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On Some Non-Linear Contractions in Modular Metric Spaces

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Abstract. Motivated by the exciting notion of modular metric spaces, in this manuscript, we positively answer an open question posed by Mitrović et al. [Ital. J. Pure Appl. Math., 41 (2019), 679-690] on the existence of fixed points of Hardy-Rogers contractions. Moreover, in the said setting, we conceive the common fixed point theorem of Jungck. As consequences of our findings, we deduce a few fixed point and common fixed point results which authenticate the novelty of the obtained theories. Finally, we construct numerical examples to validate our study.

1. Introduction

The fascinating and intensive development on the study of modulars on various linear spaces is due to Nakano [26, 27] and some of his fellow mathematicians from his school. As the modular type hypotheses involved in the theories can be comfortably verified than that of their norm or metric counterparts, such assumptions arise quite naturally in the study of integral equations, approximation theories, the electrorheological fluids, economics and in many other courses. Therefore in recent times, the notion of modulars and modular spaces are thoroughly investigated, particularly in a variety of Orlicz spaces which has a huge applicability in diversified fields [19, 20, 24, 25, 28, 29].

In recent past, Chistyakov [11, 12] coined the idea of a new kind of modular which is not too restrictive and is also consistent with the classical concept of it. Further, this construction of a novel modular notion is more functional in complying with the questions of description of multi-valued superposition operators. One of the major motivation at the back of this newly defined modular by Chistyakov is the physical interpretation of it. Precisely, while a metric defined on a non-empty set stands for the finite distances between any two points of the set, a modular on a set associates a non-negative, at times infinite valued, field of velocities with the elements. Informally, one can correspond an average velocity $\omega_\lambda(x, y)$ to any arbitrary time $\lambda > 0$ so that it takes λ time to travel the distance between points $x, y \in X$. In the wake of such modification by Chistyakov, plenty of impressive and compelling results are done in the setting of modular metric spaces [2, 3, 5, 6, 9, 15, 21, 23].

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In this manuscript, we continue to explore this remarkable abstract space and during this process, we positively respond to an open problem raised by Mitrović et al. [23]. We here make a note of the subsequent open question.

Open Problem 1.1. *Suppose that ω is a strict convex modular on X such that the modular space X^*_ω is ω-complete. Also suppose that $T : X^*_\omega \to X^*_\omega$ is a Hardy-Rogers ω-contractive mapping such that for each $\lambda > 0$, there is $x = x(\lambda) \in X^*_\omega$ with $\omega_\lambda(x, Tx) < \infty$. Then T owns a fixed point x^* in X^*_ω. Further, if the modular ω assumes only finite values on X^*_ω, then the assumption $\omega_\lambda(x, Tx) < \infty$ is redundant, and the fixed point x^* of T is unique and for each $x_0 \in X^*_\omega$, the Picard iterates $(T^n x_0)$ is modular convergent to x^*.*

To answer this question assuredly, we come up with a fixed point result related to such contractions. However, we also confirm the Jungck common fixed point theorem in the setting of a modular metric space, which improves, complements and unifies many other results from the existing literature. To endorse our findings, we furnish numerical examples and dish out some corollaries in various metric spaces.

2. Preliminaries

Beforehand, we recollect a few necessary notions, terminologies and some notable results on the modular metric spaces. Let X be any arbitrary non-empty set and $\omega : (0, \infty) \times X \times X \to [0, \infty]$ be a mapping. Now, to keep it simple, we use the following notation throughout the article:

$$\omega_\lambda (x, y) = \omega (\lambda, x, y),$$

for each $\lambda > 0$ and $x, y \in X$. Firstly, we put down the basic definition of a modular metric space.

Definition 2.1. *[11] Suppose that X is a non-empty set and also suppose that the mapping $\omega : (0, \infty) \times X \times X \to [0, \infty]$ satisfies the following:*

(i) $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0$ *if and only if* $x = y$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$;

(iii) $\omega_{\lambda+\mu}(x, z) \leq \omega_\lambda(x, y) + \omega_\mu(y, z)$;

for all $\lambda, \mu > 0$ and for all $x, y, z \in X$. Then ω is said to be a modular metric on X.

Here one can note that a modular metric ω is strict if we have $\omega(\lambda, x, y) > 0$ for all $\lambda > 0$ and for every $x, y \in X$ with $x \neq y$. Further, a modular metric ω on X is convex if it satisfies the succeeding inequality:

$$\omega_{\lambda+\mu}(x, z) \leq \frac{\lambda}{\lambda + \mu}\omega_\lambda(x, y) + \frac{\mu}{\mu + \lambda}\omega_\mu(y, z)$$

for all $\lambda, \mu > 0$ and for all $x, y, z \in X$. The ideas and results related to the convergence and Cauchy criteria for sequences and also completeness of the space can be found in detail in [11, 12]. Now, we enlist some non-trivial examples of modular metrics in the following. Suppose (X, d) be a metric space with at least two elements.

Example 2.2. *[1] Consider*

$$\omega_\lambda(x, y) = d(x, y),$$

for all $\lambda > 0$ for all $x, y \in X$. Then this is an example of a modular which is not convex. In fact one can confirm that by putting $z = y$ and $\mu = \lambda$ in Definition 2.1.

Example 2.3. *[1] Consider*

$$\omega_\lambda(x, y) = \frac{d(x, y)}{\lambda},$$

for all $\lambda > 0$ for all $x, y \in X$. It can be easily verified that the modular is convex.

Definition 2.4. *[10, 11] Given an arbitrary $x_0 \in X$ and a modular ω defined on X, the following two collections*

$$X_\omega \equiv X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}$$

are called modular spaces. However, from [11], we state that $X_\omega \subset X^*_\omega$, and further, generally, this inclusion is proper.

On the other hand, Jungck [16] established the succeeding version of the illustrious Banach fixed point result [4] in the context of a complete space and thenceforth, the theory on common fixed points was marked as a dynamic research field and the mathematicians gradually introduced many novel notions like, coincidence points, compatible mappings, weakly compatible mappings, commuting mappings [13, 17, 18] and also some other related ones [7, 8, 14, 22].

Theorem 2.5. *Let T and I be two commuting self-mappings defined on a complete metric space (X, d) such that*

$$d(Tx, Ty) \leq \lambda d(Ix, Iy)$$

holds for all $x, y \in X$, where $0 < \lambda < 1$. Also assume that I is a continuous mapping and the range of T is contained in that of I. Then T and I possess a unique common fixed point.

3. Main Results

In this section, before anything else, we put forward the notion of a Hardy-Rogers ω-contraction in the context of a modular metric space and subsequently, we come across with a fixed point result involving such contractions. Thereupon, we deliver the modular space version of Jungck common fixed point theorem for a pair of commuting self-maps. However, our findings are aptly endowed with suitable examples. Firstly, we present the definition of a Hardy-Rogers ω-contraction.

Definition 3.1. *[23] Suppose ω is a modular on a set X and X^*_ω is a modular set. A self-mapping T on X^*_ω is said to be a Hardy-Rogers ω-contraction if there exist $a, b, c, d, e \in (0, 1)$ satisfying $a + b + c + 2e < 1$ and $a + d + e + c < 1$, and $\lambda_0 > 0$ such that*

$$\omega_\lambda(Tx, Ty) \leq \omega_{\frac{\lambda}{a}}(x, y) + \omega_{\frac{\lambda}{b}}(x, Tx) + \omega_{\frac{\lambda}{c}}(y, Ty) + \omega_{\frac{\lambda}{d}}(x, Ty) + \omega_{\frac{\lambda}{e}}(y, Tx), \tag{3.1}$$

for all $0 < \lambda \leq \lambda_0$ and all $x, y \in X^*_\omega$.

The following theorem concerning the aforementioned kind of contractions confirms the existence of a fixed point of the same.

Theorem 3.2. *Let ω be a strict convex modular on X such that the modular space X^*_ω is ω-complete and suppose that $T : X^*_\omega \to X^*_\omega$ is a Hardy-Roger ω-contractive mapping with the condition that for each $\lambda > 0$, there exists $x = x(\lambda) \in X^*_\omega$ satisfying $\omega_\lambda(x, Tx) < \infty$. Then T owns a fixed point x^* in X^*_ω. Further, if ω assumes only finite values on X^*_ω, then the additional assumption $\omega_\lambda(x, Tx) < \infty$ is redundant, and so the fixed point x^* is unique.*

Proof. Let $x_0 \in X^*_\omega$ and construct the Picard iterate by $x_n = T^n x_0, n \in \mathbb{N}$. Now we put $x = x_n$ and $y = x_{n-1}$ in (3.1) and obtain,

$$\begin{aligned} \omega_\lambda(x_{n+1}, x_n) \leq& \omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) + \omega_{\frac{\lambda}{b}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{c}}(x_{n-1}, x_n) \\ &+ \omega_{\frac{\lambda}{d}}(x_n, x_n) + \omega_{\frac{\lambda}{e}}(x_{n-1}, x_{n+1}). \end{aligned} \tag{3.2}$$

Here we have,

$$\begin{aligned}\omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) =&\omega_{\lambda+\frac{\lambda(1-a)}{a}}(x_n, x_{n-1})\\ \leq&\frac{\lambda}{\frac{\lambda}{a}}\omega_\lambda(x_n, x_{n-1}) + \frac{\lambda\frac{(1-a)}{a}}{\frac{\lambda}{a}}\omega_{\lambda\frac{1-a}{a}}(x_{n-1}, x_{n-1})\\ =&a\omega_\lambda(x_n, x_{n-1}).\end{aligned}$$

This implies that

$$\omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) \leq a\omega_\lambda(x_n, x_{n-1}). \tag{3.3}$$

Following similar technique, we obtain

$$\omega_{\frac{\lambda}{b}}(x_n, x_{n+1}) \leq b\omega_\lambda(x_n, x_{n+1}), \tag{3.4}$$

$$\omega_{\frac{\lambda}{c}}(x_{n-1}, x_n) \leq c\omega_\lambda(x_{n-1}, x_n), \tag{3.5}$$

and

$$\omega_{\frac{\lambda}{e}}(x_n, x_{n-1}) \leq e\omega_\lambda(x_{n-1}, x_{n+1}). \tag{3.6}$$

Applying (3.3), (3.4), (3.5) and (3.6), in (3.2), we get as follows

$$(1-b)\omega_\lambda(x_{n+1}, x_n) \leq (a+c)\omega_\lambda(x_n, x_{n-1}) + \omega_{\frac{\lambda}{e}}(x_{n-1}, x_{n+1}). \tag{3.7}$$

Now we have

$$\begin{aligned}\omega_{\frac{\lambda}{e}}(x_{n-1}, x_{n+1}) &= \omega_{\lambda+\lambda\frac{1-e}{e}}(x_{n-1}, x_{n+1})\\ &\leq e\omega_\lambda(x_{n-1}, x_n) + (1-e)\omega_{\lambda\frac{1-e}{e}}(x_n, x_{n+1}).\end{aligned}$$

Using the same technique as in (3.3) we have

$$\omega_{\frac{\lambda}{e}}(x_{n-1}, x_{n+1}) \leq e\omega_\lambda(x_{n-1}, x_n) + e\omega_\lambda(x_{n+1}, x_n). \tag{3.8}$$

From (3.7) and (3.8) we get

$$\begin{aligned}(1-b-e)\omega_\lambda(x_{n+1}, x_n) \leq&(a+c+e)\omega_\lambda(x_n, x_{n-1})\\ \omega_\lambda(x_{n+1}, x_n) \leq&\left(\frac{a+c+e}{1-b-e}\right)\omega_\lambda(x_n, x_{n-1}).\end{aligned}$$

From the above, by mathematical induction, we have the following

$$\omega_\lambda(x_{n+1}, x_n) \leq \left(\frac{a+c+e}{1-b-e}\right)^n \omega_\lambda(x_1, x_0). \tag{3.9}$$

Now we show that (x_n) is a Cauchy sequence, we have

$$\begin{aligned}\omega_\lambda(x_m, x_n) \leq&\omega_{\frac{\lambda}{a}}(x_{m-1}, x_{n-1}) + \omega_{\frac{\lambda}{b}}(x_{m-1}, x_m) + \omega_{\frac{\lambda}{c}}(x_{n-1}, x_n)\\ &+\omega_{\frac{\lambda}{d}}(x_{m-1}, x_n) + \omega_{\frac{\lambda}{e}}(x_{n-1}, x_m).\end{aligned} \tag{3.10}$$

By the convexity of ω, we also have,

$$\begin{aligned}\omega_{\frac{\lambda}{d}}(x_{m-1}, x_n) =&\omega_{\lambda+\lambda\frac{1-d}{d}}(x_{m-1}, x_n)\\ \leq&d\omega_\lambda(x_{m-1}, x_m) + (1-d)\omega_{\lambda\frac{1-d}{d}}(x_{m-1}, x_n)\\ \leq&d\omega_\lambda(x_{m-1}, x_m) + d\omega_\lambda(x_m, x_n).\end{aligned} \tag{3.11}$$

Similarly, we obtain

$$\omega_{\frac{\lambda}{a}}(x_{m-1}, x_{n-1}) = \omega_{\lambda\frac{1-a}{2a}+\lambda+\lambda\frac{1-a}{2a}}(x_{m-1}, x_{n-1})$$

and

$$\omega_{\frac{\lambda}{e}}(x_{n-1}, x_m) \leq e\omega_\lambda(x_{n-1}, x_n) + e\omega_\lambda(x_n, x_m). \tag{3.12}$$

Therefore

$$\begin{aligned}\omega_{\frac{\lambda}{a}}(x_{m-1}, x_{n-1}) \leq &\frac{1-a}{2}\omega_{\lambda\frac{1-a}{2a}}(x_{m-1}, x_m) + a\omega_\lambda(x_m, x_n) + \frac{1-a}{2}\omega_{\lambda\frac{1-a}{2a}}(x_n, x_{n-1})\\ =&a\omega_\lambda(x_{m-1}, x_m) + a\omega_\lambda(x_m, x_n) + a\omega_\lambda(x_n, x_{n-1}).\end{aligned} \tag{3.13}$$

From (3.10)-(3.13), we have

$$\begin{aligned}(1-a-d-e)\omega_\lambda(x_m, x_n) \leq &a\omega_\lambda(x_{m-1}, x_m) + a\omega_\lambda(x_n, x_{n-1}) + b\omega_\lambda(x_{m-1}, x_m)\\ &+c\omega_\lambda(x_{n-1}, x_n) + d\omega_\lambda(x_{m-1}, x_m) + e\omega_\lambda(x_{n-1}, x_n).\end{aligned} \tag{3.14}$$

From (3.9) and (3.14), it follows that (x_n) is a Cauchy sequence. Since ω is strict and X^*_ω is ω-complete, there exists a unique limit $x^* \in X^*_\omega$ of (x_n). Now we show that x^* is a fixed point of T. Here we have

$$\begin{aligned}\omega_\lambda(Tx_n, Tx^*) =&\omega_\lambda(x_{n+1}, Tx^*)\\ \leq&\omega_{\frac{\lambda}{a}}(x_n, x^*) + \omega_{\frac{\lambda}{b}}(x_n, Tx_n) + \omega_{\frac{\lambda}{c}}(x^*, Tx^*) + \omega_{\frac{\lambda}{d}}(x_n, Tx^*) + \omega_{\frac{\lambda}{e}}(x^*, Tx_n)\\ \leq&\omega_{\frac{\lambda}{a}}(x_n, x^*) + \omega_{\frac{\lambda}{b}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{c}}(x^*, Tx^*) + d\omega_\lambda(x^*, Tx^*)\\ &+(1-d)\omega_{\lambda\frac{1-d}{d}}(x_n, x^*) + \omega_{\frac{\lambda}{e}}(x^*, x_{n+1}).\end{aligned} \tag{3.15}$$

On the other hand, one can easily verify that

$$\omega_{2\lambda}(x^*, Tx^*) \leq \frac{1}{2}\omega_\lambda(x^*, x_{n+1}) + \frac{1}{2}\omega_\lambda(x_{n+1}, Tx^*). \tag{3.16}$$

From (3.15) and (3.16) and letting $n \to \infty$, we have

$$\begin{aligned}\omega_{2\lambda}(x^*, Tx^*) \leq&\frac{1}{2}\omega_\lambda(x^*, x_{n+1}) + \frac{1}{2}\omega_{\frac{\lambda}{a}}(x_n, x^*) + \frac{1}{2}\omega_{\frac{\lambda}{b}}(x_n, x_{n+1})\\ &+\frac{1}{2}\omega_{\frac{\lambda}{c}}(x^*, Tx^*) + \frac{d}{2}\omega_\lambda(x^*, Tx^*)\\ &+\frac{(1-d)}{2}\omega_{\lambda\frac{1-d}{d}}(x_n, x^*) + \frac{1}{2}\omega_{\frac{\lambda}{e}}(x^*, x_{n+1})\\ \leq&\frac{1}{2}\omega_\lambda(x^*, x_{n+1}) + \frac{a}{2}\omega_\lambda(x_n, x^*) + \frac{b}{2}\omega_\lambda(x_n, x_{n+1})\\ &+\frac{1}{2}\omega_{\frac{\lambda}{c}}(x^*, Tx^*) + \frac{d}{2}\omega_\lambda(x^*, Tx^*)\\ &+\frac{(1-d)}{2}\omega_{\lambda\frac{1-d}{d}}(x_n, x^*) + \frac{1}{2}\omega_{\frac{\lambda}{e}}(x^*, x_{n+1})\\ \leq&\frac{1}{2}\{\omega_{\frac{\lambda}{c}}(x^*, Tx^*) + d\omega_\lambda(x^*, Tx^*)\}\\ \leq&\frac{1}{2}\{2c\omega_{2\lambda}(x^*, Tx^*) + 2d\omega_{2\lambda}(x^*, Tx^*)\}.\end{aligned}$$

So, $(1-c-d)\omega_{2\lambda}(x^*, Tx^*) \leq 0$, since $(1-c-d) > 0$, we obtain $\omega_{2\lambda}(x^*, Tx^*) = 0$. Hence $Tx^* = x^*$, and we conclude that T possesses a fixed point in X^*_ω. □

The subsequent example illustrates our previous result.

Example 3.3. *Let $X_\omega = [0,1]$ be equipped with the modular metric*

$$\omega_\lambda(x,y) = \begin{cases} \frac{x+y}{\lambda}, & \text{if } x \neq y; \\ 0, & \text{otherwise}; \end{cases}$$

for all $x, y \in X_\omega$ and $\lambda > 0$. Clearly, $(X^*_\omega, \omega_\lambda)$ is a strictly convex, ω-complete modular space. Now we define a map $T : X^*_\omega \to X^*_\omega$ by

$$Tx = \frac{x}{2}$$

for all $x \in X^*_\omega$ and take $a = b = c = d = e = \frac{1}{6}$. Then we have $a + b + c + 2e < 1$ and $a + c + d + e < 1$. Further we get,

$$\begin{aligned}\omega_\lambda(Tx, Ty) =& \frac{Tx + Ty}{\lambda} \\ =& \frac{x+y}{2\lambda} \\ \leq& \frac{4x+4y}{6\lambda} \\ =& \omega_{\frac{\lambda}{a}}(x,y) + \omega_{\frac{\lambda}{b}}(x,Tx) + \omega_{\frac{\lambda}{c}}(y,Ty) + \omega_{\frac{\lambda}{d}}(x,Ty) + \omega_{\frac{\lambda}{e}}(y,Tx).\end{aligned}$$

This shows that T is a Hardy-Rogers ω-contraction and satisfies all the assumptions of Theorem 3.2. Hence, T has a fixed point, which is $x = 0$.

The following is the definition of ω-continuity which is an essential tool for our next finding. We define it as:

Definition 3.4. *Let ω be a strict convex modular on X. A mapping $T : X^*_w \to X^*_w$ is said to be ω-continuous if for any sequence (x_n) which converges to x_0 implies that (Tx_n) converges to Tx_0.*

Now we state the common fixed point result of Jungck-type in the setting of a modular space.

Theorem 3.5. *Let ω be a strict convex modular on X such that the modular space X^*_ω is ω-complete and T, I be two self-mappings on X^*_ω such that for each $\lambda > 0$, there exists $x = x(\lambda) \in X^*_\omega$ such that $\omega_\lambda(x, Tx) < \infty$. We also assume that*

(i) $T(X^*_\omega) \subseteq I(X^*_\omega)$;
(ii) T, I are both ω-continuous;
(iii) $I(X^*_\omega)$ is a ω-complete subspace of X^*_ω;
(iv) T, I satisfy

$$\omega_\lambda(Tx, Ty) \leq \omega_{\frac{\lambda}{a}}(Ix, Iy) + \omega_{\frac{\lambda}{b}}(Ix, Tx) + \omega_{\frac{\lambda}{c}}(Iy, Ty),$$

for all $x, y \in X^*_\omega$, with $a + b + c < 1$ and $\lambda > 0$.

Then I, T possess a common fixed point x^* in X^*_ω.

Proof. Suppose that $x_0 \in X$ is an arbitrary element. Then Tx_0 and Ix_0 are well-defined. As $Tx_0 \in I(X)$, there exists $x_1 \in X$ such that $Ix_1 = Tx_0$. In general, if x_n is chosen, then we can consider an element $x_{n+1} \in X$ such that $Ix_{n+1} = Tx_n$.

Case-I: If $Tx_n = Tx_{n+1}$ for some n, then $Tx_{n+1} = Ix_{n+1} = p$, we show that p is a common fixed point of T and I. Now we have

$$Ip = I(Tx_{n+1}) = T(Ix_{n+1}) = Tp.$$

Further, we assume that, $p \neq Tp$, i.e., $\omega_\lambda(p, Tp) > 0$. Then

$$\begin{aligned}\omega_\lambda(p, Tp) =& \omega_\lambda(Tx_{n+1}, Tp) \\ \leq& \omega_{\frac{\lambda}{a}}(Ix_{n+1}, Ip) + \omega_{\frac{\lambda}{b}}(Ix_{n+1}, Tx_{n+1}) + \omega_{\frac{\lambda}{c}}(Ip, Tp) \\ \leq& \omega_{\frac{\lambda}{a}}(p, Tp) + \omega_{\frac{\lambda}{b}}(p, p) + \omega_{\frac{\lambda}{c}}(Tp, Tp) \\ \leq& a\omega_\lambda(p, Tp) \\ <& \omega_\lambda(p, Tp)\end{aligned}$$

which is a contradiction. Hence, p is a common fixed point of T and I.

Case-II: Now we suppose that $Tx_{n+1} \neq Tx_n$ for all $n \in \mathbb{N}$. Then we have,

$$\begin{aligned}
\omega_\lambda(Tx_{n+1}, Tx_n) \leq& \omega_{\frac{\lambda}{a}}(Ix_{n+1}, Ix_n) + \omega_{\frac{\lambda}{b}}(Ix_{n+1}, Tx_{n+1}) + \omega_{\frac{\lambda}{c}}(Ix_n, Tx_n) \\
=& a\omega_\lambda(Tx_n, Tx_{n-1}) + b\omega_\lambda(Tx_n, Tx_{n+1}) + c\omega_\lambda(Tx_{n-1}, Tx_n) \\
\leq& \left(\frac{a+c}{1-b}\right)\omega_\lambda(Tx_n, Tx_{n-1}) \\
\leq& \left(\frac{a+c}{1-b}\right)^n \omega_\lambda(Tx_1, Tx_0).
\end{aligned} \tag{3.17}$$

As we let $n \to \infty$ in (3.17), we have $\lim\limits_{n\to\infty} \omega_\lambda(Tx_{n+1}, Tx_n) = 0$. Next we show that (x_n) is a Cauchy sequence. Hence

$$\begin{aligned}
\omega_\lambda(Tx_m, Tx_n) \leq& \omega_{\frac{\lambda}{a}}(Ix_m, Ix_n) + \omega_{\frac{\lambda}{b}}(Ix_m, Tx_m) + \omega_{\frac{\lambda}{c}}(Ix_n, Tx_n) \\
\leq& \omega_{\frac{\lambda}{a}}(Tx_{m-1}, Tx_{n-1}) + \omega_{\frac{\lambda}{b}}(Tx_{m-1}, Tx_m) + \omega_{\frac{\lambda}{c}}(Tx_{n-1}, Tx_n).
\end{aligned} \tag{3.18}$$

Now we have,

$$\begin{aligned}
\omega_{\frac{\lambda}{a}}(Tx_{m-1}, Tx_{n-1}) =& \omega_{\lambda\frac{1-a}{2a}+\lambda+\lambda\frac{1-a}{2a}}(Tx_{m-1}, Tx_{n-1}) \\
=& \frac{1-a}{2}\omega_{\lambda\frac{1-a}{2a}}(Tx_{m-1}, Tx_m) + a\omega_\lambda(Tx_m, Tx_n) \\
&+ \frac{1-a}{2}\omega_{\lambda\frac{1-a}{2a}}(Tx_n, Tx_{n-1}).
\end{aligned} \tag{3.19}$$

Therefore, using (3.19) in (3.18), we obtain

$$\begin{aligned}
(1-a)\omega_\lambda(Tx_m, Tx_n) \leq& \omega_{\frac{\lambda}{b}}(Tx_{m-1}, Tx_m) + \frac{1-a}{2}\omega_{\lambda\frac{1-a}{2a}}(Tx_{m-1}, Tx_m) \\
&+ \omega_{\frac{\lambda}{c}}(Tx_{n-1}, Tx_n) + \frac{1-a}{2}\omega_{\lambda\frac{1-a}{2a}}(Tx_n, Tx_{n-1}).
\end{aligned}$$

Letting $n \to \infty$ in the previous inequality, we get

$$\lim_{m,n\to\infty} \omega_\lambda(Tx_m, Tx_n) = 0.$$

This shows that (Tx_n) is a Cauchy sequence. Since X_w^* is a complete modular space, (Tx_n) converges to $p \in X_w^*$. Therefore,

$$\lim_{n\to\infty} Ix_{n+1} = \lim_{n\to\infty} Tx_n = p.$$

Again by the continuity of T and I,

$$Ip = I(\lim_{n\to\infty} Tx_n) = \lim_{n\to\infty} IT(x_n) = \lim_{n\to\infty} TI(x_n) = T(\lim_{n\to\infty} Ix_n) = Tp.$$

Let us consider that $Tp = Ip = q$. If possible, assume $Tp \neq Tq$. Then, we have,

$$Tq = TIp = ITp = Iq.$$

Further,

$$\begin{aligned}
\omega_\lambda(Tq, Tp) \leq& \omega_{\frac{\lambda}{a}}(Iq, Ip) + \omega_{\frac{\lambda}{b}}(Iq, Tq) + \omega_{\frac{\lambda}{c}}(Ip, Tp) \\
\leq& \omega_{\frac{\lambda}{a}}(Tq, Tp) + \omega_{\frac{\lambda}{b}}(Tq, Tq) + \omega_{\frac{\lambda}{c}}(Tp, Tp) \\
\leq& \omega_{\frac{\lambda}{a}}(Tq, Tp) \\
\leq& a\omega_\lambda(Tq, Tp) \\
<& \omega_\lambda(Tq, Tp),
\end{aligned}$$

which is a contradiction. Similarly, we can show that $Ip = Iq$. Hence we have $Tq = Iq = q$ and q is a common fixed point of T and I. $\square$

The subsequent non-trivial numerical example validates our secured common fixed point result.

Example 3.6. *Let $X_\omega = [0,1]$ be endowed with the modular metric*

$$\omega_\lambda(x,y) = \begin{cases} \frac{x+y}{\lambda}, & \text{if } x \neq y; \\ 0, & \text{otherwise;} \end{cases}$$

for all $x, y \in X_\omega$ and $\lambda > 0$. Clearly, $(X^*_\omega, \omega_\lambda)$ is a strictly convex, ω-complete modular space. Let us define maps $T, I : X^*_\omega \to X^*_\omega$ by

$$Tx = \frac{x^2}{4} \text{ and } Ix = \frac{x}{2},$$

for all $x \in X^*_\omega$. It is clear that T and I are both ω-continuous mappings. Also, we have $T(X^*_\omega) \subseteq I(X^*_\omega)$ and $I(X^*_\omega)$ is a complete subspace of X^*_ω. Further T, I satisfy

$$\begin{aligned}
\omega_\lambda(Tx,Ty) =& \frac{x^2+y^2}{4\lambda} \\
=& \frac{x^2+y^2}{8\lambda} + \frac{x^2+y^2}{8\lambda} \\
\leq& \frac{\left(\frac{x}{2}+\frac{y}{2}\right)}{4\lambda} + \frac{x^2+y^2}{16\lambda} + \frac{x^2+y^2}{16\lambda} \\
\leq& \frac{\left(\frac{x}{2}+\frac{y}{2}\right)}{4\lambda} + \frac{\left(\frac{x}{2}+\frac{y}{2}\right)}{8\lambda} + \frac{\left(\frac{x^2}{4}+\frac{y^2}{4}\right)}{4\lambda} \\
\leq& \frac{\left(\frac{x}{2}+\frac{y}{2}\right)}{4\lambda} + \frac{\left(\frac{x}{2}+\frac{y}{2}\right)}{4\lambda} + \frac{\left(\frac{x^2}{4}+\frac{y^2}{4}\right)}{4\lambda} \\
\leq& \frac{\left(\frac{x}{2}+\frac{y}{2}\right)}{4\lambda} + \frac{\left(\frac{x}{2}+\frac{x^2}{4}\right)}{4\lambda} + \frac{\left(\frac{y}{2}+\frac{y^2}{4}\right)}{4\lambda} \\
=& \omega_{\frac{\lambda}{\frac{1}{4}}}(Ix,Iy) + \omega_{\frac{\lambda}{\frac{1}{4}}}(Ix,Tx) + \omega_{\frac{\lambda}{\frac{1}{4}}}(Iy,Ty) \\
\Rightarrow \omega_\lambda(Tx,Ty) \leq& \omega_{\frac{\lambda}{\frac{1}{4}}}(Ix,Iy) + \omega_{\frac{\lambda}{\frac{1}{4}}}(Ix,Tx) + \omega_{\frac{\lambda}{\frac{1}{4}}}(Iy,Ty).
\end{aligned}$$

Hence all the hypotheses of Theorem 3.5 are satisfied and therefore, we can conclude that I, T has a common fixed point $x = 0$.

4. Consequences

This section deals with the immediate corollaries which can be thought of as special cases of our conceived theories. Here we note down some of those. Firstly, if we consider b, c, d, e tend to 0 in Theorem 3.2, then we come up with the modular Banach contraction theorem delivered in [21].

Corollary 4.1. *Let (X, ω) be a ω-complete modular metric space. Any self-mapping T defined on X^*_ω such that it satisfies*

$$\omega_\lambda(Tx,Ty) \leq \omega_{\frac{\lambda}{a}}(x,y)$$

for all $x, y \in X^*_\omega$ with $a \in (0,1)$, $a < 1$ and $0 < \lambda \leq \lambda_0$, has a unique fixed point.

Further, if we let a, d, e tend to 0 in Theorem 3.2, then we get the modular Kannan contraction theorem.

Corollary 4.2. *Let (X, ω) be a ω-complete modular metric space. Any self-mapping T defined on X^*_ω such that it satisfies*

$$\omega_\lambda(Tx, Ty) \leq \omega_{\frac{\lambda}{b}}(x, Tx) + \omega_{\frac{\lambda}{c}}(y, Ty)$$

for all $x, y \in X^*_\omega$ with $a \in (0, 1)$, $a < 1$ and $0 < \lambda \leq \lambda_0$, has a unique fixed point.

If we let a, b, c tend to 0 in Theorem 3.2, then we get the modular Chatterjea contraction theorem.

Corollary 4.3. *Let (X, ω) be a ω-complete modular metric space. Any self-mapping T defined on X^*_ω such that it satisfies*

$$\omega_\lambda(Tx, Ty) \leq \omega_{\frac{\lambda}{d}}(x, Ty) + \omega_{\frac{\lambda}{e}}(y, Tx)$$

for all $x, y \in X^*_\omega$ with $d, e \in (0, 1)$, $d + e < 1$ and $0 < \lambda \leq \lambda_0$, has a unique fixed point.

Considering $Ix = x$ for all $x \in X^*_\omega$ in Theorem 3.5, we get the following result, which is previously obtained in Mitrović et al. [23]. These kind of contractions are called as Reich ω-contractions on a modular metric space.

Corollary 4.4. *Let ω be a strict convex modular on X such that the modular metric space X^*_ω is ω-complete and T be any self-mapping on X^*_ω such that for each $\lambda > 0$, there exists $x = x(\lambda) \in X^*_\omega$ with $\omega_\lambda(x, Tx) < \infty$. We also assume that T satisfies*

$$\omega_\lambda(Tx, Ty) \leq \omega_{\frac{\lambda}{a}}(x, y) + \omega_{\frac{\lambda}{b}}(x, Tx) + \omega_{\frac{\lambda}{c}}(y, Ty),$$

for all $x, y \in X^*_\omega$, with $a + b + c < 1$ and $\lambda > 0$. Then T possesses a fixed point x^* in X^*_ω.

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