, $F(x,0) = f(x)$ and $F(x,m) = g(x)$;
- For all $x \in X$, the induced function $F_x : [0,m]_{\mathbb{Z}} \longrightarrow Y$ defined by $F_x(t) = F(x,t)$ for all $t \in [0,m]_{\mathbb{Z}}$ is $(2,\lambda)$-continuous; and
- For all $t \in [0,m]_{\mathbb{Z}}$, the induced function $F_t : X \longrightarrow Y$ defined by $F_t(x) = F(x,t)$ for all $x \in X$ is (κ,λ)-continuous.

Then F is called a **digital** (κ,λ)**-homotopy** between f and g, denoted by $f \overset{(\kappa,\lambda)}{\simeq} g$, and f and g are said to be digital (κ,λ)-homotopic in Y by F.

Digital (κ,λ)-homotopy relation is an equivalence relation among digitally continuous functions $f : (X,\kappa) \longrightarrow (Y,\lambda)$ [3].

Let f and f' be κ-loops in the pointed digital image (X,x_0). We say f' is a trivial extension of f if there are sets of κ-paths $\{f_1, f_2, ..., f_r\}$ and $\{F_1, F_2, ..., F_p\}$ in X such that

(1) $r \leq p$;

(2) $f = f_1 * f_2 * ... * f_r$;

(3) $f_0 = F_1 * F_2 * ... * F_p$;

(4) There are indices $1 \leq i_1 < i_2 < ... < i_r \leq p$ such that $F_{i_j} = f_j$, $1 \leq j \leq r$ and $i \notin \{i_1, i_2, ..., i_r\}$ implies F_i is a trivial loop[3].

Two loops f, f' with the same base point $x_0 \in X$ belong to the same loop class $[f]_X$ if they have trivial extensions that can be joined by a homotopy that keeps the endpoints fixed. Let $\pi_1^\kappa(X,x_0)$ be the set of all such classes, $[f]_X$. The operation $*$ enables us to define an operation on $\pi_1^\kappa(X,x_0)$ via

$$[f]_X.[g]_X = [f*g]_X.$$

This operation is well defined, and makes
$\pi_1^\kappa(X, x_0)$ into a group in which the identity element is the class $[\overline{x_0}]$ of the constant loop $\overline{x_0}$ and the inverse elements are given by $[f]^{-1} = [f^{-1}]$, where $f^{-1} : [0, m]_{\mathbb{Z}} \longrightarrow X$ is the loop defined by $f^{-1}(t) = f(m - t)$[3].

Definition 2.5. ([8]) For two digital spaces (X, κ) in $\mathbb{Z}^n$ and (Y, λ) in $\mathbb{Z}^m$, a (κ, λ)-continuous map $h : X \longrightarrow Y$ is called a **local** (κ, λ)**-isomorphism** if for every $x \in X$, $h|_{N_\kappa(x;1)}$ is a (κ, λ)-isomorphism onto $N_\lambda(h(x); 1)$. If $n = m$ and $\kappa = \lambda$, then the map h is called a local κ-isomorphism.

For $n \in \mathbb{N}$, the map h is called a radius n local isomorphism [8] if the restriction map $h|_{N_\kappa(x,n)} : N_\kappa(x, n) \longrightarrow N_\lambda(h(x), n)$ is a (κ, λ)-isomorphism.

Definition 2.6. ([4, 7, 8]) Let (E, κ) and (B, λ) be digital images and $p : E \longrightarrow B$ be a (κ, λ)-continuous surjection map. The map p is called a (κ, λ)**-covering map** if and only if for each $b \in B$ there exists an index set M such that

(1) $p^{-1}(N_\lambda(b, 1)) = \bigsqcup_{i \in M} N_\kappa(e_i, 1)$ with $e_i \in p^{-1}(b)$;

(2) if $i, j \in M$, $i \neq j$, then $N_\kappa(e_i, 1) \cap N_\kappa(e_j, 1) = \emptyset$;

(3) the restriction map $p|_{N_\kappa(e_i,1)} : N_\kappa(e_i, 1) \longrightarrow N_\lambda(b, 1)$ is a (κ, λ)-isomorphism for all $i \in M$.

Moreover, $(E; p; B)$ is said to be a (κ, λ)-covering and (E, κ) is called a digital (κ, λ)-covering space over (B, λ). Also, $N_\lambda(b, 1)$ is called an elementary λ-neighborhood of b.

It is notable that in the property (1) of the original definition of a digital covering map by Han [7], there was $N_\lambda(b, \varepsilon)$, for an $\varepsilon \in \mathbb{N}$ which is simplified by Boxer [4]. Also, we can replace (κ, λ)-continuous surjection with surjection because surjective map p with the properties (1) and (3) of the definition is (κ, λ)-continuous[11].

In this paper, all the digital spaces (X, κ) are assumed to be κ-connected.

3. Coverings are Derived from Unique Path Lifting

Like in Algebraic Topology, digital covering maps have also good behavior with lifting problems. In this section, at first we list some results of the other papers about digital coverings and lifting problems which are digitization of similar results in Algebraic Topology. Then by modification of the digital path lifting and the unique path lifting, we show how digital covering maps can be derived from the unique path lifting property.

Definition 3.1. ([7]) Let (E, κ), (B, λ) and (X, μ) be digital images, let $p : E \longrightarrow B$ be a (κ, λ)-covering map, and let $f : X \longrightarrow B$ be (μ, λ)-continuous. A **lifting** of f with respect to p is a (μ, κ)-continuous map $\widetilde{f} : X \longrightarrow E$ such that $p \circ \widetilde{f} = f$.

Theorem 3.2. ([7]) *Let (E, κ) be a digital image and $e_0 \in E$. Let (B, λ) be a digital image and $b_0 \in B$. Let $p : E \longrightarrow B$ be a (κ, λ)-covering map such that $p(e_0) = b_0$. Then any λ-path $\alpha : [0, m]_{\mathbb{Z}} \longrightarrow B$ beginning at b_0 has a unique lifting to a path $\widetilde{\alpha}$ in E beginning at e_0.*

Remark 3.3. Although liftings are defined for digital (κ, λ)-covering maps, but for a (κ, λ)-continuous surjection map $p : (E, \kappa) \to (B, \lambda)$ and any digital path α in B, the lifting of α means a digital path $\widetilde{\alpha}$ in E such that $p \circ \widetilde{\alpha} = \alpha$ [8].

Definition 3.4. Let $p : (E, \kappa) \to (B, \lambda)$ be a (κ, λ)-continuous surjection map. We say that

(i) p has the **digital path lifting property** if for any digital path α in B and any $e \in p^{-1}(\alpha(0))$ there is a lifting $\widetilde{\alpha}$ of α in E such that $\widetilde{\alpha}(0) = e$;

(ii) p has the **uniqueness of digital path lifts property** if any two paths $\alpha, \beta : [0, m]_{\mathbb{Z}} \longrightarrow E$ are equal if $p \circ \alpha = p \circ \beta$ and $\alpha(0) = \beta(0)$;

(iii) p has the **unique path lifting property** (u.p.l, for abbreviation) if it has both the path lifting property and the uniqueness of path lifts property.

Example 3.5. By Theorem 3.2, every digital covering map has the u.p.l. Consider $\mathbb{Z}^2$ by 8-adjacency and $\mathbb{Z}$ by 2-adjacency. Then the $(8,2)$-continuous map $pr_1 : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ defined by $pr_1((x,y)) = x$ has the digital path lifting property, but does not have the uniqueness of digital path lifts property. For this, consider $\alpha : [0,2]_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ defined by $\alpha(k) = k$, for $k = 0,1,2$. $pr_1^{-1}(0) = \{0\} \times \mathbb{Z}$ and for every $(0,j) \in pr_1^{-1}(0)$, $\widetilde{\alpha}_j : [0,2]_{\mathbb{Z}} \longrightarrow \mathbb{Z}^2$ defined by $\widetilde{\alpha}_j(k) = (k,j)$ is a 8-path. Then $\widetilde{\alpha}_j$ is the lifting of α beginning at $(0,j)$ and hence pr_1 has the digital path lifting property. Now let

$$\begin{cases} \beta, \gamma : [0,2]_{\mathbb{Z}} \longrightarrow \mathbb{Z}^2 \\ \beta(0) = (0,0),\ \beta(1) = (1,1),\ \beta(2) = (2,0) \\ \gamma(0) = (0,0),\ \gamma(1) = (1,-1),\ \gamma(2) = (2,0). \end{cases}$$

Then $pr_1 \circ \beta = pr_1 \circ \gamma$ and $\beta(0) = \gamma(0)$, but $\beta \neq \gamma$. Therefore pr_1 does not have the uniqueness of digital path lifts property. Also, pr_1 is not a digital $(8,2)$-covering because for every $e \in \mathbb{Z}^2$, $pr_1|_{N_8(e,1)}$ is not injective.

Example 3.5 shows that a digitally continuous surjection with the path lifting property is not necessarily a digital covering map. By the following example, we show that a digitally continuous surjection with the uniqueness of digital path lifts property is not necessarily a digital covering map.

Example 3.6. Consider the map $h : \mathbb{Z}^+ \longrightarrow SC_8^{2,4} =: (c_i)_{i \in [0,3]_{\mathbb{Z}}}$ given by $h(i) = c_{i \ mod \ 4}$, where $\mathbb{Z}^+ := \{k \in \mathbb{Z} | k \geq 0\}$. Let $\gamma : [0,1]_{\mathbb{Z}} \longrightarrow SC_8^{2,4}$ be defined by $\gamma(0) = c_0$ and $\gamma(1) = c_3$. Since $0 \in h^{-1}(c_0)$ and there is no lifting of γ beginning at 0, h does not have the digital path lifting property.

Now, Let $\alpha, \beta : [0,m]_{\mathbb{Z}} \longrightarrow \mathbb{Z}^+$ be two paths in which $h \circ \alpha = h \circ \beta$ and $\alpha(0) = \beta(0) = d$. We show that $\alpha = \beta$. By contrary, suppose that there is a $s \in [0,m]_{\mathbb{Z}}$ such that $\alpha(s) \neq \beta(s)$. We may assume that s is the smallest $t \in [0,m]_{\mathbb{Z}}$ such that $\alpha(t) \neq \beta(t)$. Thus we have the following:

$$\begin{cases} \alpha(s) \neq \beta(s), \\ \alpha(t) = \beta(t),\ for\ \ all\ t \in [0, s-1]_{\mathbb{Z}}, \\ h \circ \alpha(t) = h \circ \beta(t),\ for\ \ all\ t \in [0,m]_{\mathbb{Z}}. \end{cases}$$

If $k := \alpha(s-1) = \beta(s-1)$, then we have

$$\begin{cases} \alpha(s) = k+1, \\ \beta(s) = k-1. \end{cases} \quad or \quad \begin{cases} \alpha(s) = k-1, \\ \beta(s) = k+1. \end{cases}$$

But $h \circ \alpha(s) = h \circ \beta(s)$ follows that $h(k-1) = h(k+1)$ which is a contradiction because $h(j) = h(k)$ if an only if $j = k \ mod \ 4$.

In the following proposition, we give some basic properties of maps with the u.p.l which are essential in our results and make proofs more shorter and simpler. For $x, y \in (E, \kappa)$, by $x \overset{\kappa}{\leftrightarrow} y$ it means that x and y are κ-adjacent.

Proposition 3.7. *Let $p : (E,\kappa) \to (B,\lambda)$ be a (κ,λ)-continuous surjection with the u.p.l. Then:*

(i) If $e \overset{\kappa}{\leftrightarrow} e'$ then $p(e) \neq p(e')$;

(ii) If $e \overset{\kappa}{\leftrightarrow} e'$, $e \overset{\kappa}{\leftrightarrow} e''$ and $e' \neq e''$ then $p(e') \neq p(e'')$;

(iii) If $p(e) \overset{\lambda}{\leftrightarrow} p(e')$ then there is a unique element $e'' \in p^{-1}(p(e'))$ such that $e \overset{\kappa}{\leftrightarrow} e''$;

(iv) If $p(e) \overset{\lambda}{\leftrightarrow} b$ then there is a unique $e' \in p^{-1}(b)$ such that $e \overset{\kappa}{\leftrightarrow} e'$;

(v) If $b \overset{\lambda}{\leftrightarrow} b'$ then for every $e \in p^{-1}(b)$ there is a unique element $e' \in p^{-1}(b')$ such that $e \overset{\kappa}{\leftrightarrow} e'$.

Proof. (i) Let $\alpha, \beta : [0,1]_{\mathbb{Z}} \longrightarrow E$ be defined by $\alpha(0) = \beta(0) = e$, $\alpha(1) = e$ and $\beta(1) = e'$ which are κ-pathes by assumption. If $p(e) = p(e')$ then $p \circ \alpha = p \circ \beta$ while $\alpha \neq \beta$. This is a contradiction and hence $p(e) \neq p(e')$.

(ii) Let $\alpha, \beta : [0,1]_{\mathbb{Z}} \longrightarrow E$ be defined by $\alpha(0) = \beta(0) = e$, $\alpha(1) = e'$ and $\beta(1) = e''$ which are κ-pathes by assumption. If $p(e') = p(e'')$ then $p \circ \alpha = p \circ \beta$ while $\alpha \neq \beta$. Hence $p(e') \neq p(e'')$.

(iii) Define $\alpha : [0,1]_{\mathbb{Z}} \longrightarrow B$ by $\alpha(0) = p(e)$ and $\alpha(1) = p(e')$ which is a λ-path. By the path lifting property, there is a lifting $\widetilde{\alpha} : [0,1]_{\mathbb{Z}} \longrightarrow E$ beginning at e. Since $p \circ \widetilde{\alpha} = \alpha$, $\widetilde{\alpha}(1) \in p^{-1}(\alpha(1)) = p^{-1}(p(e'))$. Now it is sufficient to let $e'' = \widetilde{\alpha}(1)$ because $\widetilde{\alpha}(0) \overset{\kappa}{\leftrightarrow} \widetilde{\alpha}(1)$. Uniqueness comes from part (ii).

(iv) The proof is similar to the proof of (iii).

(v) This is also similar to (iii) because maps with the path lifting property are surjective. □

Definition 3.8. Let $p : (E,\kappa) \to (B,\lambda)$ be a (κ,λ)-continuous map and $e \in E$. We say that e is a conciliator point for p if there exist $e', e'' \in N_\kappa(e,1)$ for which $e' \overset{\kappa}{\not\leftrightarrow} e''$ and $p(e') \overset{\lambda}{\leftrightarrow} p(e'')$.

In classical topology, every covering map has the u.p.l but every map with the u.p.l is not generally a covering map [20]. In fact, the domain and the codomain in maps with the u.p.l that are not a covering map have locally complicated behaviors and this will ruin it to be a covering map. We show that the u.p.l is enough for a digital map without a conciliator point to be a digital covering map.

Theorem 3.9. *A (κ,λ)-continuous surjection $p : (E,\kappa) \to (B,\lambda)$ is a digital (κ,λ)-covering if it has the u.p.l and has no conciliator point.*

Proof. Let $b \in B$ and $e \in p^{-1}(N_\lambda(b,1))$. We show that $e \in \bigsqcup_{j\in J} N_\kappa(e_j,1)$, where $p^{-1}(b) = \{e_j\}_{j\in J}$. If $e \in p^{-1}(b)$, then e is one of the e_j's and the assertion is obvious. If $e \notin p^{-1}(b)$, then $p(e) \overset{\lambda}{\leftrightarrow} b$ and by Proposition 3.7, there is a $e_j \in p^{-1}(b)$ such that $e \overset{\kappa}{\leftrightarrow} e_j$ which implies $e \in N_\kappa(e_j,1)$, as desired.
If $e \in \bigsqcup_{j\in J} N_\kappa(e_j,1)$, then there is a $j_0 \in J$ such that $e \in N_\kappa(e_{j_0},1)$ and hence either $e = e_{j_0}$ or $e \overset{\kappa}{\leftrightarrow} e_{j_0}$ which implies that $p(e) = b$ or $p(e) \overset{\lambda}{\leftrightarrow} b$. This means $p(e) \in N_\lambda(b,1)$ and therefore $e \in p^{-1}(N_\lambda(b,1))$.

Let $x \in N_\kappa(e_i,1) \cap N_\kappa(e_j,1)$, for $i \neq j$. Then $x \overset{\kappa}{\leftrightarrow} e_i$ and $x \overset{\kappa}{\leftrightarrow} e_j$ and by Proposition 3.7, $b = p(e_i) \neq p(e_j) = b$ which is a contradiction.

For every $j \in J$, the restriction map $p|_{N_\kappa(e_j,1)} : N_\kappa(e_j,1) \to N_\lambda(b,1)$ is injective by Proposition 3.7, parts i and ii and also is surjective by the part v. For continuity of $(p|_{N_\kappa(e_j,1)})^{-1}$, let $b', b'' \in N_\lambda(b,1)$ be two λ-adjacent points. Since $p|_{N_\kappa(e_j,1)}$ is bijective, there are $e', e'' \in N_\kappa(e_j,1)$ such thet $p(e') = b'$ and $p(e'') = b''$. If $e' \overset{\kappa}{\not\leftrightarrow} e''$ then e_j is a conciliator point of p which is a contradiction. Hence $e' \overset{\kappa}{\leftrightarrow} e''$ and so $(p|_{N_\kappa(e_j,1)})^{-1}$ is continuous. □

In the following, we give an example of a continuous surjection with the u.p.l in which is not a digital covering map. This shows that the absence of conciliator points is essential.

Example 3.10. Let $E = \mathbb{Z}$, $B = \{b_0 = (0,0), b_1 = (1,0), b_2 = (0,1)\}$ and $p : E \longrightarrow B$ be defined by $p(i) = b_{i \text{ mod } 3}$. Then p is a $(2,8)$-continuous surjection that has the unique path lifting property and also p has some conciliator points, for example $-1, 1$ are not 2-adjacent but $b_2 = p(-1)$ and $b_1 = p(1)$ are 8-adjacent. The map p is not a digital $(2,8)$-covering map because p does not satisfy the condition (3) of Definition 2.6: for example, $N_2(0,1) = \{-1,0,1\}$ and so the inverse of the restriction of p to $N_2(0,1)$ is not $(8,2)$-continuous, because it maps two 8-adjacent points b_1 and b_2 in $N_8(b_0,1) = B$ to two distinct points of $N_2(0,1) = \{-1,0,1\}$ that are not 2-adjacent.

4. Coverings are Derived from Local Isomorphisms

Like what happened to covering maps and the u.p.l in classical topology, every covering map is a local isomorphism, but every local isomorphism is not necessarily a covering map. Obviously and by definitions, every digital covering map is a local isomorphism. Han [9] gave an example showing that a continuous surjective local isomorphism is not a covering map. We will find a gap in his example and will show that in the digital topology, continuous surjective local isomorphisms are digital covering maps.

Example 4.1. ([9]) Assume

$$X = \{q_0 = (x_1, y_1), q_1 = (x_1 - 1, y_1 + 1), q_2 = (x_1 - 2, y_1 + 1), q_3 = (x_1 - 3, y_1),$$

$$q_4 = (x_1 - 3, y_1 - 1), q_5 = (x_1 - 2, y_1 - 2), q_6 = (x_1 - 1, y_1 - 1)\} \subseteq (\mathbb{Z}^2, 8)$$

and

$$Y = \{v_0 = (a, b), v_1 = (a - 1, b + 1), v_2 = (a - 2, b), v_3 = (a - 1, b - 1)\} \subseteq (\mathbb{Z}^2, 8).$$

Han claimed that the map $h : X \longrightarrow Y$ with $h(q_i) = v_{i(mod\ 4)}$ is a local 8-isomorphism and is not a $(8, 8)$-covering map because the third assumption in the definition of a covering map is not satisfied in point v_0. Although his assertion about the point v_0 is true, but h is not 8-continuous and hence is not a local isomorphism. In fact q_0 and q_6 are 8-adjacent, while $h(q_0)$ and $h(q_6)$ are not same or 8-adjacent.

Theorem 4.2. *A (κ, λ)-continuous surjection map $p : (E, \kappa) \to (B, \lambda)$ is a digital (κ, λ)-covering map if and only if it is a local isomorphism.*

Proof. By Theorem 3.9 it suffices to show that every local isomorphism has the u.p.l and has no conciliator point. Let $\alpha : [0, m]_{\mathbb{Z}} \longrightarrow B$ be a digital λ-path with $b_0 = \alpha(0)$. Since p is surjective, there exists an $e_0 \in p^{-1}(b_0)$ and by assumption, $p|_{N_\kappa(e_0,1)} : N_\kappa(e_0, 1) \to N_\lambda(b_0, 1)$ is a (κ, λ)-isomorphism. Since $\alpha(0) \overset{\lambda}{\leftrightarrow} \alpha(1)$, there exists an $e_1 \in p^{-1}(\alpha(1))$ such that $e_1 \in N_\kappa(e_0, 1)$. If $\alpha(0) = \alpha(1)$, put $e_1 = e_0$. Inductively, there is an $e_i \in p^{-1}(\alpha(i))$ such that $e_i \in N_\kappa(e_{i-1}, 1)$, for any $0 < i \leq m$. Now, define $\widetilde{\alpha} : [0, m]_{\mathbb{Z}} \longrightarrow E$ by $\widetilde{\alpha}(i) = e_i$ which is a κ-path because $\widetilde{\alpha}(i) = e_i \overset{\kappa}{\leftrightarrow} e_{i-1} = \widetilde{\alpha}(i - 1)$. Since $e_i \in p^{-1}(\alpha(i))$, $\widetilde{\alpha}$ is a lifting of α.

For the uniqueness of path lifts property, consider two paths $\alpha, \beta : [0, m]_{\mathbb{Z}} \longrightarrow E$ in which $p \circ \alpha = p \circ \beta$ and $\alpha(0) = \beta(0)$. Assume that $\alpha \neq \beta$. Since $\alpha(0) = \beta(0)$, the set $\{i \in [0, m]; \alpha(i) \neq \beta(i)\}$ has minimum l. Hence $\alpha(l) \neq \beta(l)$, while $\alpha(l - 1) = \beta(l - 1)$. But the map

$$p|_{N_\kappa(\alpha(l-1),1)} : N_\kappa(\alpha(l - 1), 1) \to N_\lambda(p \circ \alpha(l - 1), 1)$$

is an isomorphism which implies that $p \circ \alpha(l) \neq p \circ \beta(l)$, for $\alpha(l), \beta(l) \in N_\kappa(\alpha(l - 1), 1)$. This contradicts $p \circ \alpha = p \circ \beta$ and therefore $\alpha = \beta$.

Now, let $e \in E$ be a conciliator point of the map p. Then, there are $e', e'' \in N_\kappa(e, 1)$ such that $e' \overset{\kappa}{\nleftrightarrow} e''$ and $p(e') \overset{\lambda}{\leftrightarrow} p(e'')$ which implies that the restricted map $p|_{N_\kappa(e,1)} : N_\kappa(e, 1) \longrightarrow N_\lambda(p(e), 1)$ is not an isomorphism. This is a contradition and hence p has no conciliator point. □

If in the definition of digital covering map, we replace the $N_\lambda(b, 1)$ by $N_\lambda(b, n)$, for $n \in \mathbb{N}$, the map is called **radius n covering map** [10] and hence it is a radius n local isomorphism. Radius n coverings, particulary radius 2 coverings are very important in the digital covering theory because some essential theorems in Algebraic Topology are satisfied in digital topology if covering maps will be radius 2 covering maps.

Theorem 4.3. ([4, 5, 8, 10]) *Let $p : (E, \kappa) \longrightarrow (B, \lambda)$ be a (κ, λ)-covering map such that $p(e_0) = b_0$. Suppose that p is a radius 2 local isomorphism. Then:*

(1) For κ-paths $\alpha, \beta : [0, m]_{\mathbb{Z}} \longrightarrow E$ starting at e_0, if there is a λ-homotopy in B from $p \circ \alpha$ to $p \circ \beta$ that holds the endpoints fixed, then $\alpha(m) = \beta(m)$, and there is a κ-homotopy in E from α to β that holds the endpoints fixed;

(2) The induced homomorphism $p_* : \pi_1^\kappa(E, e_0) \longrightarrow \pi_1^\lambda(B, b_0)$ is a monomorphism;

(3) For a given κ'-connected space X with $x_0 \in X$, any (κ', λ)-continuous map $\varphi : (X, \kappa') \longrightarrow (B, \lambda)$ with $\varphi(x_0) = b_0$ has a digital lifting $\widetilde{\varphi} : (X, \kappa') \longrightarrow (E, \kappa)$ for which $\widetilde{\varphi}(x_0) = e_0$ if and only if $\varphi_*\big(\pi_1^{\kappa'}(X, x_0)\big) \subseteq p_*\big(\pi_1^\kappa(E, e_0)\big)$.

For more results in digital covering maps based on 2-radius property, see [4, 5, 8]. In the following, we give a loop criterion for a digital covering to be a radius n covering map. But we need the next lemma.

Lemma 4.4. *Let $p : (E, \kappa) \longrightarrow (B, \lambda)$ be a (κ, λ)-covering map, $e \in E$, $e' \in N_\kappa(e, 1)$ and $e' \neq e'' \in N_\kappa(e, 2)$. Then $p(e') \neq p(e'')$.*

Proof. Let $b = p(e)$ and assume the contrary $b' := p(e') = p(e'')$. Since $p|_{N_\kappa(e,1)}$ is an isomorphism, $e'' \in N_\kappa(e, 2) - N_\kappa(e, 1)$ and hence there exists $f \in N_\kappa(e, 1)$ such that $e'' \overset{\kappa}{\leftrightarrow} f$. Since $p(e') = p(e'') \overset{\kappa}{\leftrightarrow} p(f)$ and $p|_{N_\kappa(e,1)}$ is an isomorphism and also $p(e'), p(f) \in N_\lambda(b, 1)$, we have $f \overset{\kappa}{\leftrightarrow} e'$. Now, $p|_{N_\kappa(f,1)}$ is an isomorphism, $e', e'' \in N_\kappa(f, 1)$ and $p(e') = p(e'')$ which is a contradiction. □

Theorem 4.5. *Let $p : (E, \kappa) \longrightarrow (B, \lambda)$ be a (κ, λ)-covering map. p is a radius n covering map if and only if every lifting of any simple λ-loop with length at most $2n + 1$ is a simple κ-loop.*

Proof. Let p be a radius n covering map and in the worst conditions, $\alpha : [0, 2n+1]_\mathbb{Z} \longrightarrow B$ be a simple λ-loop with length $2n + 1$. Let $b := \alpha(0)$ and assume that $e \in p^{-1}(b)$. Consider two λ-paths $\alpha_1, \alpha_2 : [0, n]_\mathbb{Z} \longrightarrow B$ defined by $\alpha_1(k) = \alpha(k)$ and $\alpha_2(k) = \alpha(2n + 1 - k)$. In fact, α_1 is $\alpha|_{[0,n]}$ and α_2 is $(\alpha|_{[n+1,2n]})^{-1}$. Since $p|_{N_\kappa(e,n)} : N_\kappa(e, n) \longrightarrow N_\lambda(b, n)$ is a (κ, λ)-isomorphism, $\widetilde{\alpha_1} := (p|_{N_\kappa(e,n)})^{-1} \circ \alpha_1$ and $\widetilde{\alpha_2} := (p|_{N_\kappa(e,n)})^{-1} \circ \alpha_2$ are liftings of α_1 and α_2, respectively and $\widetilde{\alpha_1}(0) = \widetilde{\alpha_2}(0) = e$. Also, $\widetilde{\alpha_1}(n) \overset{\kappa}{\leftrightarrow} \widetilde{\alpha_2}(n)$ because $\alpha_1(n) \overset{\kappa}{\leftrightarrow} \alpha_2(n)$ and $p|_{N_\kappa(b,n)}$ is an isomorphism. Define $\mu : [0, 2n + 1] \longrightarrow E$ by

$$\mu(i) = \begin{cases} \widetilde{\alpha_1}(i) & 0 \le i \le n, \\ \widetilde{\alpha_2}(2n + 1 - i) & n + 1 \le i \le 2n + 1. \end{cases}$$

Obviously, μ is a simple κ-loop and $p \circ \mu = \alpha$, as desired.

For the converse, we use induction to show that $p|_{N_\kappa(e,n)}$ is an isomorphism, for every $b \in B$ and any $e \in p^{-1}(b)$.

Assume that all liftings of every simple λ-loop with length 5 are simple κ-loops. We must prove that $p|_{N_\kappa(e,2)}$ is an isomorphism, for every $b \in B$ and any $e \in p^{-1}(b)$. Since $p|_{N_\kappa(e,1)}$ is an isomorphism, if $x \in N_\lambda(b, 1)$, then there is a $y \in N_\kappa(e, 1)$ such that $p(y) = x$. Assume $x \in N_\lambda(b, 2) - N_\lambda(b, 1)$. Let $\alpha : [0, 2]_\mathbb{Z} \longrightarrow N_\lambda(b, 2)$ be a λ-path from b to x. Then there is a unique lifting $\widetilde{\alpha} : [0, 2]_\mathbb{Z} \longrightarrow N_\kappa(e, 2)$ beginning at e such that $y := \widetilde{\alpha}(2) \in N_\kappa(e, 2)$ and $p(y) = x$. Hence $p|_{N_\kappa(e,2)}$ is onto.

For injectivity, assume the contrary that there are $y, y' \in N_\kappa(e, 2)$ such that $x := p(y) = p(y')$. Since $p|_{N_\kappa(e,1)}$ is an isomorphism, $y, y' \notin N_\kappa(e, 1)$. If $y \in N_\kappa(e, 1)$ and $y' \in N_\kappa(e, 2)$, then by Lemma 4.4, $p(y) \neq p(y')$. Hence we can consider $y, y' \in N_\kappa(e, 2) - N_\kappa(e, 1)$. By the part i of Proposition 3.7, y, y' are not κ-adjacent.

There are two points $e_1, e_1' \in N_\kappa(e, 1)$ such that $e_1 \overset{\kappa}{\leftrightarrow} y$ and $e_1' \overset{\kappa}{\leftrightarrow} y'$. If $e_1 = e_1'$, then we have two liftings beginning at e_1 for the path $\delta : [0, 1]_\mathbb{Z} \longrightarrow B$, by $\delta(0) = p(e_1)$ and $\delta(1) = x$ which is a contradiction. Hence $e_1 \neq e_1'$.

If $p(e_1) \overset{\lambda}{\leftrightarrow} p(e_1')$, then $e_1 \overset{\kappa}{\leftrightarrow} e_1'$ (because $p|_{N_\kappa(e,1)}$ is an isomorphism) and since $y \in N_\kappa(e_1, 1)$ and $y' \in N_\kappa(e_1, 2)$, by Lemma 4.4 we have $p(y) \neq p(y')$. Therefore $p(e_1) \overset{\lambda}{\not\leftrightarrow} p(e_1')$. Let $b_1 = p(e_1)$ and $b_1' = p(e_1')$. Define

$$\begin{cases} \alpha : [0, 4]_\mathbb{Z} \longrightarrow B, \\ \alpha(0) = \alpha(4) = x, \\ \alpha(1) = b_1, \alpha(2) = b, \alpha(3) = b_1', \end{cases}$$

which is a simple λ-loop based at b with length 4 and hence all of its liftings are closed. Since $\gamma : [0, 4]_\mathbb{Z} \longrightarrow E$ defined by $\gamma(0) = y$, $\gamma(1) = e_1$, $\gamma(2) = e$, $\gamma(3) = e_1'$ and $\gamma(4) = y'$ is a lifting of α, $y = y'$ and therefore $p|_{N_\kappa(e,2)}$ is injective.

Obviously $p|_{N_\kappa(e,2)}$ is continuous. For the continuity of $\left(p|_{N_\kappa(e,2)}\right)^{-1}$, let $b', b'' \in N_\lambda(b, 2)$ such that $b' \overset{\lambda}{\leftrightarrow} b''$. Then

- If $b', b'' \in N_\lambda(b, 1)$, then $\left(p|_{N_\kappa(e,2)}\right)^{-1}(b') \overset{\kappa}{\leftrightarrow} \left(p|_{N_\kappa(e,2)}\right)^{-1}(b'')$ because $p|_{N_\kappa(e,1)}$ is an isomorphism.

- If $b', b'' \in N_\lambda(b,2) - N_\lambda(b,1)$, then we can define a simple λ-loop $\alpha : [0,5]_\mathbb{Z} \longrightarrow N_\lambda(b,2)$ by $\alpha(0) = b, \alpha(1) = b_1, \alpha(2) = b', \alpha(3) = b'', \alpha(4) = b'_1$ and $\alpha(5) = b$, where $b_1, b'_1 \in N_\lambda(b,1)$. By assumption, all of its liftings are closed and by the u.p.l, there exists a unique simple κ-loop $\widetilde{\alpha}$ started at e such that $p \circ \widetilde{\alpha} = \alpha$. Since $p|_{N_\kappa(e,2)}$ is bijective, $\left(p|_{N_\kappa(e,2)}\right)^{-1}(b') = \widetilde{\alpha}(2) \overset{\kappa}{\leftrightarrow} \widetilde{\alpha}(3) = \left(p|_{N_\kappa(e,2)}\right)^{-1}(b'')$, as desired.
- If $b' \in N_\lambda(b,2)$ and $b'' \in N_\lambda(b,1)$, by a similar way as in the previous item, we can define a simple λ-loop with the length 4 in $N_\lambda(b,2)$ and deduce that $\left(p|_{N_\kappa(e,2)}\right)^{-1}(b') \overset{\kappa}{\leftrightarrow} \left(p|_{N_\kappa(e,2)}\right)^{-1}(b'')$.

Therefore $p|_{N_\kappa(e,2)}$ is an isomorphism.

If $p|_{N_\kappa(e,n-1)}$ is an isomorphism and all liftings of every simple λ-loop with length $2n+1$ are simple κ-loops, a similar method shows that $p|_{N_\kappa(e,n)}$ is an isomorphism. □

In the following example we show that we can not replace 2n+1 by 2n in the Theorem 4.5.

Example 4.6. Let $B = \{b_0 = (0,1,0), b_1 = (1,0,-1), b_2 = (0,-1,-1), b_3 = (-1,-1,0), b_4 = (-1,0,1)\}$ and define simple 26-loop $\alpha : [0,5]_\mathbb{Z} \longrightarrow B$ by $\alpha(i) = b_{i \bmod 5}$. Also, define $p : \mathbb{Z} \longrightarrow B$ by $p(i) = b_{i \bmod 5}$. Readily, p is a (2,26)-covering map. As $B = SC_{26}^{3,5}$ and $|B| > 2n = 4$, there is no non-trivial simple 26-loop in B of the length 4 or less and so the hypothesis that "every lifting of any simple 26-loop with the length at most $2n$ is a simple 2-loop" is satisfied. Also, there exists a 26-loop with the length 5 such that has no closed lifting, namely α. But p is not a radius 2 covering map as it is not a radius 2 local isomorphism. For example, $N_2(0,2) = [-2,2]_\mathbb{Z}$ and so $\left(p|_{N_2(0,2)}\right)^{-1}$ maps 26-adjacent points b_2 and b_3 in $N_{26}(b_0,2) = B$ to two distinct points of $N_2(0,2) = [-2,2]_\mathbb{Z}$ that are not 2-adjacent (namely the points 2 and -2).

Although in classical topology, the induced homomorphism on fundamental groups by a covering map is a monomorphism, but this is not true in digital topology [4]. We have the following corollary by using Theorem 4.3 and Theorem 4.5.

Corollary 4.7. *Let $p : (E,\kappa) \longrightarrow (B,\lambda)$ be a (κ,λ)-covering map. The induced homomorphism $p_* : \pi_1^\kappa(E,e_0) \longrightarrow \pi_1^\lambda(B,b_0)$ is a monomorphism if every lifting of any simple λ-loop with length* 5 *is closed.*

Proof. Since every lifting of any simple λ-loop with length 5 is closed, p is a radius 2 covering map by Theorem 4.5 and so by part 2 of Theorem 4.3, p is a monomorphism □

Using Theorem 4.5, we can restate Theorem 4.3:

Corollary 4.8. *(Digital Lifting Criteria) Let $p : (E,\kappa) \longrightarrow (B,\lambda)$ be a continuous surjection map, X be a κ'-connected space with $x_0 \in X$ and $\varphi : (X,\kappa') \longrightarrow (B,\lambda)$ be a (κ',λ)-continuous map with $\varphi(x_0) = b_0$. Then the existence of a digital lifting $\widetilde{\varphi} : (X,\kappa') \longrightarrow (E,\kappa)$ for which $\widetilde{\varphi}(x_0) = e_0$ is equivalent to the algebraic assumption $\varphi_*\left(\pi_1^{\kappa'}(X,x_0)\right) \subseteq p_*\left(\pi_1^\kappa(E,e_0)\right)$ if at least one of the following conditions hold:*

(a) p has the u.p.l, has no conciliator point and every lifting of any simple λ-loop with length 5 *is closed;*

(b) p is a radius 2 local isomorphism.

Proof. (a) If p has the u.p.l and has no conciliator point, then it is a covering map, by Theorem 3.9 and since every lifting of any simple λ-loop with the length 5 is closed, p is a radius 2 covering map. Part 3 of Theorem 4.3 implies the existence of desired $\widetilde{\varphi} : (X,\kappa') \longrightarrow (E,\kappa)$.

(b) If p is a radius 2 local isomorphism, Theorem 4.2 implies that it is a radius 2 covering map and so the existence of desired $\widetilde{\varphi} : (X,\kappa') \longrightarrow (E,\kappa)$ comes from the part a. □

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