Filomat 34:12 (2020), 3943–3955
https://doi.org/10.2298/FIL2012943S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Solvability for Infinite Systems of Fractional Differential Equations in Banach Sequence Spaces ℓ_p and c_0

Ayub Samadi[a], Sotiris K. Ntouyas[b,c]

[a]*Department of Mathematics, Miyaneh Branch, Islamic Azad University, Miyaneh, Iran*
[b]*Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*
[c]*Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.P. Box 80203, Jeddah 21589, Saudi Arabia*

Abstract. This paper is devoted to an infinite system of nonlinear fractional differential equations in the Banach spaces c_0 and ℓ_p with $p \geq 1$. Existence results are obtained, by using the theory of measure of noncompactness and a new generalization of Darbo's fixed point theorem. Some examples are also included to show the efficiency of our results.

1. Introduction

The interest for studying fractional differential equations is based on the fact that the theory of fractional differential equations has been applied to various fields such as physics, chemistry, engineering and heat conduction in material with memory, see for example [14, 15]. Indeed, by applying the theory of fractional differential equations, we can find numerous applications in economics, geology, viscoelastic materials, bioengineering, fluid mechanics, chaotic dynamics and polymer science, ect. [4, 10, 13, 17, 20**?** **?**]. In recent years, ordinary and partial functional differential equations have been developed by the fractional calculus techniques and equations of fractional order are more general compared with integer order. The problem of the existence of solutions for fractional differential equations plays a significant role in the investigation of these types of equations and it is important to apply original studies in our investigations[1, 2, 5, 7, 12].

The theory of infinite systems of differential equations can be regarded as a particular case of differential equations in Banach spaces , where the infinite system can be represented as an ordinary differential equation. Recently, Mursaleen *et al.* in [11] studied a three point infinite system of fractional differential equations

$$\begin{cases} D^\alpha u_i(t) = f_i(t, u(t)), & t \in (0, T), \\ u_i(0) = 0, \quad u_i(T) = a u_i(\xi), & i = 1, 2, \dots \end{cases}$$

where D^α is the standard Riemann-Liouville fractional derivative of order $1 < \alpha < 2$ and $\xi \in (0, T)$ with $a\xi^{\alpha-1} < T^{\alpha-1}$. By using the theory of measure of noncopmactness and condensing operators they established the existence of solutions in sequence spaces c_0 and ℓ_p. In [12] Mursaleen and Rizvi studied existence results

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2010 *Mathematics Subject Classification*. Primary 47H09; Secondary 47H10, 34A34
Keywords. Measure of noncompactness; fractional differential equations; Darbo fixed point theorem; Banach spaces c_0 and ℓ_p
Received: 11 December 2019; Revised: 06 May 2020; Accepted: 19 May 2020
Communicated by Snežana Živković-Zlatanović
Email addresses: ayub.samadi@m-iau.ac.ir (Ayub Samadi), sntouyas@uoi.gr (Sotiris K. Ntouyas)

for the solution of infinite system of second order differential equations in Banach sequence spaces c_0 and ℓ_1 using the idea of Meir-Keeler condensing operators.

Motivated by the above papers, the aim of this work is to study the existence of solutions of the following infinite system of fractional differential equations

$$D^{\alpha}_{0^+}u_i(t) = F_i\Big(t, h_i(t,u(t)), (Gu)(t)\int_0^T g_i(s,u(s))ds\Big), \qquad t\in[0,T] \tag{1}$$

supplemented with three point boundary conditions

$$u_i(0)=0,\ \ u_i(T)=au_i(\xi), i=0,1,2,\dots, \tag{2}$$

where $\xi\in[0,T]$, $F_i:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $h_i,g_i:[0,T]\times D\to\mathbb{R}$ $(D\in\{c_0,\ell_p\})$ are continuous functions and $G:A\to C(I,\mathbb{R})$, $A\in\{C(I,\ell_p), C(I,c_0)\}$ is a continuous operator.

The paper is organized as follows. In Section 2, we recall some essential concepts and results which are used in the main results. In the next two sections, by applying a generalization of Darbo fixed point theorem together with the technique of measure of noncompactness the existence of solutions is studied, in sequence space ℓ_p in Section 3 and in sequence space c_0 in Section 4. Examples illustrating the obtained results are also presented.

2. Preliminaries

In this section, we firstly introduce some notations and definitions which are used throughout this paper. For a bounded subset S of a metric space X, Kuratowski [9] defined the function $\alpha(S)$, known as Kuratowski measure of noncompactness, by the formula

$$\alpha(S)=\inf\Big\{\delta>0: S=\bigcup_{i=1}^{n}S_i,\ \ diam(S_i)\le\delta\ \text{ for }\ 1\le i\le n<\infty\Big\}.$$

Another useful measure of noncompactness is the so called Hausdorff measure of noncompactness defined as

$$\chi(S)=\inf\{\varepsilon>0:\ \ S\text{ has finite }\ \varepsilon-\text{net in }\ X\}.$$

Banas and Goebel [3] have presented some basic properties of the Hausdorff measure of noncompactness χ. Now we assume that E is a real Banach space with norm $\|\cdot\|$ and zero element θ. If X is a nonempty subset of E, the closure and the closed convex hull of X will be denoted by $\overline{X}$ and Convc(X), respectively. Moreover, let us denote by M_E the family of all nonempty and bounded subsets of E and by N_E its subfamily consisting of all relatively compact sets.

Definition 2.1. *[3] A mapping $\mu: M_E\longrightarrow[0,\infty)$ is called a measure of noncompactness if it satisfies the following conditions:*

(1) The family $Ker\mu=\{X\in M_E:\mu(X)=0\}$ is nonempty and $Ker\mu\subseteq N_E$.

(2) $X\subseteq Y\Longrightarrow\mu(X)\le\mu(Y)$.

(3) $\mu(\overline{X})=\mu(X)$.

(4) $\mu(Conv(X))=\mu(X)$.

(5) $\mu(\lambda X+(1-\lambda)Y)\le\lambda\mu(X)+(1-\lambda)\mu(Y)$ for $\lambda\in[0,1]$.

(6) If $\{X_n\}$ is a sequence of closed sets from M_E such that $X_{n+1}\subseteq X_n$ for $n=1,2,\dots$ and $\lim_{n\to\infty}\mu(X_n)=0$, then $\bigcap_{n=1}^{\infty}X_n$ is nonempty.

Theorem 2.2. *(Darbo [6]). Let Q be a nonempty, closed, bounded and convex subset of a Banach space E and $F : Q \longrightarrow Q$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Q, where μ is a measure of noncompact defined in Q. Then F has a fixed point in Q.*

Samadi and Ghaemi [18, 19] proved some generalizations of Darbo fixed point theorem. Also, the first author [17] extended Darbo fixed point theorem and presented the following result which is basic for our main results.

Theorem 2.3. *Let C be a nonempty bounded, closed and convex subset of a Banach space E. Assume $T : C \longrightarrow C$ is a continuous operator satisfying*

$$\theta((\mu(X)) + f(\mu(T(X))) \leq f(\mu(X)) \tag{3}$$

for all nonempty subset X of C, where μ is an arbitrary measure of noncompactness defined in E, $F : (0,\infty) \longrightarrow \mathbb{R}$, $\theta : (0,\infty) \longrightarrow (0,\infty)$ and $(\theta, f) \in \Delta$. Then T has a fixed point in C.

In Theorem 2.3, Δ is the set of all pairs (θ, f) satisfying the following:

(Δ_1) $\theta(t_n) \nrightarrow 0$ for each strictly increasing sequence $\{t_n\}$;

(Δ_2) f is strictly increasing function;

(Δ_3) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} f(\alpha_n) = -\infty$.

(Δ_4) If $\{t_n\}$ is a decreasing sequence such that $t_n \to 0$ and $\theta(t_n) < f(t_n) - f(t_{n+1})$, then we have $\sum_{n=1}^{\infty} t_n < \infty$.

The following essential definitions and auxiliary facts in fractional calculus will be needed in our main results.

Definition 2.4. *[8] The fractional order integral of the function $y \in L^1([a,b],\mathbb{R})$ of order $q \in \mathbb{R}^+$ is defined by*

$$I_a^q y(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} y(s) ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.5. *[8] The Riemann-Liouville derivative of order α with the lower limit zero for a function $f : [0,\infty) \longrightarrow \mathbb{R}$ can be written as*

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \times \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \qquad t > 0, n-1 < \alpha < n.$$

The following lemma is the main tool in our investigation.

Lemma 2.6. *[11] Let $f \in C([0,T],\mathbb{R})$ be a given function and $1 < \alpha < 2$. Then the unique solution of*

$$D_{0^+}^{\alpha} u(t) = f(t), \ \ u(0) = 0, \ u(T) = au(\xi), \xi \in [0,T]$$

is given by

$$u(t) = \int_0^T k(t,s) f(s) ds$$

where $k(t,s)$ is the Green's function, given by $k(t,s) = \begin{cases} k_1(t,s), \ 0 \leq t \leq \xi, \\ k_2(t,s), \ \xi \leq t \leq T, \end{cases}$ *and*

$$k_1(t,s) = \begin{cases} (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}], \ 0 \leq s \leq t, \\ -t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}], \ t \leq s \leq \xi, \\ -(t(T-s)^{\alpha-1}), \ \xi \leq s \leq T, \end{cases}$$

$$k_2(t,s) = \begin{cases} (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}], \ 0 \leq s \leq \xi, \\ (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - (t(T-s))^{\alpha-1}, \ \xi < s \leq t, \\ -(t(T-s))^{\alpha-1}, \ t < s \leq T. \end{cases}$$

3. Solution in sequence space ℓ_p

In this section we investigate the solution of the infinite system (1)-(2) in the sequence space ℓ_p, the space of all absolutely p-summable series

$$\ell_p = \Big\{x \in \omega : \sum_{n=1}^{\infty} |x_n|^p < \infty\Big\}, \ \ 1 \leq p < \infty,$$

where ω is the space of all complex sequences $x = \{x_n\}_{n=1}^{\infty}$. Clearly , ℓ_p is a Banach space with norm

$$\|x\|_{\ell_p} = \|(x_n)\|_{\ell_p} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}, \ \ 1 \leq p < \infty.$$

Let us denote by M_{ℓ_p} the families of all nonempty bounded subsets of ℓ_p. It is well known that in the space $(\ell_p, \|\cdot\|_{\ell_p})$, the Hausdorff measure of noncompactnes χ is defined by the formula [3]:

$$\chi(B) = \lim_{n\longrightarrow\infty}\Big\{\sup_{x\in B}\Big\{\sum_{k\geq n} |x_k|^p\Big\}^{\frac{1}{p}}\Big\}, \tag{4}$$

where $B \in M_{\ell_p}$ and $x(t) = (x_i(t))_{i=1}^{\infty} \in \ell_p$.

By applying Theorem 2.3, the existence of solutions for the infinite system (1)-(2) is studied in the Banach space $(\ell_p, \|\cdot\|_{\ell_p})$. We list the following conditions:

(H_1) The functions $F_i : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $h_i : [0,T]\times\ell_p \to \mathbb{R}$ are continuous and there exists a positive real number $\tau > 0$ such that:

$$\begin{aligned} |F_i(t,x_1,x_2) - F_i(t,y_1,y_2)|^p &\leq e^{-\tau}(|x_1-y_1|^p + |x_2-y_2|^p),\\ |h_i(t,u(t))|^p &\leq e^{-\tau}|u_i(t)|^p,\\ |h_i(t,u(t)) - h_i(t,v(t))|^p &\leq e^{-\tau}|u_i(t)-v_i(t)|^p, \end{aligned}$$

for $t \in [0,T]$, $x_1,x_2,y_1,y_2 \in \mathbb{R}$ and $u(t) = (u_i(t)), v(t) = (v_i(t)) \in \ell_p$ where $i = 1,2,3,\ldots$.

(H_2) The function $t \to F_i(t,0,0)$ is bounded on $[0,T]$ such that:

$$N_1 = \sum_{i=1}^{\infty}\int_0^T |F_i(s,0,0)|^p ds, \quad \lim_{n\longrightarrow\infty}\sum_{k\geq n}\int_0^T |F_k(s,0,0)|^p ds = 0.$$

(H_3) $G : C(I,\ell_p) \to C(I,\mathbb{R})$ is a continuous operator such that:

$$|(Gx)(t) - (Gu)(t)| \leq \|x(t)-u(t)\|_{\ell_p}, \ \ |(Gx)(t)| \leq a + b\|x(t)\|_{\ell_p}^p,$$

for $x,u \in C(I,\ell_p)$ and $t \in [0,T]$, where a,b are positive real numbers.

(H_4) $g_i : [0,T]\times\ell_p \to \mathbb{R}$ are continuous and there exist continuous functions $b_i : [0,T] \to \mathbb{R}$ such that

$$|g_i(s,u(s))| \leq |b_i(s)|, \ \ q = \sup\{|b_i(s)|; s \in [0,T], i \geq 1\},$$

for $u \in C(I,\ell_p)$. Furthermore the series $\sum_{i=1}^{\infty} b_i(t)$ is uniformly convergent with

$$b(t) = \sum_{i=1}^{\infty} b_i(t), \ \ B = \sup\{b(t), t \in [0,T]\}.$$

(H_5) There exist B_k such that

$$B_k = \sup\Big\{\sum_{n\geq k}\Big|\int_0^T g_n(s,u(s))ds\Big|^p : s\in[0,T]\Big\}.$$

Also, as $k\longrightarrow\infty$, $B_k\longrightarrow 0$ and $\sup_k B_k = B_0$.

(H_6) There exists a positive solution r_0 of the inequality

$$\Big(2^pT^{\frac{p}{q}}M^pN_1 + 2^pT^{\frac{p}{q}}M^pe^{-2\tau}r^pT + e^{-\tau}(a+br^p)B_0T2^pT^{\frac{p}{q}}M^p\Big)^{\frac{1}{p}} \leq r.$$

Moreover, assume that $(T^{\frac{p}{q}})^{\frac{1}{p}}M^pT2^p < 1$.

Remark 3.1. *For each $i\in\mathbb{N}$, the infinite system (1)-(2) has a solution if and only if the integral equation*

$$u_i(t) = \int_0^T k(t,s)F_i\Big(s, h_i(s,u(s)), (Gu)(s)\int_0^T g_i(s,u(s))ds\Big)ds,$$

has a solution, where $k_i(t,s) = k(t,s)$ described in Lemma 2.6.

In the following we put $M = \max_{t,s\in[0,T]}|k(t,s)|$.

Theorem 3.2. *Under the assumptions* $(H_1)-(H_6)$, *infinite system (1)-(2) has at least one solution $u(t) = \{u_i(t)\}_{i=1}^{\infty}$ such that $u(t)\in\ell_p$ for all $t\in[0,T]$.*

Proof. Let us consider the operator F defined on the space $C(I,\ell_p)$ by the formula

$$(Fu)(t) = \Big(\int_0^T k(t,s)F_i\Big(s, h_i(s,u(s)), (Gu)(s)\int_0^T g_i(s,u(s))ds\Big)ds\Big),$$

for all $t\in[0,T]$, where $C(I,\ell_p)$ is the space of all continuous functions on the interval $[0,T]$ with values in the space ℓ_p and equipped with the norm

$$\|u\| = \sup\{\|u(t)\|_{\ell_p} :\ t\in[0,T]\}.$$

Let u be an arbitrary element of $C(I,\ell_p)$ where $u(t) = \{u_i(t)\}_{i=1}^{\infty}\in\ell_p$ for all $t\in[0,T]$. Keeping in mind our assumptions, for any $t\in I$ we deduce that

$$\begin{aligned}
&\|(Fu)(t)\|_{\ell_p}^p\\
= &\sum_{i=1}^{\infty}\Big|\int_0^T k(t,s)F_i\Big(s, h_i(s,u(s)), (Gu)(s)\int_0^T g_i(s,u(s))ds\Big)ds\Big|^p\\
\leq &\sum_{i=1}^{\infty}\int_0^T |k(t,s)|^p\Big|F_i\Big(s, h_i(s,u(s)), (Gu)(s)\int_0^T g_i(s,u(s))ds\Big)\Big|^p ds\Big(\int_0^T ds\Big)^{\frac{p}{q}},
\end{aligned}$$

where $q > 1$ is a positive number such that $\frac{1}{p} + \frac{1}{q} = 1$. Consequently,

$$\begin{aligned}
&\|(Fu)(t)\|^p_{\ell_p}\\
&\leq T^{\frac{p}{q}} \textstyle\sum_{i=1}^{\infty} \int_0^T |k(t,s)|^p \Big| F_i\Big(s, h_i(s,u(s)), (Gu)(s) \int_0^T g_i(s,u(s))ds\Big)\Big|^p ds\\
&\leq 2^p T^{\frac{p}{q}} M^p \textstyle\sum_{i=1}^{\infty} \int_0^T \Big| F_i\Big(s, h_i(s,u(s)), (Gu)(s) \int_0^T g_i(s,u(s))ds\Big) - F_i(s,0,0))\Big|^p ds\\
&+2^p M^p T^{\frac{p}{q}} \textstyle\sum_{i=1}^{\infty} \int_0^T |F_i(s,0,0)|^p ds\\
&\leq 2^p T^{\frac{p}{q}} M^p \textstyle\sum_{i=1}^{\infty} \int_0^T \Big\{ e^{-\tau}|h_i(s,u(s))|^p + e^{-\tau}\Big|(Gu)(s)\int_0^T g_i(s,u(s))ds\Big|^p \Big\} ds + 2^p T^{\frac{p}{q}} M^P N_1\\
&\leq 2^p T^{\frac{p}{q}} M^p \textstyle\sum_{i=1}^{\infty} \int_0^T \Big\{ e^{-2\tau}\|u_i(s)\|^p + e^{-\tau}(a + b\|u(s)\|^p_{l_p})\Big| \int_0^T g_i(s,u(s))ds\Big|^p \Big\} ds\\
&+2^p T^{\frac{p}{q}} M^P N_1\\
&\leq 2^p T^{\frac{p}{q}} M^p e^{-2\tau}\|u\|^p T + e^{-\tau}(a + b\|u\|^p) B_0 T 2^p T^{\frac{p}{q}} M^p + 2^p T^{\frac{p}{q}} M^p N_1.
\end{aligned} \tag{5}$$

From the above estimate we have

$$\|Fu\| \leq \Big(2^p T^{\frac{p}{q}} M^p e^{-2\tau}\|u\|^p T + e^{-\tau}(a + b\|u\|^p) B_0 T 2^p T^{\frac{p}{q}} M^p + 2^p T^{\frac{p}{q}} M^p N_1\Big)^{\frac{1}{p}}. \tag{6}$$

Now, we show that F is continuous on $[0,T]$. Let $t_0 \in [0,T]$ and $\epsilon > 0$ be arbitrary. In view of the continuity of $k(t,s)$ there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies that

$$|k(t,s) - k(t_0,t)|^p < \frac{\epsilon^p}{\Big(2^p T^{\frac{p}{q}} N_1 + 2^p T^{\frac{p}{q}} e^{-2\tau} r^p T + e^{-\tau}(a + br^p) B_0 T 2^p T^{\frac{p}{q}}\Big)}. \tag{7}$$

From (5) and (7) we get

$$\|(Fu)(t) - (Fu)(t_0)\|_{\ell_p} < \epsilon.$$

Moreover, due to (6) we conclude that F is bounded in the classical supremum norm on $C(I,\ell_p)$ and transforms $C(I,\ell_p)$ into itself. Due to the last inequality we conclude that F maps the ball $\overline{B_{r_0}}$ into itself where r_0 is the existing constant in the assumption (H_6) and

$$\overline{B_{r_0}} = \{u \in C(I,\ell_p); \|u\|_{C(I,\ell_p)} \leq r,\ u(0) = 0, u(T) = au(\xi)\}.$$

Next we show that F is continuous on the ball $\overline{B_{r_0}}$. Let $u, v \in \overline{B_{r_0}}$ and $\varepsilon > 0$ such that $\|u - v\|_{C(I,\ell_p)} < \varepsilon$. For all $t \in [0,T]$, we have

$$\begin{aligned}
&\|(Fu)(t) - (Fv)(t)\|^p_{\ell_p}\\
= &\sum_{i=1}^{\infty} \Big| \int_0^T k(t,s)\Big[F_i\Big(s, h_i(s,u(s)), (Gu)(s)\int_0^T g_i(s,u(s))ds\Big)\\
&-F_i\Big(s, h_i(s,v(s)), (Gv)(s)\int_0^T g_i(s,v(s))ds\Big)\Big]ds\Big|^p\\
\leq\ & T^{\frac{p}{q}} \int_0^T \sum_{i=1}^{\infty} |k(t,s)|^p \Big| F_i\Big(s, h_i(s,u(s)), (Gu)(s)\int_0^T g_i(s,u(s))ds\Big)\\
&-F_i\Big(s, h_i(s,v(s)), (Gv)(s)\int_0^T g_i(s,v(s))ds\Big)\Big|^p ds\\
\leq\ & T^{\frac{p}{q}} M^p \int_0^T \sum_{i=1}^{\infty} \Big[e^{-\tau}\Big(|h_i(s,u(s)) - h_i(s,v(s))|^p
\end{aligned}$$

$$
\begin{aligned}
& +\Big|(Gu)(s)\int_0^T g_i(s,u(s))ds-(Gv)(s)\int_0^T g_i(s,v(s))ds\Big|^p\Big)\Big]ds \\
\leq\;& T^{\frac{p}{q}}M^p\int_0^T\Big[e^{-\tau}\sum_{i=1}^{\infty}e^{-\tau}|v_i(s)-u_i(s)|^p \\
& +2^pe^{-\tau}|(Gv)(s)|^p\Big(\int_0^T\sum_{i=1}^{\infty}|g_i(s,v(s))-g_i(s,u(s))|ds\Big)^p \\
& +2^pe^{-\tau}|(Gv)(s)-(Gu)(s)|^p\Big(\int_0^T\sum_{i=1}^{\infty}|g_i(s,u(s))|ds\Big)^p\Big]ds \\
\leq\;& T^{\frac{p}{q}}M^p\int_0^T\Big[e^{-2\tau}\|v-u\|^p_{C(I,\ell_p)}+e^{-\tau}2^p(a+b\|v\|^p)^p\times \\
& \Big(\lim_{k\longrightarrow\infty}\times\int_0^T\sum_{i=1}^{k}|g_i(s,v(s))-g_i(s,u(s))|ds\Big)^p \\
& +2^pe^{-\tau}\Big(\|v-u\|_{C(I,\ell_p)}\int_0^T\sum_{i=1}^{\infty}|b_i(s)|ds\Big)^p\Big]ds.
\end{aligned}
$$

As a consequence of Lebesgue dominated convergence theorem, from the above inequality and applying the continuity of g on $[0,T]\times\ell_p$ we insert that

$$
\begin{aligned}
& \|(Fu)(t)-(Fv)(t)\|^p_{l_p} \\
\leq\;& T^{\frac{p}{q}}M^pT\Big\{e^{-2\tau}\varepsilon+2^pe^{-\tau}(a+b\|v\|^p)^p\Big(\int_0^T\delta_1(\varepsilon)ds\Big)^p+2^pe^{-\tau}(\varepsilon TB)^p\Big\},
\end{aligned}
$$

where

$$\delta_1(\varepsilon)=\sup\{|g_i(t,v)-g_i(t,u)|:u,v\in\ell_p,\|u-v\|_{C(I,\ell_p)}\leq\varepsilon,t\in I,i=1,2,3,\ldots\},$$

and $\delta_1(\varepsilon)\longrightarrow 0$ as $\varepsilon\to 0$. Hence F is continuous on the ball $\overline{B_{r_0}}$. Now let X be a nonempty subset of $\overline{B_{r_0}}$. Then, taking into account our assumptions, for arbitrary fixed $t\in I$ we have

$$
\begin{aligned}
& \chi_{\ell_p}((FX)(t)) \\
=\;& \lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\sum_{i\geq n}|Fu_i(t)|^p\Big)^{\frac{1}{p}}\Big\} \\
\leq\;& \lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\sum_{i\geq n}\Big|\int_0^T k(t,s)\times F_i\Big(s,h_i(s,u(s)),(Gu)(s)\int_0^T g_i(s,u(s))ds\Big)ds\Big|^p\Big)^{\frac{1}{p}}\Big\} \\
\leq\;& (T^{\frac{p}{q}})^{\frac{1}{p}}2^p\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\sum_{i\geq n}\int_0^T|k(t,s)|^p\Big|F_i\Big(s,h_i(s,u(s)),(Gu)(s)\int_0^T g_i(s,u(s))ds\Big) \\
& -F_i(s,0,0)\Big|^p+|F_i(s,0,0)|^p\Big)ds\Big)^{\frac{1}{p}}\Big\} \\
\leq\;& (T^{\frac{p}{q}})^{\frac{1}{p}}M^p2^p\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\sum_{i\geq n}\int_0^T\Big(e^{-\tau}|h_i(s,u(s))|^p \\
& +e^{-\tau}\Big|(Gu)(s)\int_0^T g_i(s,u(s))ds\Big|^p ds\Big)^{\frac{1}{p}}
\end{aligned}
$$

$$\begin{aligned}
&\leq 2^p(T^{\frac{p}{q}})^{\frac{1}{p}}M^p \lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\int_0^T\Big(e^{-\tau}\sum_{i\geq n}|h_i(s,u(s))|^p\\
&\quad +e^{-\tau}(a+b\|u\|^p)\int_0^T\sum_{i\geq n}|b_i(s)|ds\Big)^p ds\Big)^{\frac{1}{p}}\Big\}\\
&\leq 2^p(T^{\frac{p}{q}})^{\frac{1}{p}}TM^pe^{-2\tau}\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\sum_{i\geq n}|u_i(t)|^p\Big)^{\frac{1}{p}}\Big\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sup_{t\in I}\chi_{\ell_p}((FX)(t)) &= \chi_{C(I,\ell_p)}(FX)\\
&\leq \sup_{t\in I}e^{-2\tau}M^p(T^{\frac{p}{q}})^{\frac{1}{p}}T2^p\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\sum_{i\geq n}|u_i(t)|^p\Big)^{\frac{1}{p}}\Big\}.
\end{aligned}$$

By passing to logarithms, we get

$$\ln(\chi_{C(I,\ell_p)}(FX))+2\tau\leq\ln(\chi(X)).$$

Now by applying Theorem 2.3 with $f(t)=\ln(t)$ and $\theta(t)=2\tau$, we obtain that F has a fixed point and the proof is completed. □

Example 3.3. *Now, we investigate the following infinite system of fractional differential equations*

$$\begin{cases}
D^{5/4}u_n(t)=\dfrac{\Big(e^{-t-\tau-n}\Big)^{\frac{1}{p}}}{2}\sin\Big(\dfrac{\Big(e^{-t-\tau-n}\Big)^{\frac{1}{p}}\cos\Big(|u_n(t)|\Big)}{2}\\
\qquad +\cos\Big(\dfrac{1}{1+|x(t)|_{\ell_p}}\Big)\displaystyle\int_0^T\arctan\Big(\dfrac{\frac{1}{2^n}e^{-s}}{8+|u_n(s)|}|\Big)ds\Big), t\in[0,T],\\
u_n(0)=0,\quad u_n(T)=\sqrt[4]{2}u_n\Big(\dfrac{T}{3}\Big);\ \ n=1,2,3,\dots
\end{cases}\tag{8}$$

Let us observe that the system (8) is a special case of system (1)-(2) if we put

$$\begin{aligned}
F_n(t,x,y) &= \frac{\Big(e^{-t-\tau-n}\Big)^{\frac{1}{p}}\sin\Big(x+y\Big)}{2},\quad (Gx)(t)=\cos\Big(\frac{1}{1+|x(t)|_{\ell_p}}\Big)\\
h_n(t,u(t)) &= \frac{\Big(e^{-t-\tau-n}\Big)^{\frac{1}{p}}\cos\Big(|u_n(t)|\Big)}{2},\quad g_n(s,u(s))=\arctan\Big(\frac{\frac{1}{2^n}e^{-s}}{8+|u_n(s)|}\Big).
\end{aligned}$$

Suppose $t\in[0,T]$, $x_1,x_2,y_1,y_2\in\mathbb{R}$ and $u,v\in\ell_p$. From the definition of F_n and h_n, we conclude that

$$\begin{aligned}
|F_n(t,x_1,y_1)-F_n(t,x_2,y_2)|^p &\leq e^{-\tau}\Big(|x_1-y_1|^p+|x_2-y_2|^p\Big),\\
|h_n(t,u(t))|^p &\leq e^{-\tau}|u_n(t)|^p,\\
|h_n(t,u(t))-h_n(t,v(t))|^p &\leq e^{-\tau}|u_n(t)-v_n(t)|^p.
\end{aligned}$$

Consequently F and h satisfy the assumption (H_1). *Moreover, $|F_i(s,0,0)|=0$ and condition* (H_2) *is clearly satisfied. On the other hand the function*

$$(Gx)(t)=\cos\Big(\frac{1}{1+|x(t)|_{\ell_p}}\Big)$$

verifies assumption (H$_3$) *with* $a = 1$ *and* $b = 0$. *To justify assumption* (H$_4$), *let* $s \in [0, T]$ *and* $u \in C(I, \ell_p)$. *Then we have*

$$\begin{aligned} |g_n(s, u(s))| &= \left|\arctan\left(\frac{\frac{1}{2^n}e^{-s}}{8 + |u_n(s)|}\right)\right| < \frac{e^{-s}}{2^n} = b_n(s) < e^{-s}, \\ q &= \sup\{|b_n(s)|; n \geqslant 1, s \in [0, T]\} \leq 1. \end{aligned}$$

Furthermore, applying the above inequality we infer that the series $\sum_{i=1}^{\infty} b_n(t)$ *is uniformly convergent on I. Again we have*

$$B_k = \sup\Big\{\sum_{n\geq k}\Big|\int_0^T g_n(s, u(s))ds\Big|^p : s \in [0, T]\Big\} \leq \sup\Big\{(1 - e^{-T})\sum_{n\geq k}\frac{1}{2^n}\Big\}^p.$$

As $k \longrightarrow \infty$ *we get* $\sum_{k\geq n}\frac{1}{2^n} \longrightarrow 0$. *Thus,* $B_k \longrightarrow 0$. *Finally the existing inequality in assumption* (H$_6$) *has the form*

$$2^p T^{\frac{p}{q}} M^p e^{-2\tau} r^p T + e^{-\tau} B_0 T 2^p T^{\frac{p}{q}} M^p \leq r^p.$$

Thus, for the number r_0 *we can take* $r_0 = e^{-\tau}B_0T2^pT^{\frac{p}{q}}M^p\Big/(1-2^pT^{\frac{p}{q}}M^pe^{-2\tau}T)$. *Consequently all conditions of Theorem 3.2 are satisfied and thus the system of fractional differential equation (8) has at least one solution in the space* $C(I, \ell_p)$.

4. Solution in sequence space c_0

Now we investigate the existence of solutions for the infinite system (1)-(2) in the space c_0, the space of sequences converging to zero, equipped with the norm $\|x\|_{c_0} = \sup\{|x_i| : i = 1, 2, \ldots\}$. Let us denote by M_{c_0} the families of all nonempty bounded subsets of c_0. For the Banach space $(c_0, \|\cdot\|_{c_0})$, the Hausdorff measure of noncompactnes χ is given by (cf. [3]):

$$\chi(B) = \lim_{n\longrightarrow\infty}\Big\{\sup_{u\in B}\Big\{\max_{k\geq n}|u_k|\Big\}\Big\}, \tag{9}$$

where $B \in M_{c_0}$ and $x(t) = (x_i(t))_{i=1}^{\infty} \in c_0$.

We need the following assumptions in the sequel:

(A$_1$) The functions $F_n : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h_n : [0, T] \times c_0 \to \mathbb{R}$ are continuous functions and there exist positive real numbers $\tau > 0$ such that

$$\begin{aligned} |F_n(t, x_1, x_2) - F_n(t, x_1, x_2)| &\leq e^{-\tau}(|x_1 - y_1| + |x_2 - y_2|), \\ |h_n(t, u(t))| &\leq e^{-\tau}\sup_{n\geq 1}\{|u_i(s)|; i \geq n\}, \\ |h_n(t, u(t)) - h_n(t, v(t))| &\leq e^{-\tau}\sup_{n\geq 1}\{|u_i(s) - v_i(s)|; i \geq n\}. \end{aligned}$$

for $t \in [0, T]$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $u(t) = (u_i(t)), v(t) = (v_i(t)) \in c_0$ where $i = 1, 2, 3, \ldots$.

(A$_2$) The function $t \to F_i(t, 0, 0)$ is bounded on $[0, T]$ i.e

$$M_1 = \sup\{|F_i(t, 0, 0)|; t \in [0, T], i \geqslant 1\}.$$

Moreover, $\lim_{i\longrightarrow\infty} F_i(t, 0, 0) = 0$.

(A$_3$) $G : C(I, c_0) \to C(I, \mathbb{R})$ is a continuous operator such that

$$|(Gx)(t) - (Gu)(t)| \leq \|x(t) - u(t)\|_{c_0}, \quad |(Gx)(t)| \leq a + b\|x(t)\|_{c_0},$$

for $x, u \in C(I, c_0)$, $t \in [0, T]$, where a, b are positive real numbers.

(A_4) $g_n : [0,T] \times c_0 \to \mathbb{R}$ is continous and there exist continuous functions $b_i : [0,T] \to \mathbb{R}$ such that

$$|g_n(s,u(s))| \leq |b_n(s)|, \ \ q = \sup\{|b_n(s)|; s \in [0,T]\},$$

for all $s \in [0,T]$ and $u \in C(I,c_0)$. Moreover, $\lim_{n\longrightarrow\infty} \int_0^T |b_n(s)|ds = 0$.

(A_5) There exists a positive solution r_0 of the inequality

$$e^{-2\tau} r + e^{-\tau}(a+br)qT + M_1 \leq r.$$

Moreover, assume that $TM < 1$.

Theorem 4.1. *Under assumptions* $(A_1) - (A_5)$, *infinite system (1)-(2) has at least one solution u such that $u(t) \in c_0$ for all $t \in [0,T]$.*

Proof. Let us define the operator F on the space $C(I,c_0)$ by

$$(Fu)(t) = \left(\int_0^T k(t,s)F_i\left(t, h_i(s,u(s)), (Gu)(s)\int_0^T g_i(s,u(s))ds\right)ds\right)$$

for all $t \in [0,T]$, where $C(I,c_0)$ is the space of all continuous functions on the interval $[0,T]$ with values in space c_0 and eqipped with the norm $\|u\| = \sup\{\|u(t)\|_{c_0} : \ t \in [0,T]\}$. We show that $(Fu)(t) \in c_0$. For arbitrarily fixed $t \in [0,T]$, we have

$$\begin{aligned}
&\|(Fu)(t)\|_{c_0}\\
&= \sup_{n\geq 1}\left|\int_0^T k(t,s)F_n\left(s, h_n(s,u(s)), (Gu)(s)\int_0^T g_n(s,u(s))ds\right)ds\right|\\
&\leq \sup_{n\geq 1}\int_0^T |k(t,s)|\left(\left|F_n\left(s, h_n(s,u(s)), (Gu)(s)\int_0^T g_n(s,u(s))ds\right) - F_n(s,0,0)\right|\right.\\
&\left.+|F_n(s,0,0)|\right)ds\Big| \leq \sup_{n\geq 1}\int_0^T |k(t,s)|\left(e^{-\tau}|h_n(s,u(s))| + e^{-\tau}\left|(Gu)(s)\int_0^T g_n(s,u(s))ds\right|\right.\\
&\left.+|F_n(s,0,0)|\right)ds \leq e^{-\tau}\sup_{n\geq 1}\int_0^T M|h_n(s,u(s))|ds\\
&+Me^{-\tau}\sup_{n\geq 1}\left|(Gu)(s)\int_0^T g_n(s,u(s))ds\right| + M\sup_{n\geq 1}|F_n(s,0,0)|\\
&\leq Me^{-2\tau}\int_0^T \sup\{|u_i(s)| : i \geq n\}ds + Me^{-\tau}(a+b\|u\|)qT + MM_1T.
\end{aligned} \tag{10}$$

Consequently,

$$\|Fu\| \leq Te^{-2\tau}M\|u\| + Me^{-\tau}(a+b\|u\|)qT + MM_1T. \tag{11}$$

We show that Fu continuously transforms the interval $[0,T]$ in to the space c_0. Let $t_{t_0} \in [0,T]$ and $\epsilon > 0$ be arbitrary. By the continuity $k(t,s)$, there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies that

$$|k(t,s) - k(t_0,s)| < \frac{\epsilon}{Te^{-2\tau}r + e^{-\tau}(a+br)qT + TM_1}. \tag{12}$$

Consequently, from (10) and (12) we conclude that $\|(Fu)(t) - (Fu)(t_0)\|_{c_0} < \epsilon$. Moreover, as a consequence of (11), the function F maps the ball $\overline{B_{r_0}}$ into itself where r_0 is the existing constant in assumption (A_5) and

$$\overline{B_{r_0}} = \{u \in C(I,c_0); \|u\|_{C(I,c_0)} \leq r, u(0) = 0, u(T) = au(\xi)\}.$$

Now, we prove that F is a continuous operator on $\overline{B_{r_0}}$. To do this, let us fix $\varepsilon > 0$ and take arbitrary $u, v \in \overline{B_{r_0}}$ such that $\|u - v\|_{C(I,c_0)} < \varepsilon$. Then, for $t \in [0, T]$, we have

$$\begin{aligned}
\|(Fu)(t) - (Fv)(t)\|_{c_0} &\leq \sup_{n\geq 1} \int_0^T k(t,s)\Big[e^{-\tau}\sup\{|u_i(s) - v_i(s)|; i \geqslant n\} \\
&\quad + e^{-\tau}|(Gu)(s) - (Gv)(s)| \int_0^T |g_n(s, u(s)))|ds \\
&\quad + e^{-\tau}|(Gv)(s)| \int_0^T |g_n(s, u(s)) - g_n(s, v(s))|ds\Big]ds \\
&\leq \sup_{n\geq 1}\Big[\int_0^T |k(t,s)|\Big(e^{-\tau}|u - v|_{C(I,c_0)} \\
&\quad + qTe^{-\tau}|u - v|_{C(I,c_0)} + e^{-\tau}|(Gv)(s)|\Big|\int_0^T \omega_{r_0}^T(g, \varepsilon)ds\Big|\Big)ds\Big],
\end{aligned}$$

where

$$\omega_{r_0}^T(g_i, \varepsilon) = \sup\{|g_i(t, u) - g_i(t, v)| : t \in [0, T], u, v \in c_0, \|u - v\|_{C(I,c_0)} < \varepsilon\}.$$

Moreover, in light of the continuity of g_i on $[0, T] \times c_0$, we have $\omega_{r_0}^T(g_i, \varepsilon) \to 0$. By applying this remark and the previous inequality, the continuty of F is followed. Now let X be a nonempty subset of $\overline{B_{r_0}}$. In view of the formula (9) and our assumptions, we have

$$\begin{aligned}
&\chi_{c_0}(FX)(t) \\
= &\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\max_{i\geq n}\Big|\int_0^T k(t,s)F_i\Big(s, h_i(s, u(s)), (Gu)(s)\int_0^T g_i(s, u(s))ds\Big)ds\Big|\Big)\Big\} \\
\leq &\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big[\max_{i\geq n}\int_0^T |k(t,s)|\Big(e^{-\tau}|h_i(s, u(s))| \\
&+ e^{-2\tau}\Big|(Gu)(s)\int_0^T g_i(s, u(s))ds\Big| + |F_i(s, 0, 0)|\Big)ds\Big]\Big\} \\
\leq &\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big[\max_{i\geq n}\int_0^T |k(t,s)|\Big(e^{-\tau}\sup_{i\geq n}|u_i(s)| \\
&+ e^{-\tau}(a + b\|u(s)\|_{c_0})\int_0^T |b_i(s)|ds + \sup_{i\geq n}|F_i(s, 0, 0)|\Big)ds\Big]\Big\}.
\end{aligned}$$

Consequently,

$$\chi_{C(I,c_0)}(FX) \leq TMe^{-2\tau}\sup_{t\in I}\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\max_{i\geq n}|u_i(t)|\Big)\Big\}.$$

As, $MT < 1$, by passing to logarithms, we have

$$2\tau + \ln(\chi_{c(I,c_0)})(FX) \leq \chi_{c(I,c_0)}(X).$$

Thus all conditions of Theorem 2.3 hold true with $f(t) = \ln(t)$ and $\theta(t) = 2\tau$ and Theorem 2.3 implies that F has a fixed point in the space $C(I, c_0)$, which is a solution of the system (1) − (2). The proof is complete. □

Example 4.2. *In order to show the applicability of Therem 4.1, the fractional differential system*

$$
\begin{aligned}
D^{5/4}u_n(t) &= e^{-\tau-t-n}\sqrt[3]{e^{-\tau-n-t}\sqrt[5]{\sum_{k\geq n}\frac{|u_k(t)|}{(k^2+1)10^n}}+\sqrt[7]{\frac{1}{100}\frac{1}{1+\sum_{k\geq n}\frac{|u_k(t)|}{1+k^2}}\int_0^T g_n(s,u(s))ds}} \\
u_n(0) &= 0, \quad u_n(T)=\frac{1}{2}u_n\Big(\frac{T}{3}\Big); \quad n=1,2,3,\ldots
\end{aligned}
\tag{13}
$$

is included, where

$$
\begin{aligned}
F_n(t,x,y) &= e^{-\tau-n-t}\sqrt[3]{\sqrt[5]{x}+\sqrt[7]{y}}, \quad h_n(t,u(t)) = e^{-\tau-n-t}\sum_{k\geq n}\frac{|u_k(t)|}{(k^2+1)10^n} \\
(Gx)(t) &= \frac{1}{100}\times\frac{1}{1+\sum_{k\geq n}\frac{|x_k(t)|}{(1+k^2)}}, \quad g_n(s,u(s)) = \arctan\left(\frac{e^{-s}2^{-n}}{8+\sum_{k=n}^{\infty}\frac{|u_k(s)|}{(1+k^2)n^2}}\right),
\end{aligned}
$$

For all $x_1,x_2,y_1,y_2\in\mathbb{R}$, $u(t)\in c_0$ and $t\in[0,T]$, we have

$$
\begin{aligned}
|F_n(t,x_1,y_1)-F_n(t,x_2,y_2)| &= e^{-\tau-n-t}\Big[|\sqrt[3]{\sqrt[5]{x_1}+\sqrt[7]{y_1}}-\sqrt[3]{\sqrt[5]{x_1}+\sqrt[7]{y_1}}|\Big] \\
&\leq e^{-\tau}\Big[\sqrt[3]{|\sqrt[5]{x_1}+\sqrt[7]{y_1}-\sqrt[5]{x_2}-\sqrt[7]{y_2}|}\Big] \\
&\leq e^{-\tau}\Big[\sqrt[3]{\sqrt[5]{|x_1-x_2|}+\sqrt[7]{|y_1-y_2|}}\Big] \\
&\leq e^{-\tau}\Big[|x_1-x_2|+|y_1-y_2|\Big],
\end{aligned}
$$

$$
\begin{aligned}
|h_n(t,u(t))| &\leq e^{-\tau}\frac{\pi^2}{6\times 10^n}\sup_{n\geq 1}\{|u_i(t)|; i\geq n\} \\
&\leq e^{-\tau}|u(t)|_{c_0}, \\
|h_n(t,u(t))-h_n(t,v(t))| &\leq e^{-\tau}\frac{\pi^2}{6\times 10^n}|u(t)-v(t)|_{c_0}.
\end{aligned}
$$

Consequently, the hypothesis (A_1) *is fulfilled. Furthermore,*

$$M_1=\sup\{|F_n(t,0,0)|; t\in[0,T], n\geq 1\}=0,$$

and $\lim_{i\longrightarrow\infty}F_i(t,0,0)=0$. *On the other hand, for all $x,u\in C(I,c_0)$ and $t\in[0,T]$, we have*

$$
\begin{aligned}
|(Tx)(t)|\leq 1, \quad |(Tx)(t)-(Tu)(t)| &= \frac{1}{100}\left(\left|\frac{1}{1+\sum_{k\geq n}\frac{|x_k(t)|}{(1+k^2)}}-\frac{1}{1+\sum_{k\geq n}\frac{|u_k(t)|}{(1+k^2)}}\right|\right) \\
&\leq \sup_{n\geq 1}\{|x_k(t)-u_k(t)|; k\geq n\}=|x(t)-u(t)|_{c_0}.
\end{aligned}
$$

Thus the operator G satisfies condition (A_3) *with $a=1, b=0$. In this example*

$$g_n(s,u(s))=\arctan\left(\frac{e^{-s}2^{-n}}{8+\sum_{k=n}^{\infty}\frac{|u_k(s)|}{(1+k^2)n^2}}\right)$$

verifies condition (A_4) *with $b_n(s)=\frac{e^{-s}}{2^n}$ and $q=\frac{1}{2}$. Inconsequently, the existent inequality in condition* (A_5) *has the form*

$$e^{-2\tau}r+e^{-\tau}\frac{T}{2}\leq r.$$

Obviously, the last inequality has a positive solution. Thus, all conditions of Theorem 4.1 are satisfied and thus the system (13) has at least one solution in the space $C(I,c_0)$.

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