Filomat 34:12 (2020), 3995–4004
https://doi.org/10.2298/FIL2012995C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalizations Of Second Submodules

Seçil Çeken Gezen[a], Ünsal Tekir[b], Suat Koç[b], Ajlin Kelleqi[a]

[a]*Trakya University, Department of Mathematics, Edirne, Turkey*
[b]*Marmara University, Department of mathematics, Ziverbey, Goztepe, 34722, Istanbul, Turkey*

Abstract. In this paper, we introduce and study some new generalizations of second submodules via a function φ on the set of all submodules of a module. Let R be a ring with non-zero identity, M be an R-module and $\varphi : S(M) \longrightarrow S(M)$ be a function where $S(M)$ is the set of all submodules of M. A non-zero submodule N of M is said to be a φ-second submodule if, for any element a of R and a submodule K of M, $aN \subseteq K$ and $a\varphi(N) \nsubseteq K$ imply either $N \subseteq K$ or $a \in ann_R(N)$. Let $n \geq 2$ be an integer and $\varphi_n : S(M) \longrightarrow S(M)$ be the function defined by $\varphi_n(L) = (L :_M ann_R(L)^{n-1})$ for every $L \in S(M)$. Then a φ_n-second submodule of M is said to be an n-almost second submodule of M. We determine various algebraic properties of these submodules and investigate their relationships with other known submodule classes such as second, prime and semisimple submodules. We study the structure of n-almost second submodules of modules over ZPI-rings and Dedekind domains. We also give some characterizations of modules and submodules by using n-almost second submodules.

1. Introduction

In commutative ring theory, it is an important matter whether all proper ideals of a ring can be written as a product (or intersection) of some special ideals. This is because the structure of an ideal can be more easily determined by using favorable properties of the ideals in the product (or intersection). The most basic tools used in research on this subject are prime ideals and their various generalizations. For example, Dedekind domains are the integral domains in which every proper ideal is a product of prime ideals and this ring class is a fundamental tool for many studies in algebraic geometry and algebraic number theory. As another example, Laskerian rings are the rings in which every proper ideal is a finite intersection of primary ideals and they have an important role in commutative algebra since they generalize Noetherian rings. Therefore, in order to expand the scope of prime ideals and to study with larger ring classes, many different generalizations of prime ideals have been introduced and the rings in which all proper ideals are a product (or intersection) of these generalized prime ideals have been tried to be characterized (see [2], [3], [12], [19], [21]). In this paper we use the concept of ϕ-prime ideal which was defined in [2]. This concept generalizes prime ideals by the following way. Let $\mathcal{I}(R)$ denote the set of all ideals of R and $\phi : \mathcal{I}(R) \longrightarrow \mathcal{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal I of R is called a *ϕ-prime ideal* of R if, for $x, y \in R$, $xy \in I \backslash \phi(I)$ implies $x \in I$ or $y \in I$ [2]. Let $\phi_n : \mathcal{I}(R) \longrightarrow \mathcal{I}(R)$ be the function defined by $\phi_n(I) = I^n$ for an

2010 *Mathematics Subject Classification*. 13C13; 13C99
Keywords. Second submodule, φ-second submodule, n-almost second submodule, almost second submodule
Received: 16 December 2019; Revised: 30 August 2020; Accepted: 09 September 2020
Communicated by Dijana Mosić
Email addresses: cekenseci l@gmail.com (Seçil Çeken Gezen), utekir@marmara.edu.tr (Ünsal Tekir), suat.koc@marmara.edu.tr (Suat Koç), ajlin.03@hotmail.com (Ajlin Kelleqi)

integer $n > 1$. Any ϕ_n-ideal of R is called an *n-almost prime ideal* of R [2]. In particular, for $n = 2$, a 2-almost prime ideal of R is called an *almost prime ideal* of R [12].

Prime submodules are the module theoretic versions of prime ideals. The class of prime submodules has an important role in commutative ring theory as it gives characterizations of important ring classes such as Dedekind domains, Prüfer domains, arithmetical rings. The concept of prime submodule was first introduced in 1965 by E. H. Feller and E. W. Swokowski [23]. Let R be a commutative ring with non-zero identity. A proper submodule P of M is called a prime submodule if whenever $rm \in P$, where $r \in R$, $m \in M$, we have either $r \in (P :_R M)$ or $m \in P$. If P is a *prime submodule* of M, then $p = (P :_R M)$ is a prime ideal of R and in this case P is called a p–prime submodule [27]. If (0) is a prime submodule of M, then M is called a prime module. It can be easily seen that M is a prime module if and only if $ann_R(N) = ann_R(M)$ for every non-zero submodule N of M [27].

Besides the prime submodules, module theoretic versions of generalizations of prime ideals have been investigated since the begining of 2000s (see for example [11], [20], [25], [28], [29], [32]). In [32], the author introduced the concept of φ-prime submodule as follows. Let R be a commutative ring with non-zero identity, M be an R-module, $S(M)$ be the set of all submodules of M, and $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule P of M is called a *φ-prime submodule* if $a \in R$ and $x \in M$ with $ax \in P\backslash\varphi(P)$ implies that $a \in (P :_R M)$ or $x \in P$. Let $n \geq 2$ be an integer and $\varphi_n : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be the function defined by $\varphi_n(N) = (N :_R M)^{n-1}N$ for every $N \in S(M)$. Then a φ_n-prime submodule of M is said to be an *n-almost prime submodule* of M. In particular, for $n = 2$, a 2-almost prime submodule of M is called an *almost prime submodule* of M. In this article we introduce and study the dual notion of these submodule classes.

Second submodules of modules over commutative rings were introduced in [31] as the dual notion of prime submodules. According to this definition a non-zero submodule N of an R-module M is said to be a *second submodule* if for all $r \in R$, either $rN = 0$ or $rN = N$. If N is a second submodule of M, then $p = ann_R(N)$ is a prime ideal of R. In this case, N is called a p-second submodule of M [31]. In recent years, second submodules have attracted the attention of many researchers and it has been understood that this submodule class has an important role in determining characterizations of modules and rings (see for example [9], [13], [14], [15]). Along with the increased work on the second submodules, generalization of these submodules has begun to be investigated and it has been seen that these generalized second submodules also have interesting and important algebraic properties (see for example [7], [16], [17], [18], [22]).

In this paper, we study some generalizations of second submodules via a function φ on the set of all submodules of a module. We investigate φ-second and n-almost second submodules which were defined in [22]. We determine various algebraic properties of these submodules and give some characterizations of modules and submodules by using these submodule classes (see Theorems 3.2, 3.5, 3.8, 3.14). We characterize φ-second submodules of a comultiplication module (see Proposition 2.6). We investigate the relationships of φ-second and n-almost second submodules with other known submodule classes such as second, prime and semisimple submodules (see Theorems 2.3, 3.3 and Proposition 3.1). We study the structure of n-almost second submodules of modules over ZPI-rings and Dedekind domains (see Theorem 3.13). We introduce the notion of almost second radical of a submodule as a generalization of the second radical of a submodule which was defined in [14]. We also define the concept of almost m^*-system and we give a characterization of almost second radical of a module via almost m^*-systems (see Theorem 3.19).

Throughout this paper all rings will be commutative with non-zero identity and all modules will be unital left modules. Unless otherwise stated R will denote a ring.

2. φ-Second Submodules

In this section we study φ-second submodules where φ is a function on the set of all submodules of a module. In the rest of the paper $S(M)$ will denote the set of all submodules of an R-module M and $\varphi : S(M) \to S(M)$ will be a function.

Definition 2.1. *[22, Definition 2.1] Let M be an R-module and N be a non-zero submodule of M. If, for any element a of R and a submodule K of M, $aN \subseteq K$ and $a\varphi(N) \nsubseteq K$ imply either $N \subseteq K$ or $a \in ann_R(N)$, then N is said to be a*

φ-second submodule of M.

Let $\varphi_M : S(M) \longrightarrow S(M)$ be the function defined by $\varphi_M(L) = M$ for every $L \in S(M)$. Then a φ_M-second submodule of M is said to be a weak second submodule of M.

Let $n \geq 2$ be an integer and $\varphi_n : S(M) \longrightarrow S(M)$ be the function defined by $\varphi_n(L) = (L :_M ann_R(L)^{n-1})$ for every $L \in S(M)$. Then a φ_n-second submodule of M is said to be an n-almost second submodule of M. In particular, for $n = 2$, a 2-almost second submodule of M is called an almost second submodule of M.

Let M be an R-module and N be a submodule of M. Since $\varphi(N)\backslash N = (\varphi(N) \cup N)\backslash N$, without loss of generality, throughout this paper we will assume that $N \subseteq \varphi(N)$.

It is clear from the definition that every R-module M is a φ-second submodule of itself for any function $\varphi : S(M) \longrightarrow S(M)$. But not every R-module is a second submodule of itself. For example, $\mathbb{Z}$ is not a second $\mathbb{Z}$-submodule of itself but $\mathbb{Z}$ is φ-second $\mathbb{Z}$-submodule of itself.

Theorem 2.2. *[22, Theorem 2.16] Let M be an R-module and Q be a non-zero submodule of M. Then the following are equivalent.*

(1) Q is a φ-second submodule of M.

(2) $(X :_R Q) = ann_R(Q) \cup (X :_R \varphi(Q))$ for every submodule X of M with $Q \not\subseteq X$.

(3) $(X :_R Q) = ann_R(Q)$ or $(X :_R Q) = (X :_R \varphi(Q))$ for every submodule X of M with $Q \not\subseteq X$.

(4) For any ideal I of R and any submodule L of M, $IQ \subseteq L$ and $I\varphi(Q) \not\subseteq L$ imply either $I \subseteq ann_R(Q)$ or $N \subseteq L$.

(5) $IQ = Q$ or $IQ = I\varphi(Q)$ for every ideal I of R with $I \not\subseteq ann_R(Q)$.

Theorem 2.3. *Let N be a φ-second submodule of M such that $N = M_1 + ... + M_k$ where $M_1,...,M_k$ are submodules of M such that $ann_R(M_i)$ is a maximal ideal of R for each i $(1 \leq i \leq k)$. Then either N is second or $N = \varphi(N)$.*

Proof. Without loss of generality we may assume that $N \not\subseteq \sum_{\substack{j=1\\j\neq i}}^{k} M_j$ for any i $(1 \leq i \leq k)$. By Theorem 2.2, $(\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N) = ann_R(N)$ or $(\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N) = (\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R \varphi(N))$. If the first case holds for some i $(1 \leq i \leq k)$, then $ann_R(M_i) \subseteq (\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N) = ann_R(N)$ and so $ann_R(M_i) = ann_R(N)$. This implies that N is a second submodule as $ann_R(N)$ is a maximal ideal of R.

Now suppose that for each i $(1 \leq i \leq k)$, $(\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N) \neq ann_R(N)$. So we have $(\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N) = (\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R \varphi(N))$ for each i $(1 \leq i \leq k)$. If $ann_R(M_1) \subseteq ann_R(N)$, then $ann_R(N)$ is a maximal ideal and so N is a second submodule. So we may assume that $ann_R(M_1) \not\subseteq ann_R(N)$. Take an element $r \in ann_R(M_1)\backslash ann_R(N)$. Then $r \notin ann_R(M_i)$ for some i $(2 \leq i \leq k)$. Since $ann_R(M_i)$ is a maximal ideal, $Rr + ann_R(M_i) = R$. On the other hand we have $r \in ann_R(M_1) \subseteq (\sum_{j=2}^{k} M_j :_R N) = (\sum_{j=2}^{k} M_j : \varphi(N))$. Therefore, $R = (\sum_{j=2}^{k} M_j :_R \varphi(N)) + ann_R(M_i)$. It follows that $\varphi(N) \subseteq \sum_{j=2}^{k} M_j + ann_R(M_i)\varphi(N) \subseteq \sum_{j=2}^{k} M_j + (\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N)\varphi(N)$. Since $(\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N) \not\subseteq ann_R(N)$ and $(\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N)N \neq N$, Theorem 2.2 implies that $(\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N)\varphi(N) = (\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N)N$. It follows that $\varphi(N) \subseteq \sum_{j=2}^{k} M_j + (\sum_{\substack{j=1\\j\neq i}}^{k} M_j :_R N)N \subseteq \sum_{j=2}^{k} M_j + \sum_{\substack{j=1\\j\neq i}}^{k} M_j = M_1 + ... + M_k = N$. Thus we get that $N = \varphi(N)$. □

Recall that an R-module M is called co-semisimple if each proper submodule of M is an intersection of maximal submodules [30].

Corollary 2.4. *Let M be an R-module and N be a φ-second submodule of M. If one of the following holds, then either N is second or $N = \varphi(N)$.*

(1) *M is a finitely generated semisimple R-module.*

(2) *M is a finitely cogenerated co-semisimple R-module.*

Proof. (1) This follows from the fact that $ann_R(S)$ is a maximal ideal of R for every simple submodule S of M.

(2) A finitely cogenerated co-semisimple module M is semisimple by [30, 23.1]. M is also finitely generated by [1, Proposition 10.6]. So the result follows from part (1). □

Recall from [4] that an R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. It also follows that M is a comultiplication module if and only if $N = (0 :_M ann_R(N))$ for every submodule N of M [4].

In [22], the authors gave a characterization of φ-second submodules of a comultiplication module as follows.

Proposition 2.5. *[22, Theorem 2.9] Let $\psi : \mathcal{I}(R) \longrightarrow \mathcal{I}(R) \cup \{\emptyset\}$ be a function and N be a non-zero submodule of an R-module M.*

(1) *If $ann_R(\varphi(N)) \subseteq \psi(ann_R(N))$ and N is a φ-second submodule of M, then $ann_R(N)$ is a ψ-prime ideal of R.*

(2) *Let $ann_R(\varphi(N)) = \psi(ann_R(N))$ and M be a comultiplication R-module. Then N is a φ-second submodule of M if and only if $ann_R(N)$ is a ψ-prime ideal of R.*

Let M be an R-module and N, K be submodules of M. The coproduct of N and K is defined by $(0 :_M ann_R(N)ann_R(K))$ and it is denoted by $C(NK)$.

In the following proposition we give a characterization of φ-second submodules of a comultiplication module via coproduct.

Proposition 2.6. *Let M be a comultiplication R-module and N be a non-zero submodule of M. Then the following are equivalent.*

(1) *N is a φ-second submodule of M.*

(2) *If K and L are two submodules of M such that $N \subseteq C(KL)$ and $\varphi(N) \nsubseteq C(KL)$, then $N \subseteq K$ or $N \subseteq L$.*

Proof. (1) $\Longrightarrow$ (2) Let $N \subseteq C(KL) = (0 :_M ann_R(K)ann_R(L))$ and $\varphi(N) \nsubseteq C(KL)$. Then $ann_R(L)N \subseteq (0 :_M ann_R(K)) = K$ and $ann_R(L)\varphi(N) \nsubseteq (0 :_M ann_R(K)) = K$. Since N is φ-second, $ann_R(L) \subseteq ann_R(N)$ or $N \subseteq K$. Thus $N = (0 :_M ann_R(N)) \subseteq (0 :_M ann_R(L) = L$ or $N \subseteq K$, as desired.

(2) $\Longrightarrow$ (1) Let I be an ideal of R and K be a submodule of M such that $IN \subseteq K$ and $I\varphi(N) \nsubseteq K$. Since M is comultiplication, $K = (0 :_M J)$ for some ideal J of R. Then $JIN = 0$ and $JI\varphi(N) \neq 0$. It follows that $N \subseteq (0 :_M IJ) = C((0 :_M I)K)$ and $\varphi(N) \nsubseteq C((0 :_M I)K)$. By (2), we have either $N \subseteq K$ or $N \subseteq (0 :_M I)$. Thus $N \subseteq K$ or $I \subseteq ann_R(N)$ as needed. □

Proposition 2.7. *Let M be an R-module and K be a submodule of M. Let $\varphi^K : S(K) \longrightarrow S(K)$ be the function defined by $\varphi^K(L) = \varphi(L) \cap K$ for every $L \in S(K)$. Then the following hold for a submodule N of K.*

(1) *If N is a φ-second submodule of M, then N is a φ^K-second submodule of K.*

(2) *Let $\varphi(N) \subseteq K$. Then N is a φ^K-second submodule of K if and only if N is a φ-second submodule of M.*

(3) *If $K \subseteq \varphi(N)$ and N is a φ-second submodule of M, then N is a weak second submodule of K.*

(4) *If $\varphi(N) \subseteq \varphi(K)$, K is a φ-second submodule of M and N is a weak second submodule of K, then N is a φ-second submodule of M.*

Proof. (1) Let $aN \subseteq L$ and $a\varphi^K(N) \nsubseteq L$ for $a \in R$ and a submodule L of K. Then $a(\varphi(N) \cap K) \nsubseteq L$ and so $a\varphi(N) \nsubseteq L$. Thus $a \in ann_R(N)$ or $N \subseteq L$, as needed.

(2) If $\varphi(N) \subseteq K$, then $\varphi^K(N) = \varphi(N)$ and the result follows.

(3) If $K \subseteq \varphi(N)$, then $\varphi^K(N) = K$ and the result follows.

(4) Let $aN \subseteq L$ and $a\varphi(N) \nsubseteq L$ for an ideal I of R and a submodule L of M. Then $a\varphi(K) \nsubseteq L$ by the hypothesis. If $aK \subseteq L$, then $a \in ann_R(K) \subseteq ann_R(N)$ or $N \subseteq K \subseteq L$ and we are done. If $aK \nsubseteq L$, then $aK = a\varphi_K(N) \nsubseteq L \cap K$ and $aN \subseteq L \cap K$ imply that $a \in ann_R(N)$ or $N \subseteq L$ as needed. □

Let R_i be a ring and M_i be an R_i-module for $i = 1, ..., n$. Denote $R := R_1 \times ... \times R_n$ and $M := M_1 \times ... \times M_n$. Then M is an R-module and it is well-known that each R-submodule N of M is of the form $N = N_1 \times ... \times N_n$ for some submodules N_i of M_i $(1 \leq i \leq n)$.

Proposition 2.8. *Let R_i be a ring and M_i be an R_i-module for $i = 1, 2$. Denote $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Suppose that N_1 is a weak second submodule of M_1 such that $\varphi(N_1 \times \{0\}) \subseteq M_1 \times \{0\}$. Then $N_1 \times \{0\}$ is a φ-second submodule of M.*

Proof. Clearly, $N_1 \times \{0\} \neq \{0_M\}$. Let $(r_1, r_2) \in R$ and $K_1 \times K_2$ be a submodule of M such that $(r_1, r_2)(N_1 \times \{0\}) \subseteq K_1 \times K_2$ and $(r_1, r_2)\varphi(N_1 \times \{0\}) \nsubseteq K_1 \times K_2$. Then $(r_1, r_2)(M_1 \times \{0\}) \nsubseteq K_1 \times K_2$ and so $r_1 M_1 \nsubseteq K_1$. Since N_1 is a weak second submodule of M_1, we have $r_1 \in ann_{R_1}(N_1)$ or $N_1 \subseteq K_1$. This implies that $(r_1, r_2) \in ann_{R_1}(N_1) \times R_2 = ann_R(N_1 \times \{0\})$ or $N_1 \times \{0\} \subseteq K_1 \times K_2$ as desired. □

Let M be an R-module. We define the function $\varphi_\omega : S(M) \longrightarrow S(M)$ as $\varphi_\omega(L) = \sum_{i \in \mathbb{Z}^+}(L :_M ann_R(L)^i)$ for every $L \in S(M)$.

Let M be an R-module, $\varphi : S(M) \longrightarrow S(M)$ and $\psi : S(M) \longrightarrow S(M)$ be two functions. If $\varphi(N) \subseteq \psi(N)$ for every $N \in S(M)$, the we will write $\varphi \leq \psi$.

Corollary 2.9. *Let R_i be a ring, M_i be an R_i-module for $i = 1, 2$. Denote $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Suppose that $\varphi : S(M) \longrightarrow S(M)$ is a function with $\varphi \leq \varphi_\omega$. Then, $N_1 \times \{0\}$ is a φ-second submodule of M for any weak second submodule of N_1 of M_1.*

Proof. $\varphi(N_1 \times \{0\}) \subseteq \varphi_\omega(N_1 \times \{0\}) = \sum_{i \in \mathbb{Z}^+}(N_1 \times \{0\} :_M ann_R(N_1 \times \{0\})^i) = \sum_{i \in \mathbb{Z}^+}(N_1 :_{M_1} ann_{R_1}(N_1)^i) \times (0 :_{M_2} R_2) = \sum_{i \in \mathbb{Z}^+}(N_1 :_{M_1} ann_{R_1}(N_1)^i) \times \{0\} \subseteq M_1 \times \{0\}$. Thus the result follows from Proposition 2.8. □

3. n-Almost Second Submodules and Almost Second Radical of a Module

In this section we investigate n-almost second submodules and we give a generalization of the second radical of a submodule which was defined in [14].

Proposition 3.1. *Let M be a prime R-module and N be a proper almost second submodule of M. Then N is a prime submodule of M.*

Proof. Let $r \in R$, $m \in M$ and $rm \in N$. Since M is a prime R-module, $ann_R(N) = ann_R(M)$. If $r \in ann_R(N)$, then $r \in (N :_R M)$ and we are done. So we may assume that $r \notin ann_R(N)$. Then, by Theorem 2.2, we have either $rN = N$ or $rN = (N :_M ann_R(N)) = (N :_M ann_R(M)) = M$. If $rN = N$, then $rm = rn$ for some $n \in N$. It follows that $r(m - n) = 0$. Since M is a prime R-module and $r \notin ann_R(M)$, we have $m = n \in N$. If $rN = M$, then $rm \in rN$. This implies that $m \in N$ as in the previous case. Thus N is a prime submodule of M. □

Theorem 3.2. *Let M be an R-module. If there exist maximal ideals $P_1, ..., P_n$ of R such that $P_1 \cap ... \cap P_n \subseteq ann_R(M)$, then for any ideal I of R, $(0 :_M I) = 0$ or $(0 :_M I)$ is an almost second submodule of M.*

Proof. Suppose that $(0 :_M I) \neq 0$. First we show that $(0 :_M I) = (0 :_M I^2)$. Clearly, $(0 :_M I) \subseteq (0 :_M I^2)$. Let $m \in (0 :_M I^2)$ and $a \in I$. Without loss of generality, we may assume that there exists t $(0 \leq t \leq n)$ such that $a \in P_1 \cap ... \cap P_t$ and $a \notin P_{t+1} \cup ... \cup P_n$. Now, for any $t + 1 \leq i \leq n$, $R = P_i + Ra$ and so $1 = x_i + \sum_{l=1}^{n_i} r_{il}a$ where $x_i \in P_i$ and $r_{il} \in R$. Thus there exists $b \in I$ such that $1 = x_{t+1}...x_n + b$ and hence $a = ax_{t+1}...x_n + ab$. Since $ax_{t+1}...x_n \in ann_R(M)$ and $abm = 0$, we have $am = 0$. Thus $Im = 0$ and so $m \in (0 :_M I)$. Hence $(0 :_M I) = (0 :_M I^2)$.

Now, clearly, $(0 :_M I) \subseteq ((0 :_M I) :_M ann_R(0 :_M I))$. On the other hand, $((0 :_M I) :_M ann_R(0 :_M I)) \subseteq (0 :_M I^2) = (0 :_M I)$. Thus $(0 :_M I) = ((0 :_M I) :_M ann_R(0 :_M I))$ and so $(0 :_M I)$ is an almost second submodule of M. □

Theorem 3.3. *Let M be an R-module, $a \in R$, $(0 :_M a) \neq (0)$ and $(0 :_M a) = aM$. Then $(0 :_M a)$ is an almost second submodule of M if and only if it is a second submodule of M.*

Proof. If $(0 :_M a)$ is a second submodule, then clearly it is an almost second submodule. Suppose that $(0 :_M a)$ is an almost second submodule of M. Let $b(0 :_M a) \subseteq K$ for $b \in R$ and a submodule K of M. If $b((0 :_M a) :_M ann_R(0 :_M a)) = bM \nsubseteq K$, then we are done. So we may assume that $bM \subseteq K$. Now, $(b + a)(0 :_M a) \subseteq K$. If $(b + a)M \nsubseteq K$, then we have $(0 :_M a) \subseteq K$ or $b \in ann_R(0 :_M a)$ as $(0 :_M a)$ is almost second. So assume that $(b + a)M \subseteq K$. $bM \subseteq K$ gives that $aM \subseteq K$. Since $(0 :_M a) \subseteq aM$, we have $(0 :_M a) \subseteq K$ which shows that $(0 :_M a)$ is a second submodule of M. □

Recall from [8] that an R-module M is said to be fully coidempotent if $N = C(N^2)$ for every submodule N of M.

Lemma 3.4. *An R-module M is fully coidempotent if and only if* $N = (N :_M ann_R(N)^m)$ *for every submodule N of M and positive integer m.*

Proof. Suppose that M is a fully coidempotent R-module. Let N be a submodule of M and m be a positive integer. It is sufficient to show that $N = (N :_M ann_R(N))$. We have $N = C(N^2) = (0 :_M ann_R(N)^2)$. Also, $N \subseteq (0 :_M ann_R(N))$ implies that $(N :_M ann_R(N)) \subseteq ((0 :_M ann_R(N)) :_M ann_R(N)) = (0 :_M ann_R(N)^2) = N$ and so $(N :_M ann_R(N)) \subseteq N$. Since the other inclusion always holds we have $(N :_M ann_R(N)) = N$ and hence $N = (N :_M ann_R(N)^m)$ for all $m \geq 1$.

Conversely, suppose that $N = (N :_M ann_R(N)^m)$ for every submodule N of M and positive integer m. Then $N = (N :_M ann_R(N))$. We have

$C(N^2) = (0 :_M ann_R(N)^2) \subseteq (N :_M ann_R(N)^2) = ((N :_M ann_R(N)) :_M ann_R(N)) = (N :_M ann_R(N)) = N$. Thus we get that $C(N^2) \subseteq N$ and so $N = C(N^2)$. □

Theorem 3.5. *Let* $R = R_1 \times ... \times R_m$ *and* $M = M_1 \times ... \times M_m$ *where* R_i *is a ring,* $0 \neq M_i$ *is an* R_i*-module for all* $i \in \{1, ..., m\}$ *and* $n, m \geq 2$*. Then every non-zero submodule of M is n-almost second if and only if M is a fully coidempotent R-module.*

Proof. Suppose that every non-zero submodule of M is n-almost second. So every non-zero submodule of M is almost second. We will show that M_i is fully coidempotent for each $i \in \{1, ..., m\}$, hence M will be fully coidempotent. Suppose on the contrary that M_1 is not fully coidempotent. Then there exists a non-zero submodule N_1 of M_1 such that $(N_1 :_{M_1} ann_{R_1}(N_1)) \nsubseteq N_1$. We have $(1, 0, ..., 0)(N_1 \times M_2 \times ... \times M_m) \subseteq N_1 \times 0 \times ... \times 0$ and $(1, 0, ..., 0)(N_1 \times M_2 \times ... \times M_m :_M ann_R(N_1 \times M_2 \times ... \times M_m)) \nsubseteq N_1 \times 0 \times ... \times 0$. By hypothesis, $N_1 \times M_2 \times ... \times M_m$ is almost second. So $(1, 0, ..., 0) \in ann_R(N_1 \times M_2 \times ... \times M_m)$ or $N_1 \times M_2 \times ... \times M_m \subseteq N_1 \times 0 ... \times 0$ which are both contradictions. Similarly, M_i is fully coidempotent for each $i \in \{2, ..., m\}$. The converse is clear. □

Corollary 3.6. *Let* $R = R_1 \times ... \times R_m$ *and* $M = M_1 \times ... \times M_m$ *where* R_i *is a ring,* $0 \neq M_i$ *is an* R_i*-module for all* $i \in \{1, ..., m\}$ *and* $n, m \geq 2$*. Then every non-zero submodule of M is n-almost second if and only if every non-zero submodule of M (n+1)-almost second.*

Proof. Suppose that every non-zero submodule of M is n-almost second. Then M is fully coidempotent by Theorem 3.5. Since $n + 1 > 2$, every non-zero submodule of M is (n+1)-almost second again by Theorem 3.5. □

Lemma 3.7. *Let M be an R-module, N be a submodule of M and I be an ideal of R. Suppose that* $(0 :_M I) \neq (N :_M I)$ *and* $(N :_M I) \neq N$*. Then* $K := (N :_M I)$ *is an almost second submodule of M if and only if* $K = (K :_M ann_R(K))$*.*

Proof. If $K = (K :_M ann_R(K))$, then clearly, K is an almost second submodule of M. Conversely, assume that K is an almost second submodule of M. If $I \subseteq ann_R(K)$, then we have $(0 :_M I) = (N :_M I)$, a contradiction. Thus $I \nsubseteq ann_R(K)$. By Theorem 2.2, $IK = K$ or $IK = I(K :_M ann_R(K))$. If $IK = K$, then $K = IK = I(N :_M I) \subseteq N$ and hence $N = K$, a contradiction. Therefore, $IK = I(K :_M ann_R(K)) \subseteq N$ and hence $(K :_M ann_R(K)) \subseteq (N :_M I) = K$. Thus $K = (K :_M ann_R(K))$ as desired. □

Theorem 3.8. *Let M be an Artinian R-module,* $I \subseteq Jac(R)$ *and N be a submodule of M such that* $(N :_R M) = 0$ *and* $(0 :_M I) \neq (N :_M I)$*. Then* $(N :_M I)$ *is not an n-almost second submodule of M for any integer* $n > 1$*.*

Proof. We have $(N :_M I) \neq N$, otherwise, if $(N :_M I) = N$ then, by [24, Proposition 3.5], $N = M$ which is a contradiction. Note that $(N :_M I) \subseteq (N :_M ann_R(N :_M I))$. If $(N :_M I)$ is n-almost second, then it is almost second and Lemma 3.7 implies that $(N :_M I) = ((N :_M I) :_M ann_R(N :_M I))$. It follows that $(N :_M I^2) = ((N :_M I) :_M I) \subseteq ((N :_M ann_R(N :_M I)) :_M I) = ((N :_M I) :_M ann_R(N :_M I)) = (N :_M I)$. Since the other inclusion always holds, we have $(N :_M I) = (N :_M I^2) = ((N :_M I) :_M I)$. By [24, Proposition 3.5], we get that $(N :_M I) = M$ and so $I \subseteq (N :_R M) = 0$. This is a contradiction since $(0 :_M I) \neq (N :_M I)$. □

Lemma 3.9. *Let I be an ideal of R and M be an R-module. Then $(0 :_M (ann_R(0 :_M I))^n) = (0 :_M I^n)$ for every integer $n > 1$. In particular, $((0 :_M I) :_M (ann_R(0 :_M I))^{n-1}) = (0 :_M I^n)$.*

Proof. Since $I \subseteq ann_R(0 :_M I)$, we have $I^n \subseteq (ann_R(0 :_M I))^n$ and hence $(0 :_M (ann_R(0 :_M I))^n) \subseteq (0 :_M I^n)$. Let $m \in (0 :_M I^n)$. Take an element $r \in ann_R(0 :_M I)$. Then $r(0 :_M I) = 0$. Since $I^{n-1}m \subseteq (0 :_M I)$, we have $rI^{n-1}m \subseteq r(0 :_M I) = 0$. Thus $rm \in (0 :_M I^{n-1})$. Since $I^{n-2}rm \subseteq (0 :_M I)$, we have $r^2I^{n-2}m \subseteq r(0 :_M I) = 0$ and so $r^2m \in (0 :_M I^{n-2})$. Repeating this process, we get that $r^nm \in (0 :_M R) = 0$. Thus $(ann_R(0 :_M I))^n m = 0$ and so $m \in (0 :_M (ann_R(0 :_M I))^n)$. Therefore, $(0 :_M I^n) \subseteq (0 :_M (ann_R(0 :_M I))^n)$. For the second part,
$((0 :_M I) :_M (ann_R(0 :_M I))^{n-1}) = ((0 :_M (ann_R(0 :_M I))^{n-1}) :_M I) = ((0 :_M I^{n-1}) :_M I) = (0 :_M I^n)$. □

It is well-known that a commutative ring R is a Von Neumann regular ring if and only if $I^2 = I$ for each ideal I of R [1, p. 176]. Lemma 3.9 implies that if M is a module over a Von Neumann regular ring R and I is an ideal of R with $(0 :_M I) \neq 0$, then $(0 :_M I)$ is an n-almost second submodule of M.

Recall from [6] that an R-module M is called a strong comultiplication module if M is a comultiplication R-module and $ann_R(0 :_M I) = I$ for every ideal I of R.

Proposition 3.10. *Let M be a strong comultiplication R-module and I be an ideal of R. Then, I is an n-almost prime ideal of R if and only if $(0 :_M I)$ is an n-almost second submodule of M.*

Proof. By Lemma 3.9, $\varphi_n((0 :_M I)) = ((0 :_M I) :_M (ann_R(0 :_M I))^{n-1}) = (0 :_M I^n)$. Since M is strong comultiplication, $ann_R(0 :_M I^n) = I^n$ and $ann_R(0 :_M I) = I$. Thus $ann_R(\varphi_n((0 :_M I))) = \phi_n(ann_R(0 :_M I))$. By Proposition 2.5-(2), I is an n-almost prime ideal of R if and only if $(0 :_M I)$ is an n-almost second submodule of M. □

Example 3.11. *Let $R = F[[X^3, X^4, X^5]]$ where F is a field, and $I = RX^3 + RX^4$. Then I is an almost prime ideal of R which is not a 3-almost prime ideal by [2, Example 11]. Let M be a strong comultiplication R-module. By Proposition 3.10, $(0 :_M I)$ is an almost second submodule of M which is not a 3-almost second submodule of M.*

Example 3.12. *Let M be a co-semisimple R-module. Then [7, Theorem 4.8] and [10, Theorem 2.3] imply that $(0 :_M I) = (0 :_M I^n)$ for every ideal I of R and an integer $n > 1$. By Lemma 3.9, $(0 :_M I)$ is an n-almost second submodule of M for each integer $n > 1$.*

A ring R is called a *ZPI-ring* if every proper ideal of R can be written as a product of prime ideals of R [26].

Theorem 3.13. *Let $n > 1$ be an integer, M be an R-module and I be an ideal of R with $(0 :_M I) \neq 0$.*

(*i*) *If R is a ZPI-ring and $(0 :_M I)$ is an n-almost second submodule of M, then $(0 :_M I) = (0 :_M I^n)$ or $(0 :_M I) = (0 :_M P)$ for some prime ideal P of R.*

(*ii*) *If R is a Dedekind domain, then $(0 :_M I)$ is an n-almost second submodule of M if and only if $(0 :_M I) = (0 :_M I^n)$ or $(0 :_M I)$ is a second submodule of M.*

(*iii*) *If $(R, \mathfrak{m})$ is a local ZPI-ring and $(0 :_M I)$ is finitely cogenerated, then $(0 :_M I)$ is an n-almost second submodule of M if and only if $(0 :_M I) = M$ or $(0 :_M I) = (0 :_M \mathfrak{m})$.*

Proof. (*i*) Since R is a ZPI-ring, $I = P_1^{t_1}...P_k^{t_k}$ for some distinct prime ideals of R and positive integers t_i. Suppose that $(0 :_M I) \neq (0 :_M P)$ for every prime ideal P of R. We have $(0 :_M I) = (0 :_M P_1^{t_1}...P_k^{t_k})$. Without loss of generality we may assume that $(0 :_M I) \neq (0 :_M P_1^{t_1-1}...P_k^{t_k})$ and $(t_1 - 1) + t_2 + ... + t_k > 0$. Put $N := (0 :_M P_1^{t_1-1}...P_k^{t_k})$ and $K := (0 :_M I)$. Then $K = (N :_M P_1)$, $(0 :_M P_1) \neq K$ and $K \neq N$. By Lemma 3.7, we have $K = (K :_M ann_R(K)^{n-1})$, that is $(0 :_M I) = ((0 :_M I) :_M (ann_R(0 :_M I))^{n-1})$. By Lemma 3.9, $(0 :_M I) = (0 :_M I^n)$.

(*ii*) Suppose that $(0 :_M I)$ is an n-almost second submodule of M. By part (*i*), $(0 :_M I) = (0 :_M I^n)$ or $(0 :_M I) = (0 :_M P)$ for some prime ideal P of R. If $(0 :_M I) = (0 :_M P)$ where P is a prime ideal of R, then $P = 0$ or P is a maximal ideal of R. Clearly, $P = 0$ implies that $(0 :_M 0) = (0 :_M I) = M$ and so $(0 :_M I^n) = M$. If P is a maximal ideal of R, then $ann_R(0 :_M P) = ann_R(0 :_M I) = P$ since $(0 :_M I) \neq 0$. This implies that $(0 :_M I)$ is a second submodule of M.

For the converse, suppose that $(0 :_M I)$ is a second submodule of M or $(0 :_M I) = (0 :_M I^n)$. If $(0 :_M I)$ is a second submodule, clearly, it is n-almost second. If $(0 :_M I) = (0 :_M I^n)$, then the result follows from Lemma 3.9.

(*iii*) If $(0 :_M I) = (0 :_M \mathfrak{m})$, then $(0 :_M I)$ is a second submodule since $ann_R(0 :_M I) = \mathfrak{m}$ is a maximal ideal of R. If $(0 :_M I) = M$, then clearly it is n-almost second.

Conversely, suppose that $(0 :_M I)$ is an n-almost second submodule of M. By [26, Theorem 9.10], R is a Noetherian ring. If $\mathfrak{m} = \mathfrak{m}^2$, Nakayama's Lemma implies that $\mathfrak{m} = 0$ and so R is a field. Thus $(0 :_M I) = (0 :_M 0) = M$. Now let $\mathfrak{m}^2 \neq \mathfrak{m}$. Take an element $a \in \mathfrak{m}\backslash\mathfrak{m}^2$. Then $\mathfrak{m}^2 \subsetneq \mathfrak{m}^2 + Ra \subsetneq \mathfrak{m}$. By [26, Theorem 9.10], there are no ideals of R strictly between $\mathfrak{m}^2$ and $\mathfrak{m}$. So $\mathfrak{m}^2 + Ra = \mathfrak{m}$ and by Nakayama's Lemma $\mathfrak{m} = Ra$. Now let P be a non-zero prime ideal of R and $0 \neq b \in P$. By the Krull's intersection theorem, we have $\cap_{i=1}^{\infty}\mathfrak{m}^i = 0$. Thus there is a positive integer k such that $b \in \mathfrak{m}^k$ and $b \notin \mathfrak{m}^{k+1}$. Since $b \in \mathfrak{m}^k = Ra^k$, there exists an element $u \in R$ such that $b = ua^k$ and since $b \notin \mathfrak{m}^{k+1}$, $u \notin \mathfrak{m}$. Thus u is a unit in R. Hence $a^k = u^{-1}b \in P$ and so $a \in P$, that is $\mathfrak{m} = P$. Thus $\mathfrak{m}$ is the only non-zero prime ideal of R. Now by part (*i*), $(0 :_M I) = (0 :_M I^n)$ or $(0 :_M I) = (0 :_M 0) = M$ or $(0 :_M I) = (0 :_M \mathfrak{m})$. If $(0 :_M I) = (0 :_M I^n)$, then
$(0 :_M I)/(0 :_M I) = ((0 :_M I) :_M I^{n-1})/(0 :_M I) = ((0 :_M I) :_{M/(0:_M I)} I^{n-1})$. By [24, Proposition 3.5], we have $(0 :_M I) = M$. □

Let S be a multiplicatively closed subset of R and P be a prime ideal of R with $S \cap P = \emptyset$. Recall from [29] that S is called P-essential, if $S \cap J \neq \emptyset$, for each ideal J with $J \nsubseteq P$.

Theorem 3.14. *Let N be an n-almost second submodule of an R-module M and $I = ann_R(N)$. Then $S = [(R\backslash I) \cup ann_R(N :_M I^{n-1})]\backslash P$ is a P-essential multiplicatively closed subset of R for each prime ideal P of R.*

Proof. First, we show that S is a multiplicatively closed subset of R. Let $r, s \in S$. If $r \in ann_R(N :_M I^{n-1})$ or $s \in ann_R(N :_M I^{n-1})$ then $rs \in ann_R(N :_M I^{n-1})\backslash P$ and so $rs \in S$. So we may assume that $r, s \notin ann_R(N :_M I^{n-1})$. Suppose on the contrary that $rs \notin S$. Then $rs \in I$. Also, $rs \notin P$, for if $rs \in P$, $r \in S \cap P$ or $s \in S \cap P$, a contradiction. Since $rs \notin ann_R(N :_M I^{n-1})$, we have $r(N :_M ann_R(N)^{n-1}) \nsubseteq (0 :_M s)$ and $rN \subseteq (0 :_M s)$. Since N is n-almost second, we have $r \in I$ or $s \in I$, a contradiction. Thus $rs \in S$ and so S is a multiplicatively closed subset of R. Now we show that S is P-essential. Let J be an ideal of R with $J \nsubseteq P$. If $I \subseteq P$, then $S = R\backslash P$ and clearly S is P-essential. Suppose that $J \cap S \neq \emptyset$. Then one can see that $J \cap [(R\backslash I) \cup ann_R(N :_M I^{n-1})] \subseteq P$ and $J \subseteq I \cup P$. Therefore, $J \subseteq I$. On the other hand, $N \subseteq (0 :_M I)$ implies that $(N :_M I^{n-1}) \subseteq (0 :_M I^n)$ and so $I^n \subseteq ann_R(0 :_M I^n) \subseteq ann_R(N :_M I^{n-1})$. Hence $J^n \subseteq I^n \cap J \subseteq J \cap ann_R(N :_M I^n) \subseteq P$ and so $J \subseteq P$, a contradiction. Thus $J \cap S \neq \emptyset$ and hence S is P-essential. □

Let M be an R-module and N be a submodule of M. The sum of all second submodules of N is called the *second radical* of N and denoted by $sec(N)$. If there is no second submodule of N, then we define $sec(N) = 0$ [14]. A subset $S \subsetneq M\backslash\{0\}$ is called an *m^*-system* if, for each ideal A of R and for all submodules $K, L \leq M$, $(0 :_{K\cap L} A) \cup S \neq M$ and $(K \cap L)A \cup S \neq M$ imply that $(K \cap L) \cup S \neq M$. In [14, Theorem 2.13] it was proved that $sec(N) = \{x \in N :$there is an m^*-system S such that $x \notin S$ and $N \cup S = M\}$. As generalizations of these concepts we define almost second radical of a submodule and almost m^*-system as follows. Then we give a characterization of almost second radical of a submodule via almost m^*-systems.

Definition 3.15. *Let M be an R-module, N be a submodule of M and*
$T := \{Q \leq N : Q$ is almost second and $(N :_M ann_R(N)) = (Q :_M ann_R(Q))\}$.
Then almost second radical of N is defined as the submodule a-sec$(N) := \sum_{Q\in T} Q$ if $T \neq \emptyset$. If $T = \emptyset$, then a-sec(N) is defined as (0).

Definition 3.16. *Let M be an R-module and S be a proper subset of M. If, for any submodules K, L of M and any ideal I of R, $(K \cap L) \cup S \neq M$, $(K \cap (0 :_M I)) \cup S \neq M$ and $(S^c :_M ann_R(S^c)) \nsubseteq (L :_M I)$ imply that $(K \cap (L :_M I)) \cup S \neq M$, then S is called an almost m^*-system.*

Proposition 3.17. *Let M be an R-module, Q be a non-zero submodule of M. Then, Q is an almost second submodule of M if and only if $S := M\backslash Q$ is an almost m^*system.*

Proof. Let Q be an almost second submodule of M. Suppose that $(K \cap L) \cup S \neq M$, $(K \cap (0 :_M I)) \cup S \neq M$ and $(S^c :_M ann_R(S^c)) \nsubseteq (L :_M I)$ where K, L are submodules of M and I is an ideal of R. Assume that $(K \cap (L :_M I)) \cup S = M$. Then $Q \subseteq K \cap (L :_M I)$. So $IQ \subseteq L$ and $Q \subseteq K$. Since Q is almost second, we have $I \subseteq ann_R(Q)$ or $Q \subseteq L$. If $I \subseteq ann_R(Q)$, then $Q \subseteq (0 :_M I)$ and so $Q \subseteq K \cap (0 :_M I)$, this contradicts with $(K \cap (0 :_M I)) \cup S \neq M$. If $Q \subseteq L$, then $Q \subseteq K \cap L$ and this contradicts with $(K \cap L) \cup S \neq M$. Therefore, $(K \cap (L :_M I)) \cup S \neq M$ and so S is an almost m^*-system.

Conversely, suppose that S is an almost m^*-system. Let $IQ \subseteq K$ and $I(Q :_M ann_R(Q)) \nsubseteq K$ for an ideal I of R and a submodule K of M. Assume that $Q \nsubseteq K$ and $I \nsubseteq ann_R(Q)$. Then $K \cup S \neq M$ and $(0 :_M I) \cup S \neq M$. Since S is an almost m^*-system $(K :_M I) \cup S \neq M$ and so $Q \nsubseteq (K :_M I)$ which is a contradiction. Therefore Q is an almost second submodule of M. □

Theorem 3.18. *Let S be an almost m^*-system in M and Q be a non-zero submodule of M minimal with respect to the properties that $Q \cup S = M$ and $(Q :_M ann_R(Q)) = (S^c :_M ann_R(S^c))$. Then Q is an almost second submodule of M.*

Proof. Let $IQ \subseteq L$ and $I(Q :_M ann_R(Q)) \nsubseteq L$ where L is a submodule of M and I is an ideal of R. Suppose that $Q \nsubseteq L$ and $I \nsubseteq ann_R(Q)$. Since $Q \nsubseteq L$, $Q \cap L \subsetneq Q$. We claim that $(Q \cap L) \cup S \neq M$. Otherwise, $S^c \subseteq Q \cap L$ and then $(Q \cap L :_M ann_R(Q \cap L)) \subseteq (Q :_M ann_R(Q \cap L)) \subseteq (Q :_M ann_R(Q)) = (S^c :_M ann_R(S^c)) \subseteq (Q \cap L :_M ann_R(S^c)) \subseteq (Q \cap L :_M ann_R(Q \cap L))$. So we have $(Q \cap L :_M ann_R(Q \cap L)) = (S^c :_M ann_R(S^c))$, a contradiction with the minimality of Q. Thus $(Q \cap L) \cup S \neq M$. Similarly, since $I \nsubseteq ann_R(Q)$, we have $(Q \cap (0 :_M I)) \cup S \neq M$. Since S is an almost m^*-system, we have $(Q \cap (L :_M I)) \cup S \neq M$. Since $IQ \subseteq L$, we have $Q \subseteq (L :_M I)$ and so $Q \cup S \neq M$, a contradiction. □

Theorem 3.19. *Let M be an R-module and N be a submodule of M. If there exists an almost second submodule Q of N with $(N :_M ann_R(N)) = (Q :_M ann_R(Q))$, then*
a-sec$(N) = \{x \in N : x \notin S, N \cup S = M$ and $(N :_M ann_R(N)) = (S^c :_M ann_R(S^c))$ for some almost m^*-system S in $M\}$.

Proof. Denote $\sqrt[a-s]{N} := \{x \in N : x \notin S, N \cup S = M$ and $(N :_M ann_R(N)) = (S^c :_M ann_R(S^c))$ for some almost m^*-system S in $M\}$. Take an element $x \in \sqrt[a-s]{N}$. Consider the set $\Psi = \{Q \subseteq N : Q \cup S = M, (Q :_M ann_R(Q)) = (S^c :_M ann_R(S^c))\}$. Since $N \in \Psi$, we have $\Psi \neq \emptyset$. Ψ is a partially ordered set with respect to reverse inclusion. Let $\{Q_i\}_{i \in I}$ be a chain in Ψ. Clearly, $(\cap_{i\in I} Q_i) \cup S = M$. Also, $(\cap_{i\in I} Q_i :_M ann_R(\cap_{i\in I} Q_i)) = \cap_{i\in I}(Q_i :_M ann_R(\cap_{i\in I} Q_i)) \subseteq \cap_{i\in I}(Q_i :_M ann_R(Q_i)) = (S^c :_M ann_R(S^c)) \subseteq (\cap_{i\in I} Q_i :_M ann_R(S^c)) \subseteq (\cap_{i\in I} Q_i :_M ann_R(\cap_{i\in I} Q_i))$ and so $(S^c :_M ann_R(S^c)) = (\cap_{i\in I} Q_i :_M ann_R(\cap_{i\in I} Q_i))$. Thus $\cap_{i\in I} Q_i \in \Psi$ and it is an upper bound for Ψ. By Zorn's Lemma, Ψ has a minimal element Q with respect to inclusion. By Theorem 3.19, Q is an almost second submodule of M. Since $x \in Q \subseteq$ a-sec(N), we have $\sqrt[a-s]{N} \subseteq$ a-sec(N). Let Q be an almost second submodule of N with $(Q :_M ann_R(Q)) = (N :_M ann_R(Q))$. By Proposition 3.17, $S := M \backslash Q$ is an m^*-system. Also, $N \cup S = M$ and $x \notin S$ for every $x \in Q$. Thus $Q \subseteq \sqrt[a-s]{N}$ and so a-sec$(N) \subseteq \sqrt[a-s]{N}$. □

References

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules. New York: Springer-Verlag, 1974.
[2] D. D. Anderson, M. Bataineh, Generalizations of prime ideals, Comm. Algebra, (2008) 36 (2), 686–696.
[3] D. D. Anderson, E. Smith, Weakly prime ideals, Houston J. of Math, (2003) 29: 831–840.
[4] H. Ansari-Toroghy, F. Farshadifar, The dual notion of multiplication modules, Taiwanese J. Math. (2007) 11 (4), 1189–1201.
[5] H. Ansari-Toroghy, F. Farshadifar, Product and dual product of submodules, Far East J. Math. Sci. (2007) 25 (3), 447–455.
[6] H. Ansari-Toroghy, F. Farshadifar, Strong comultiplication modules, CMU. J. Nat. Sci. (2009). 8 (1), 105–113.
[7] H. Ansari-Toroghy, F. Farshadifar, The dual notions of some generalizations of prime submodules, Comm. Algebra, (2011) 39 (7), 2396-2416.
[8] H. Ansari-Toroghy and F. Farshadifar, Fully idempotent and coidempotent modules, Bull. Iranian Math. Soc. 38 (2012), no. 4, 987–1005.
[9] H. Ansari-Toroghy, F. Farshadifar, On the dual notion of prime submodules, Algebra Colloq., 19 (1) (2012), 1109-1116.
[10] M. Behboodi, O. A. S. Karamzadeh, and H. Koohy, Modules Whose Certain Submodules Are Prime, Vietnam Journal of Mathematics 32 (3), (2004) 303–317.
[11] P. Karimi Beiranvand and R. Beyranvand, Almost prime and weakly prime submodules, Journal of Algbera and Its Applications Vol. 18, No. 07, 1950129 (2019).

[12] S. M. Bhatwadekar, P. K. Sharma, Uniqe factorization and birth of almost primes, Comm. Algebra, (2005) 33, 43–49.
[13] S. Çeken, M. Alkan and P. F. Smith, Second modules over noncommutative rings, Communications in Algebra, 41 (1) (2013), 83-98.
[14] S. Çeken, M. Alkan and P. F. Smith, The dual notion of the prime radical of a module, Journal of Algebra, 392 (2013), 265–275.
[15] S. Çeken, M. Alkan, On Second Submodules, Contemporary Mathematics, 634, 67-77, 2015.
[16] S. Çeken, M. Alkan, On the Weakly Second Spectrum of a Module, Filomat, (2016) 30 (4), 1013-1020.
[17] S. Çeken, M. Alkan, On Strongly 2-Absorbing Second Submodules, AIP Conference Proceedings 1991, 020003 (2018); https://doi.org/10.1063/1.5047876.
[18] S. Çeken, On The Interior of a Submodule With Respect to a Set of Ideal, Quaestiones Mathematicae, (2019) 42 (4), 431-446.
[19] M. Ebrahimpour, R. Nekooei, On generalizations of prime ideals, Comm. Algebra (2012) 40 (4), 1268–1279.
[20] M. Ebrahimpour, R. Nekooei, On generalizations of prime submodules, Bull. Iranian Math. Soc. (2013) 39 (5), 919–939.
[21] M. Ebrahimpour, On generalizations of prime ideals (II), Comm. Algebra (2014) 42 (9), 3861–3875.
[22] F. Farshadifar, H. Ansari-Toroghy, ψ-second submodules of a module, Beitr Algebra Geom (2020) 61, 571–578.
[23] E. H. Feller, E. W. Swokowski, Prime modules, Canad. J. Math., (1965) 17, 1041-1052.
[24] N. S. Karamzadeh, O. A. S. Karamzadeh, On Artinian Modules Over Duo Rings, Communications in Algebra, (2010) 38, 3521–3531.
[25] H. A. Khashan, On almost prime submodules, Acta Math. Sci. B (2012) 32 (2), 645–651.
[26] M. D. Larsen, P. J. McCarthy, Multiplicative Theory of Ideals. Pure and Applied Mathematics, Vol. 43. New York/London: Academic Press, 1971.
[27] C. P. Lu, Prime submodules of modules, Comment. Math. Univ. St. Paul, (1984) 33 (1), 61-69.
[28] S. Moradi, A. Azizi, On n-almost prime submodules, Indian J. Pure and Appl. Math., 44 (5) (2013), 605–619
[29] S. Moradi, A. Azizi, Generalizations of prime submodules, Hacettepe Journal of Mathematics and Statistics, (2015) 44 (3), 587-595.
[30] R. Wisbauer, Foundations of Modules and Rings Theory, Philadelphia: Gordon and Breach, 1991.
[31] S. Yassemi, The dual notion of prime submodules, Arch. Math (Brno) 37 (2001), 273–278.
[32] N. Zamani, Φ-prime submodules, Glasg. Math. J. (2010) 52 (2), 253-259.