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On Statistical and Strong Convergence with Respect to a Modulus Function and a Power Series Method

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Abstract. This paper introduces and focuses on two pairs of concepts in two main sections. The first section aims to examine the relation between the concepts of strong J_p-convergence with respect to a modulus function f and J_p-statistical convergence, where J_p is a power series method. The second section introduces the notions of f-J_p-statistical convergence and f-strong J_p-convergence and discusses some possible relations among them.

1. Introduction and Preliminaries

The concept of statistical convergence was initially presented by Fast [10] and Steinhaus [25] indepently and it has received much attention over the last three decades. Especially the papers [6, 8, 12–14, 16, 22, 24] has provided major contributions on this concept to be an important field of occupation for the researchers. In fact, the idea of statistical convergence is based on density of subsets of natural numbers. More details including some new kinds of densities and corresponded types of statistical convergence can be found in several studies, for instance in [2–4, 9, 17, 20, 21].

Strong Cesàro convergence with respect to a modulus function was introduced by Maddox [19]. Connor [7] extended this idea by replacing Cesàro matrix with a nonnegative regular matrix A and he proved that A-statistical convergence involves strong A-summability with respect to a modulus and further these notions are equivalent for bounded sequences. Connor also established the relationship between statistical convergence and strong Cesàro convergence in his earlier paper [6]: A real sequence is strongly Cesàro convergent if and only if it is statistical convergent and bounded. Khan and Orhan [15] improved this result by repla-
cing the boundedness condition with a strictly weaker condition called uniform integrability. Ünver and Orhan [27] has recently introduced the notions of statistical convergence, strong convergence and uniform integrability of a sequence defined by a power series method and established the similar relationship in the power series method setting.

By using the modulus functions, Aizpuru et al. [1] introduced the concept of f-statistical convergence which depends on the other new concept of f-density of natural numbers (where f is a modulus function).

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It is shown that statistical convergence encompasses f-statistical convergence. León-Saavedra et. al. [18] defined the notion of f-strongly convergence by means of modulus functions. They proved that if a sequence is f-strongly convergent then it is f-statistically convergent and uniformly integrable, and the converse statement is true when f is compatible modulus function. Such type of modulus functions are those for which the concepts of statistical convergence and f-statistical convergence are equivalent.

The present paper is motivated by the above-mentioned papers and it is divided into two main sections. In both of them, we will consider the power series method J_p, that is a sequence-to-function transformation.

The second section introduces the concept of strong J_p-convergence with respect to a modulus function and examines its relation with J_p -statistical convergence. We show that J_p-statistical convergence strictly includes strong J_p-convergence with respect to a modulus f, and these two concepts are equivalent in the context of f-J_p -uniformly integrable sequences.

In the third section we first define the concepts of f-J_p-density, f -J_p-statistical convergence and f-strong J_p-convergence. We prove some relations between them. For instance, we prove that f-J_p -statistical convergence (f-strong J_p-convergence) implies J_p -statistical convergence (strong J_p-convergence) and converse statements are true when f is a compatible modulus function. Also we will prove that when f is compatible, any real sequence is f-strongly J_p -convergent if and only if it is f-J_p-statistically convergent and J_p-uniformly integrable. Our methods are in line with a variation of that used by Aizpuru et al. [1] and León-Saavedra et. al. [18] with some changes.

Now let us recall the basic concepts and facts used throughout the paper.

Let $\mathbb{N}_0$ be set of non-negative integers. Suppose throughout that the sequence (p_k), $k \in \mathbb{N}_0$, is a sequence of non-negative numbers with $p_0 > 0$, that

$$P_n = \sum_{k=0}^{n} p_k \to \infty \ \ (n \to \infty) \tag{1}$$

and that

$$p(t) = \sum_{k=0}^{\infty} p_k t^k < \infty \quad \text{for } \ 0 < t < 1 \tag{2}$$

(in other words $p(t)$ has radius of convergence $R = 1$). Let $x = (x_k)$, $k \in \mathbb{N}_0$, be a sequence of real numbers. Then the power series method J_p is defined as follows:

$x_k \to L\left(J_p\right)$, that is (x_k) is summable to the number L by the power series method J_p (or (x_k) is said to be J_p-convergent to L) if

$$p_x(t) = \sum_{k=0}^{\infty} p_k t^k x_k$$

is convergent for $0 < t < 1$ and

$$\lim_{t\to 1^-} \frac{p_x(t)}{p(t)} = L.$$

We say that the J_p-method is regular if $x_k \to L$ implies $x_k \to L\left(J_p\right)$. It is known that the condition (1) or equivalently the condition $p(t) \to \infty$ (as $t \to 1^-$) ensures the regularity of the method J_p (see, [5]). So, by the assumption (1), we only consider regular J_p-methods.

A set $E \subset \mathbb{N}_0$ is said to have usual (or natural) density $\delta(E)$, if the limit

$$\delta(E) = \lim_{n\to\infty} \frac{|E(n)|}{n+1}$$

exists, where $E(n) = \{k \leq n : k \in E\}$ and $|E|$ denotes the cardinality of the set E [11]. The number sequence (x_k) is said to be statistically convergent to the number L, and denoted by $st\text{-}\lim x = L$, if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

i.e. $\delta(E_\varepsilon) = 0$, where $E_\varepsilon = \{k \in \mathbb{N}_0 : |x_k - L| \geq \varepsilon\}$ and hereafter this set will always be denoted by E_ε.

The ideas of strong convergence, density and statistical convergence with respect to general power series methods, namely, in the case $p(t) = \sum_{k=0}^{\infty} p_k t^k$ has radius of convergence $R \in (0, \infty]$, are introduced by Ünver and Orhan [27] and they called them as P_p-strong convergence, P_p-density and P_p-statistical convergence, respectively. Note that if $0 < R < \infty$ then it is sufficient to consider the case $R = 1$, since we may replace (p_k) with $\left(p_k R^k\right)$ (see [5], Remark 3.6.3). For the sake of simplicity, in this paper, we only deal with the case $R = 1$, and note that similar ideas can be adapted to the case $R = \infty$. So we will use the notation J_p instead of P_p.

A real sequence $x = (x_k)$ is said to be strongly J_p-convergent to the number L if

$$\lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k |x_k - L| = 0.$$

Denote the set of all strongly J_p-convergent sequences by $w\left(J_p\right)$, and by $w_0\left(J_p\right)$ if $L = 0$.

Let $E \subset \mathbb{N}_0$ be any set. If the limit

$$\delta_{J_p}(E) = \lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k\in E} p_k t^k$$

exits, then $\delta_{J_p}(E)$ is called J_p-density of E. From the definition it is clear that if $\delta_{J_p}(E)$ exists, then $0 \leq \delta_{J_p}(E) \leq 1$ and $\delta_{J_p}(E) = 1 - \delta_{J_p}(\mathbb{N}_0 \backslash E)$. If E is finite, then $\delta_{J_p}(E) = 0$. Also if $E_1 \subset E_2$ and $\delta_{J_p}(E_i)$ $(i = 1, 2)$ exist, then $\delta_{J_p}(E_1) \leq \delta_{J_p}(E_2)$. Note that J_p-density and natural density of any $E \subset \mathbb{N}_0$ need not to be equal to each other. For instance, let $(p_k) = (1, 0, 1, 0, \ldots)$. Then $p(t) = \sum_{k=0}^{\infty} t^{2k} = 1/\left(1 - t^2\right)$ for $0 < t < 1$. Now if $E = \{2k + 1 : k \in \mathbb{N}_0\}$, then $\delta_{J_p}(E) = 1/2$ but $\delta(E) = 0$ (see [27]). Also note that in case $p_k = 1$ for all k, J_p-density is called Abel density introduced by Ünver in [26].

The sequence $x = (x_k)$ is said to be J_p-statistically convergent to L if for any $\varepsilon > 0$

$$\lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k\in E_\varepsilon} p_k t^k = 0,$$

that is $\delta_{J_p}(E_\varepsilon) = 0$ for any $\varepsilon > 0$. In this case, we write $st_{J_p}\text{-}\lim x = L$. The set of all J_p-statistically convergent sequences will be denoted by st_{J_p}. Note that regularity of J_p-method requires the regularity of J_p-statistical convergence, i.e. $\lim x = L$ implies $st_{J_p}\text{-}\lim x = L$. However, the converse is not true in general. For example, let $(p_k) = (1, 0, 1, 0, \ldots)$ and $(x_k) = (0, 1, 0, 1, \ldots)$, then $st_{J_p}\text{-}\lim x = 0$, but x is not convergent. On the other hand, statistical convergence and J_p-statistical convergence are incompatible methods.

Ünver and Orhan also defined the concept of uniform integrability of sequences with respect to a power series method: The sequence (x_k) is J_p-uniformly integrable if there exists $t_0 \in [0, 1)$ such that

$$\lim_{c\to\infty} \sup_{t\in[t_0,1)} \frac{1}{p(t)} \sum_{|x_k|\geq c} p_k t^k |x_k| = 0.$$

Any bounded sequence is J_p-uniformly integrable but not conversely (see [27], Example 2). This notion and the following result will play a key role to obtain more general results in the second and third sections.

Theorem 1.1. *[27] Let $x = (x_k)$ be a real sequence. Then the following are equivalent.*
(i) x is strongly J_p-convergent to L.
(ii) x is J_p-statistically convergent to L and J_p-uniformly integrable.

Recall that a modulus function ([23]) f is a function from $[0,+\infty)$ to $[0,+\infty)$ such that (i) $f(x)=0$ if and only if $x=0$, (ii) $f(x+y)\leq f(x)+f(y)$ for all $x,y\geq 0$, (iii) f is increasing, and (iv) f is continuous from the right at zero. A modulus function can be bounded or unbounded. Some examples of modulus functions are $f(x)=x^p$ $(0<p\leq 1)$, $f(x)=\log(x+1)$, $f(x)=x+\log(x+1)$ and $f(x)=x/(1+x)$.

2. Strong J_p-Convergence with respect to a Modulus and J_p-Statistical Convergence

In this section, we first extend the notion of strong J_p-convergence by using a modulus function in the same way as Connor [7]. Then we present a relationship between this notion and the notion of J_p-statistical convergence.

Definition 2.1. Let f be a modulus function and $x=(x_k)$ be a sequence of real numbers. The sequence x is said to be strongly J_p-convergent with respect to the modulus function f if

$$\lim_{t\to 1^-}\frac{1}{p(t)}\sum_{k=0}^{\infty}p_k t^k f(|x_k-L|)=0.$$

The set of all strongly J_p-convergent sequences with respect to the modulus function f is denoted by $w\left(J_p,f\right)$. In particular, when $L=0$, we prefer to write $w_0\left(J_p,f\right)$ instead of $w\left(J_p,f\right)$.

Note that if $f(x)=x$, then the sets $w\left(J_p,f\right)$ and $w_0\left(J_p,f\right)$ are reduced to $w\left(J_p\right)$ and $w_0\left(J_p\right)$, respectively.

Theorem 2.2. *For any modulus f, strongly J_p-convergence implies strongly J_p-convergence with respect to f (to the same limit), i.e. $w\left(J_p\right)\subset w\left(J_p,f\right)$.*

Proof. Assume that $x\in w\left(J_p\right)$ with limit L. Then

$$p_x(t)=\frac{1}{p(t)}\sum_{k=0}^{\infty}p_k t^k|x_k-L|\to 0\ \ (t\to 1^-).$$

Let $\varepsilon>0$. By the continuity of f from right at $t=0$, we can select a number δ with the property $0<\delta<1$ such that $f(t)<\varepsilon$ for all $0<t\leq\delta$. Let

$$y_k:=|x_k-L|\ \text{ and }\ p_x(f,t):=\frac{1}{p(t)}\sum_{k=0}^{\infty}p_k t^k f(y_k).$$

Then

$$p_x(f,t)=\frac{1}{p(t)}\sum_{\substack{k=0\\ y_k\leq\delta}}^{\infty}p_k t^k f(y_k)+\frac{1}{p(t)}\sum_{\substack{k=0\\ y_k>\delta}}^{\infty}p_k t^k f(y_k)=:\Sigma_1+\Sigma_2.$$

If $y_k\leq\delta$, then $f(y_k)<\varepsilon$ and hence $\Sigma_1<\varepsilon$. Now let $y_k>\delta$ and $[t]$ be the integral part of the number t. Since $y_k<(y_k/\delta)<[(y_k/\delta)+1]$, we have

$$f(y_k)\leq\left[\frac{y_k}{\delta}+1\right]f(1)\leq 2f(1)\frac{y_k}{\delta}.$$

Then from the properties of the modulus function (iii) and (ii) we obtain $\Sigma_2\leq 2f(1)\delta^{-1}p_x(t)$. Hence, we get

$$p_x(f,t)<\varepsilon+2f(1)\delta^{-1}p_x(t).$$

Letting $t\to 1^-$ in this inequality, we conclude that $x\in w\left(J_p,f\right)$. □

The following characterization concerning the ideals in ℓ_∞, where as usual ℓ_∞ is the set of all bounded sequences, was given in [7], and it will be useful for the proof of our next result.

Lemma 2.3. Let $x \in \ell_\infty$ and M be an ideal in ℓ_∞. Then x belongs to the closure of M if and only if $\chi_{E_{\varepsilon,0}} \in M$ for all $\varepsilon > 0$, where χ_E denotes the characteristic function of the set E and $E_{\varepsilon,0} := \{k \in \mathbb{N}_0 : |x_k| \geq \varepsilon\}$.

Lemma 2.4. Let f be any modulus function. Then $w_0\left(J_p, f\right) \cap \ell_\infty$ is an ideal in ℓ_∞. In particular, $w_0\left(J_p\right) \cap \ell_\infty$ is an ideal in ℓ_∞.

Proof. Let $x \in w_0\left(J_p, f\right)$ and $y \in \ell_\infty$. Since $y \in \ell_\infty$, there is a $M \in \mathbb{Z}^+$ such that $\left|y_k\right| \leq M$ for each $k \in \mathbb{N}_0$. Hence, we have $f\left(\left|x_k y_k\right|\right) \leq f\left(M\left|x_k\right|\right) \leq Mf\left(\left|x_k\right|\right)$ for all k, thus we obtain

$$\frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k f\left(\left|x_k y_k\right|\right) \leq \frac{M}{p(t)} \sum_{k=0}^{\infty} p_k t^k f\left(\left|x_k\right|\right).$$

Letting $t \to 1^-$ in this inequality, we conclude that $xy \in w_0\left(J_p, f\right)$. This completes the proof of lemma. □

Lemma 2.5. $w_0\left(J_p\right) \cap \ell_\infty$ is a closed ideal in ℓ_∞.

Proof. From the Lemma 2.4, it is enough to prove that $w_0\left(J_p\right) \cap \ell_\infty$ is closed in ℓ_∞. Let $x = (x_k)$ be any sequence in the closure of $w_0\left(J_p\right) \cap \ell_\infty$. Then there exists a sequence (x^n) in $w_0\left(J_p\right) \cap \ell_\infty$ such that

$$\left\|x^{(n)} - x\right\|_\infty = \sup_k \left|x_k^{(n)} - x\right| \to 0 \ (n \to \infty).$$

For any $\varepsilon > 0$, choose any $N \in \mathbb{N}$ such that $\left\|x^{(N)} - x\right\|_\infty < \varepsilon$. Then we have

$$\begin{aligned} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k \ \left|x_k\right| &\leq \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k \left|x_k^{(N)} - x_k\right| + \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k \left|x_k^{(N)}\right| \\ &\leq \varepsilon + \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k \left|x_k^N\right|. \end{aligned}$$

Hence we get $x \in w_0\left(J_p\right)$ by letting $t \to 1^-$ in this inequality. So $w_0\left(J_p\right) \cap \ell_\infty$ is a closed ideal in ℓ_∞. □

Theorem 2.6. *Let f be any modulus function. Then $w\left(J_p, f\right) \cap \ell_\infty = w\left(J_p\right) \cap \ell_\infty$.*

Proof. It is sufficient to prove that $w_0\left(J_p, f\right) \cap \ell_\infty = w_0\left(J_p\right) \cap \ell_\infty$. We have $w_0\left(J_p\right) \cap \ell_\infty \subset w_0\left(J_p, f\right) \cap \ell_\infty$ from Theorem 2.2. Now let $x \in w_0\left(J_p, f\right) \cap \ell_\infty$ and $\varepsilon > 0$. Define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} \frac{1}{x_k} & , \left|x_k\right| \geq \varepsilon \\ 0 & , \text{otherwise.} \end{cases}$$

Since $y \in \ell_\infty$, we have $xy = \chi_{E_{\varepsilon,0}} \in w_0\left(J_p, f\right) \cap \ell_\infty$ from Lemma 2.4. Thus

$$\lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k f\left(\chi_{E_{\varepsilon,0}}(k)\right) = 0. \tag{3}$$

Since

$$\frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k f\left(\chi_{E_{\varepsilon,0}}(k)\right) = \frac{f(1)}{p(t)} \sum_{k=0}^{\infty} p_k t^k \chi_{E_{\varepsilon,0}}(k), \tag{4}$$

according to the definition of modulus function and characteristic function, we have $\chi_{E_{\varepsilon,0}} \in w_0\left(J_p\right)$ from (3) and (4). Hence $\chi_{E_{\varepsilon,0}} \in w_0\left(J_p\right) \cap \ell_\infty$. Thus, we have $x \in w_0\left(J_p\right) \cap \ell_\infty$ by Lemma 2.3 and Lemma 2.5. This completes the proof. □

Motivated by [27] we can give the following definition.

Definition 2.7. *Let f be any modulus function. Then a sequence (x_k) is said to be f-J_p-uniformly integrable if there exists $t_0 \in [0,1)$ such that*

$$\lim_{c\to\infty} \sup_{t\in[t_0,1)} \frac{1}{p(t)} \sum_{f(|x_k|)\geq c} p_k t^k f(|x_k|) = 0.$$

The following theorem is an extension of Theorem 1.1 and it characterizes strongly J_p-convergence with respect to a modulus via J_p-statistically convergence.

Theorem 2.8. *Let f be any modulus function and $x = (x_k)$ be a real sequence. Then the following are equivalent.*
(i) x is strongly J_p-convergent to L with respect to f.
(ii) x is J_p-statistically convergent to L and f-J_p-uniformly integrable.

Proof. $(i) \Rightarrow (ii)$. Let $x \in w\left(J_p, f\right)$ with limit L, that is

$$\lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k f(|x_k - L|) = 0.$$

Then for any given $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k f(|x_k - L|) &\geq \frac{1}{p(t)} \sum_{k\in E_\varepsilon} p_k t^k f(|x_k - L|) \\ &\geq \frac{f(\varepsilon)}{p(t)} \sum_{k\in E_\varepsilon} p_k t^k, \end{aligned}$$

since f is increasing. If we take the limit for $t \to 1^-$ in this inequality, we get $st_{J_p}\text{-}\lim x = L$. Letting $y_k := f(|x_k|)$, one can get from Theorem 1.1 that x is f-J_p-uniformly integrable.
$(ii) \Rightarrow (i)$. Assume that $st_{J_p}\text{-}\lim x = L$ and x is f-J_p-uniformly integrable. Let $\varepsilon > 0$. First observe that if $|x_k - L| \geq \varepsilon$ implies that $f(|x_k - L|) \geq f(\varepsilon)$. On the other hand, $\lim_{\varepsilon\to 0^+} f(\varepsilon) = 0$ since f is continuous at zero. This implies that any J_p-statistically convergent sequence satisfies the condition

$$\lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k\in E'_\varepsilon} p_k t^k = 0 \tag{5}$$

where $E'_\varepsilon = \{k \in \mathbb{N}_0 : f(|x_k - L|) \geq f(\varepsilon)\}$. Now, f-J_p-uniformly integrability and (5) imply by Theorem 1.1 that x is strongly J_p-convergent to L with respect to f. This completes the proof. □

Remark 2.9. The condition of f-J_p-uniformly integrability can not be omitted in Theorem 2.8. Indeed, let $f(x) = x$ and define (p_k) and an unbounded sequence $x = (x_k)$ by

$$p_k = \begin{cases} \frac{1}{k} & , k = 2j+1 \\ 1 & , k = 2j \end{cases} \quad , j = 0,1,2,\ldots$$

and

$$x_k = \begin{cases} k & , k = 2j+1 \text{ or } k = 0,\ j = 0,1,2,\ldots \\ \frac{1}{k} & , k = 2j,\ j = 1,2,\ldots \end{cases}$$

respectively. In this case,

$$p(t)=\sum_{k=0}^{\infty}p_k t^k=\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)+\frac{1}{1-t^2}$$

for $0<t<1$ and then we get $\delta_{J_p}(E_1)=1$ and $\delta_{J_p}(E_2)=0$ for the sets $E_1:=\{2j:j\in\mathbb{N}_0\}$ and $E_2:=\{2j+1:j\in\mathbb{N}_0\}$. Since

$$\{k\in\mathbb{N}_0:|x_k|\geq\varepsilon\}\subset E_2\cup\{\text{finite set}\}$$

for all $\varepsilon>0$, we have $\delta_{J_p}(\{k\in\mathbb{N}_0:|x_k|\geq\varepsilon\})=0$. Hence $st_{J_p}\text{-}\lim x=0$. However, since

$$\begin{aligned}\lim_{t\to1^-}\frac{1}{p(t)}\sum_{k=0}^{\infty}p_k t^k|x_k| &=\lim_{t\to1^-}\frac{1}{p(t)}\left(\sum_{k\in E_1}p_k t^k|x_k|+\sum_{k\in E_2}p_k t^k|x_k|\right)\\&=\lim_{t\to1^-}\frac{1}{p(t)}\left(-\ln\left(1-t^2\right)+\frac{t}{1-t^2}\right)=1\neq0,\end{aligned}$$

x is not strongly J_p-convergent to the number $L=0$ with respect to f. On the other hand, for any $t_0\in[0,1)$ we have

$$\begin{aligned}\sup_{t\in[t_0,1)}\frac{1}{p(t)}\sum_{|x_k|\geq c}p_k t^k|x_k| &= \sup_{t\in[t_0,1)}\frac{1}{p(t)}\left(\sum_{|x_{2k}|\geq c}p_{2k}t^{2k}|x_{2k}|+\sum_{|x_{2k+1}|\geq c}p_{2k+1}t^{2k+1}|x_{2k+1}|\right)\\&\geq \sup_{t\in[t_0,1)}\frac{c}{p(t)}\left(-\ln\left(1-t^2\right)+\frac{t}{1-t^2}\right)\\&= \lim_{t\to1^-}\frac{c}{p(t)}\left(-\ln\left(1-t^2\right)+\frac{t}{1-t^2}\right)=c\to\infty,\quad(c\to\infty)\end{aligned}$$

where we can replace *sup* with $\lim_{t\to1^-}$, since the ratio $\frac{c}{p(t)}\left(-\ln\left(1-t^2\right)+\frac{t}{1-t^2}\right)$ is an increasing function of t on the interval $[t_0,1)$. Thus (x_k) is not f-J_p-uniformly integrable.

3. f-Strong J_p-Convergence and f-J_p-Statistical Convergence

Let f be any unbounded modulus function. The f-density of a set $E\subset\mathbb{N}_0$ is defined by

$$\delta^f(E)=\lim_{n\to\infty}\frac{f(|E(n)|)}{f(n+1)}$$

if the limit exists. A sequence $x=(x_k)$ is said to be f-statically convergent to L if for each $\varepsilon>0$

$$\delta^f(E_\varepsilon)=\lim_{n\to\infty}\frac{f(|\{k\leq n:|x_k-L|\geq\varepsilon\}|)}{f(n+1)}=0$$

(see, [1]). It is also known from [1] that any f-statistically convergent sequence is also statistically convergent but not conversely. We also recall that a sequence $x=(x_k)$ is said to be f-strongly Cesàro convergent to L if

$$\lim_{n\to\infty}\frac{f\left(\sum_{k=0}^{n}|x_k-L|\right)}{f(n+1)}=0$$

(see, [18]). We remark here that if f is bounded modulus function, then these definitions hold only for trivial cases (for empty set and constant sequences).

Throughout this section, we only consider the unbounded modulus functions as in [1] and [18]. We first define the concept of f-J_p-density of subsets of $\mathbb{N}_0$ and f-J_p-statistically convergence for any real sequence. After that some inclusion relations will be investigated.

Definition 3.1. Let f be an unbounded modulus function and $E \subset \mathbb{N}_0$. If the limit

$$\delta_{J_p}^{f}(E) := \lim_{t \to 1^-} \frac{1}{f(p(t))} f\left(\sum_{k \in E} p_k t^k\right)$$

exits, then $\delta_{J_p}^{f}(E)$ is called f-J_p-density of E.

Definition 3.2. Let f be an unbounded modulus function and $x = (x_k)$ be a sequence of real numbers. The sequence (x_k) is said to be f-J_p-statistically convergent to L if for any $\varepsilon > 0$,

$$\lim_{t \to 1^-} \frac{1}{f(p(t))} f\left(\sum_{k \in E_\varepsilon} p_k t^k\right) = \lim_{t \to 1^-} \frac{1}{f(p(t))} f\left(\sum_{k=0}^{\infty} p_k t^k \chi_{E_\varepsilon}(k)\right) = 0,$$

that is $\delta_{J_p}^{f}(E_\varepsilon) = 0$ for each $\varepsilon > 0$. In this case we write f-st_{Jp}-$\lim x = L$.

If $f(x) = x$ in these definitions, then we have the concepts of J_p-density and J_p-statistical convergence, respectively. It is clear that $0 \le \delta_{J_p}^{f}(E) \le 1$ for any $E \subset \mathbb{N}_0$. If E is any finite set, then $\delta_{J_p}^{f}(E) = 0$. For this, let $E = \{n(j) : j = 1, 2, \ldots, k;\ \text{for some } k \in \mathbb{N}\}$. Then $f\left(\sum_{j \in E} p_j t^j\right) \le \sum_{j=1}^{k} f\left(p_{n(j)} t^{n(j)}\right) \le \sum_{j=1}^{k} f\left(p_{n(j)}\right)$ for $0 < t < 1$. From this, we have

$$0 \le \frac{1}{f(p(t))} f\left(\sum_{j \in E} p_j t^j\right) \le \frac{1}{f(p(t))} \sum_{j=1}^{k} f\left(p_{n(j)}\right) \to 0 \quad (\text{as } t \to 1^-).$$

Hence $\delta_{J_p}^{f}(E) = 0$. Therefore, if $\lim x_k = L$, then f-st_{Jp}-$\lim x_k = L$. In other words, f-J_p-statistical convergence is regular.

If $\delta_{J_p}^{f}(E) = 0$, then $\delta_{J_p}^{f}(\mathbb{N}_0 \backslash E) = 1$. Indeed, since

$$1 = \frac{f\left(\sum_{k=0}^{\infty} p_k t^k\right)}{f(p(t))} \le \frac{f\left(\sum_{k \in E} p_k t^k\right)}{f(p(t))} + \frac{f\left(\sum_{\mathbb{N}_0 \backslash E} p_k t^k\right)}{f(p(t))} \le \frac{f\left(\sum_{k \in E} p_k t^k\right)}{f(p(t))} + 1,$$

by taking limit as $t \to 1^-$, we deduce that $\delta_{J_p}^{f}(\mathbb{N}_0 \backslash E) = 1$. On the other hand, analogously to f-density, the converse is not true in general. For instance, let $f(x) = \log(1 + x)$, $(p_k) = (1, 1, 1, \ldots)$ and $E = \{2k : k \in \mathbb{N}_0\}$. Then

$$\delta_{J_p}^{f}(\mathbb{N}_0 \backslash E) = \lim_{t \to 1^-} \frac{\log\left(1 + t/(1 - t^2)\right)}{\log(1 + 1/(1 - t))} = 1$$

and

$$\delta_{J_p}^{f}(E) = \lim_{t \to 1^-} \frac{\log\left(1 + 1/(1 - t^2)\right)}{\log(1 + 1/(1 - t))} = 1.$$

This also means that the sequence $(\chi_E(k))$ is not f-J_p-statistical convergent.

The following example exhibits that the concepts of f-J_p-statistical convergence and f-statistical convergence can not be compared.

Example 3.3. Let $f(x) = \log(1+x)$, J_p-method be determined by the sequence

$$p_k = \begin{cases} 1 & , k = n^2 \\ 0 & , \text{otherwise} \end{cases}, n \in \mathbb{N}_0$$

and consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 0 & , k = n^2 \\ 1 & , \text{otherwise} \end{cases}, n \in \mathbb{N}_0.$$

Then for any $\varepsilon > 0$, observe that

$$\lim_{t\to 1^-} \frac{1}{f(p(t))} f\left(\sum_{|x_k|\geq\varepsilon} p_k t^k\right) = 0.$$

Hence, f-st_{Jp}-$\lim x = 0$. Also we know from Example 2.1 in [1] that x is not f-statistically convergent. On the other hand, for the same (p_k), if we take $f(x) = x$ and consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1 & , k = n^2 \\ 0 & , \text{otherwise} \end{cases}, n \in \mathbb{N}_0.$$

we see that x is f-statistically convergent to 0, but not f-J_p-statistically convergent.

Note that for any unbounded modulus f and $E \subset \mathbb{N}_0$, $\delta^f_{J_p}(E) = 0$ implies $\delta_{J_p}(E) = 0$. Indeed, if $\delta^f_{J_p}(E) = 0$ then for each $n \in \mathbb{N}$ we can choose δ_n with $0 < \delta_n < 1$ such that if $0 < t < 1 - \delta_n$, then

$$f\left(\sum_{k\in E} p_k t^k\right) < \frac{1}{n} f(p(t)) \leq \frac{1}{n} n f\left(\frac{1}{n} p(t)\right) = f\left(\frac{1}{n} p(t)\right). \tag{6}$$

From this, we get $\sum_{k\in E} p_k t^k \leq (1/n) p(t)$ for the same t's, hence $\delta_{J_p}(E) = 0$. This observation leads naturally to the following corollary.

Corollary 3.4. *Let f, g be unbounded modulus functions and (x_k) be a sequence of real numbers. Then,*
(i) f-J_p-statistical convergence implies J_p-statistical convergence with the same limit.
(ii) f-J_p-statistical limit is unique whenever it exists.
(iii) If f-st_{Jp}-$\lim x_k = L$ and g-st_{Jp}-$\lim x_k = M$ then $L = M$.

Definition 3.5. A sequence (x_k) of real numbers is said to be f-strongly J_p-convergent to L if

$$\lim_{t\to 1^-} \frac{1}{f(p(t))} f\left(\sum_{k=0}^{\infty} p_k t^k |x_k - L|\right) = 0.$$

Theorem 3.6. *If (x_k) is f-strongly J_p-convergent to L, then (x_k) is strongly J_p-convergent to L.*

Proof. Assume that (x_k) is f-strongly J_p-convergent to L. Then for each $n \in \mathbb{N}$, there exists an $\delta = \delta(n)$ with $0 < \delta < 1$ such that if $0 < t < 1 - \delta$ then

$$f\left(\sum_{k=0}^{\infty} p_k t^k |x_k - L|\right) < \frac{1}{n} f(p(t)) \leq f\left(\frac{1}{n} p(t)\right).$$

from (6). Since f is increasing, we have

$$\sum_{k=0}^{\infty} p_k t^k |x_k - L| \leq \frac{1}{n} p(t) \tag{7}$$

for all $t \in (1-\delta, 1)$. Now for any $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$ such that $(1/n_0) < \varepsilon$. Since the inequality is valid for all $n \in \mathbb{N}$, it is also valid for n_0. Hence

$$\sum_{k=0}^{\infty} p_k t^k |x_k - L| \leq \frac{1}{n_0} p(t) < \varepsilon p(t)$$

for all $t \in (1-\delta, 1)$, where δ depends on n_0 and so depends on ε. From this, we obtain that (x_k) is strongly J_p-convergent to L. This completes the proof. □

Now as in [18] we define the idea of compatible modulus in a slightly modified form.

Definition 3.7. [18] Let f be a modulus function. We say that f is compatible provided for any $\varepsilon > 0$ there exist $\tilde{\varepsilon} > 0$ and $x_0 = x_0(\varepsilon)$ such that $\frac{f(x\tilde{\varepsilon})}{f(x)} < \varepsilon$ for all $x \geq x_0$.

For example, $f(x) = x + \log(x+1)$, $g(x) = x/\sqrt{1+x}$ and $h(x) = x/(\log x + e^2)$ are unbounded compatible modulus functions, where logarithm is to the natural base e. Indeed, for the last one, let $\varepsilon > 0$ and choose any $\tilde{\varepsilon} > 0$ such that $2\tilde{\varepsilon} < \varepsilon$. Since

$$\lim_{x\to\infty} \frac{x\tilde{\varepsilon}/(\log(x\tilde{\varepsilon}) + e^2)}{x/(\log x + e^2)} = \tilde{\varepsilon},$$

there exist $x_0 = x_0(\varepsilon)$ such that $\frac{h(x\tilde{\varepsilon})}{h(x)} < \varepsilon$ for all $x \geq x_0$. On the other hand the unbounded modulus function $f(x) = \log(x+1)$ is not compatible modulus (see [18]). Here, we present a new example of unbounded modulus function that is not compatible. Consider the function $f(x) = \log(\log(x+e))$ defined on the interval $[0, \infty)$. The modulus function properties hold for this function. In particular, (ii) property of subadditivity can be checked by showing that $f(x)/x$ is decreasing on $[0, \infty)$. Now, let $0 < \varepsilon < 1$. Then, since $\lim_{x\to\infty} \frac{f(x\tilde{\varepsilon})}{f(x)} = 1$ for all $\tilde{\varepsilon} > 0$, we can not find any $\tilde{\varepsilon} > 0$ such that $\frac{f(x\tilde{\varepsilon})}{f(x)} < \varepsilon$ for sufficiently large x. So we obtain that f is not compatible.

Theorem 3.8. *Let f be a compatible modulus function. If (x_k) is J_p-statistically convergent to L, then (x_k) is f-J_p-statistically convergent to L.*

Proof. Let f be a compatible modulus function and st_{J_p}-$\lim x = L$. Since f is compatible for any given $\varepsilon > 0$, there exist $\tilde{\varepsilon} > 0$ and $t_0 = t_0(\varepsilon)$ such that $\frac{f(t\tilde{\varepsilon})}{f(t)} < \varepsilon$ for all $t > t_0$. Also the assumption $p(t) \to \infty$ $(t \to 1^-)$ implies that there exists $\delta_1 = \delta_1(t_0)$ (thus $\delta_1 = \delta_1(\varepsilon)$) such that for all $t \in (1-\delta_1, 1)$ we have $p(t) > t_0$. Hence, we obtain that $\frac{f(p(t)\tilde{\varepsilon})}{f(p(t))} < \varepsilon$ for all $t \in (1-\delta_1, 1)$. Now, let $\sigma > 0$ and fix $\tilde{\varepsilon}$. Since st_{J_p}-$\lim x = L$, there exists $\delta_2 > 0$ such that

$$\sum_{k=0}^{\infty} p_k t^k \chi_{E_\sigma(k)} < p(t)\,\tilde{\varepsilon}$$

for all $t \in (1-\delta_2, 1)$. Since f is increasing, we get

$$\frac{1}{f(p(t))} f\left(\sum_{k=0}^{\infty} p_k t^k \chi_{E_\sigma(k)}\right) < \frac{f(p(t)\,\tilde{\varepsilon})}{f(p(t))} < \varepsilon$$

for all $t \in (1-\delta_0, 1)$, where $\delta_0 = \min\{\delta_1, \delta_2\}$. Thus, f-st_{J_p}-$\lim x = L$ and this completes the proof. □

With the same manner we can prove the following.

Theorem 3.9. *Let f be a compatible modulus function. If (x_k) is strongly J_p-convergent to L, then (x_k) is f-strongly J_p-convergent to L.*

Theorem 3.10. *Let f be a modulus function.*
(i) *If all J_p-statistically convergent sequences are f-J_p-statistically convergent, then f must be compatible.*
(ii) *If all strongly J_p-convergent sequences are f-strongly J_p-convergent, then f must be compatible.*

Proof. Let (x_k) be any sequence such that it is J_p-statistically convergent to L , but not f-J_p-statistically convergent to L. Then there exists $\varepsilon_0 > 0$ and a constant $\alpha > 0$ such that

$$\limsup_{t\to 1^-} \frac{1}{f(p(t))} f\left(\sum_{k=0}^{\infty} p_k t^k \chi_{E_\varepsilon}(k)\right) \geq \alpha.$$

Thus, there exists a sequence (t_n) with $t_n \in (0,1)$ for all n and $t_n \to 1^-$ such that

$$\frac{1}{f(p(t_n))} f\left(\sum_{k=0}^{\infty} p_k t_n^k \chi_{E_\varepsilon(k)}\right) \geq \alpha.$$

On the other hand, by the assumption st_{J_p}-$\lim x = L$, for all $\varepsilon > 0$ there exist $\delta > 0$ such that

$$\sum_{k=0}^{\infty} p_k t^k \chi_{E_\varepsilon(k)} < p(t)\,\varepsilon$$

for all $t \in (1-\delta, 1)$. Since f is increasing, we have

$$f\left(\sum_{k=0}^{\infty} p_k t^k \chi_{E_\varepsilon(k)}\right) < f(p(t)\,\varepsilon)$$

for all $t \in (1-\delta, 1)$. In particular for all $t_n \in (1-\delta, 1)$, we have

$$0 < \alpha \leq \frac{1}{f(p(t_n))} f\left(\sum_{k=0}^{\infty} p_k t_n^k \chi_{E_\varepsilon(k)}\right) < \frac{f(p(t_n)\,\varepsilon)}{f(p(t_n))}.$$

Thus, f is not compatible and this completes proof of (i). Since the proof of (ii) is similar to that of (i), we omit it. □

Corollary 3.11. *Let f be an unbounded modulus. Then the following statments are equivalent.*
(i) *All J_p-statistically convergent sequences are f-J_p-statistically convergent.*
(ii) *For any $E \subset \mathbb{N}_0$ if $\delta_{J_p}(E) = 0$, then $\delta^f_{J_p}(E) = 0$.*
(iii) *f is compatible.*

Theorem 3.12. *Let $x = (x_k)$ be a real sequence and f be a compatible modulus. Then the following are equivalent.*
(i) *x is f-strongly J_p-convergent to L.*
(ii) *x is f-J_p-statistically convergent to L and J_p-uniformly integrable.*

Proof. $(ii) \Rightarrow (i)$. Let x be f-J_p-statistically convergent to L and J_p-uniformly integrable. Then from Corollary 3.4, x is J_p-statistically convergent to L. By Theorem 1.1, x is strongly J_p-convergent to L. Finally, since f is a compatible modulus, x is f-strongly J_p-convergent to L by Theorem 3.9.
$(i) \Rightarrow (ii)$. Assume that x is f-strongly J_p-convergent to L. Then applying Theorem 3.6 and Theorem 1.1 we obtain that x is J_p-uniformly integrable. Now prove that x is f-J_p-statistically convergent to L. Let $\varepsilon > 0$ and choose any $n \in \mathbb{N}$ such that $(1/n) < \varepsilon$. Since $E_\varepsilon \subset E_{1/n}$ we have

$$\frac{1}{f(p(t))} f\left(\sum_{k\in E_\varepsilon} p_k t^k\right) \leq \frac{1}{f(p(t))} f\left(\sum_{k\in E_{1/n}} p_k t^k\right)$$

and so it is enough to prove that

$$\lim_{t\to 1^-}\frac{1}{f(p(t))}f\left(\sum_{k\in E_{1/n}}p_k t^k\right)=0 \tag{8}$$

for any $n\in\mathbb{N}$. Hence for any $n\in\mathbb{N}$, we can write

$$\begin{aligned} f\left(\sum_{k=0}^{\infty}p_k t^k|x_k-L|\right) &\geq f\left(\sum_{k\in E_{1/n}}p_k t^k|x_k-L|\right)\geq f\left(\frac{1}{n}\sum_{k\in E_{1/n}}p_k t^k\right)\\ &\geq \frac{1}{n}f\left(\sum_{k\in E_{1/n}}p_k t^k\right). \end{aligned}$$

From this, we have

$$\frac{1}{f(p(t))}f\left(\sum_{k\in E_{1/n}}p_k t^k\right)\leq\frac{n}{f(p(t))}f\left(\sum_{k=0}^{\infty}p_k t^k|x_k-L|\right)$$

Thus, by the assumption we obtain (8) and this completes the proof. □

Note that in the second part of proof, the modulus function f need not to be compatible. This part is valid for any unbounded modulus function. On the other hand, Remark 2.9 also shows that the condition of J_p-uniform integrability cannot be omitted in Theorem 3.12.

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