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Existence and Multiplicity of Positive Solutions for a Singular Riemann-Liouville Fractional Differential Problem

Rodica Luca[a]

[a]*Department of Mathematics, Gh. Asachi Technical University, Iasi 700506, Romania*

Abstract. We investigate the existence and multiplicity of positive solutions for a nonlinear Riemann-Liouville fractional differential equation with a nonnegative singular nonlinearity, subject to Riemann-Stieltjes boundary conditions which contain fractional derivatives. In the proofs of our main results, we use an application of the Krein-Rutman theorem and some theorems from the fixed point index theory.

1. Introduction

We consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha}x(t) + f(t, x(t)) = 0, \quad t \in (0,1), \tag{1}$$

with the integral-differential boundary conditions

$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad D_{0+}^{\beta_0}x(1) = \sum_{i=1}^{m} \int_0^1 a_i(t) D_{0+}^{\beta_i}x(t)\, dH_i(t), \tag{2}$$

where $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, n, $m \in \mathbb{N}$, $n \geq 3$, $\beta_i \in \mathbb{R}$ for all $i = 0, \ldots, m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m < \alpha - 1$, $1 \leq \beta_0 < \alpha - 1$, D_{0+}^{k} denotes the Riemann-Liouville derivative of order k (for $k = \alpha, \beta_0, \beta_1, \ldots, \beta_m$), the integrals from the boundary conditions (BC) are Riemann-Stieltjes integrals with H_i, $i = 1, \ldots, m$, functions of bounded variation, the functions $a_i \in C(0,1) \cap L^1(0,1)$, $i = 1, \ldots, m$, and the nonlinearity f is nonnegative and it may be singular at the points $t = 0$, $t = 1$ and/or $x = 0$.

We will present conditions for the data of problem (1),(2) connected to the spectral radii of some associated linear operators such that this problem has at least one or two positive solutions ($x(t) > 0$ for all $t \in (0,1]$). In the proof of the main existence theorems we use an application of the Krein-Rutman theorem in the space $C[0,1]$ and the fixed point index theory. Our assumptions are different than those used in [1], where the authors use various height functions of the sign-changed nonlinearity defined on special bounded sets and two theorems from the fixed point index theory to prove the existence of multiple

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Email address: rluca@math.tuiasi.ro (Rodica Luca)

positive solutions for problem (1),(2) with $a_i(t) = 1$ for all $t \in [0,1]$ and $i = 1, \ldots, m$. The equation (1) with a positive parameter λ, subject to the boundary conditions

$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \ \ D_{0+}^{p} x(1) = \sum_{i=1}^{m} a_i D_{0+}^{q} x(\xi_i),$$

where $\xi_i \in \mathbb{R}$, $i = 1, \ldots, m$, $0 < \xi_1 < \cdots < \xi_m < 1$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$, was investigated in [12]. In paper [12], the nonlinearity f changes sign and it is singular only at $t = 0$ and/or $t = 1$, and there the authors apply the Guo-Krasnosel'skii fixed point theorem to prove the existence of positive solutions when the parameter belongs to various intervals. In the paper [32], the authors prove the existence of at least three positive solutions for equation (1) with the boundary conditions

$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \ \ D_{0+}^{\beta} x(1) = \lambda \int_0^{\eta} h(t) D_{0+}^{\beta} x(t)\, dt,$$

where $\beta \geq 1$, $\alpha - \beta - 1 > 0$, $\eta \in (0,1]$, $0 \leq \lambda \int_0^{\eta} h(t) t^{\alpha-\beta-1}\, dt < 1$, $h \in L^1[0,1]$ is nonnegative and may be singular at $t = 0$ and $t = 1$, and the function f is nonnegative and may be singular at the points $t = 0$, $t = 1$ and $x = 0$. In [32], the authors use different height functions of the nonlinear term on special bounded sets, the Krasnosel'skii theorem and the Leggett-Williams fixed point index theorem. We also mention the paper [26], where the author investigates the fractional differential equation (1) supplemented with the boundary conditions

$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \ \ D_{0+}^{\beta} x(1) = \int_0^{\eta} a(t) D_{0+}^{\gamma} x(t)\, dV(t),$$

where $\beta \in (0,1)$, $\gamma \in [0, \alpha - 1)$, $\eta \in (0,1]$, $a(t) \in L^1[0,1] \cap C(0,1)$, and the function $f(t,x)$ is nonnegative and it may be singular at $t = 0$, $t = 1$ and $x = 0$. The author proves in [26] some existence and multiplicity results which are closely associated with the relationship between 1 and the spectral radii corresponding to the relevant linear operators. For some recent results on the existence, nonexistence and multiplicity of positive solutions for fractional differential equations and systems of fractional differential equations with various boundary conditions, we refer the reader to the monographs [11], [16], [33] and the papers [2]-[6], [8]-[10], [13]-[15], [18]-[23], [25], [27], [28], [30].

The paper is organized as follows. Section 2 contains some auxiliary results which investigate a nonlocal boundary value problem for linear fractional differential equations, and the theorems used in the proofs of the main results. In Section 3, we give the existence and multiplicity theorems for the positive solutions of problem (1),(2). Finally in Section 4, an example is presented to illustrate our main results.

2. Auxiliary results

In this section we present some auxiliary results that we will use in the proof of the main theorems. We consider the fractional differential equation

$$D_{0+}^{\alpha} x(t) + y(t) = 0, \ \ t \in (0,1), \tag{3}$$

with the boundary conditions (2), where $y \in C(0,1) \cap L^1(0,1)$. We denote by

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_0)} - \sum_{i=1}^{m} \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_i)} \int_0^1 s^{\alpha-\beta_i-1}\, a_i(s)\, dH_i(s).$$

Lemma 2.1. *([1]) If $\Delta \neq 0$, then the unique solution $x \in C[0,1]$ of problem (3),(2) is given by*

$$x(t) = \int_0^1 G(t,s) y(s)\, ds, \ \ t \in [0,1], \tag{4}$$

where

$$G(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^{m} \left(\int_0^1 a_i(\tau) g_{2i}(\tau,s)\, dH_i(\tau) \right), \tag{5}$$

and

$$\begin{aligned} g_1(t,s) &= \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta_0-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta_0-1}, & 0 \le t \le s \le 1, \end{cases} \\ g_{2i}(t,s) &= \frac{1}{\Gamma(\alpha-\beta_i)} \begin{cases} t^{\alpha-\beta_i-1}(1-s)^{\alpha-\beta_0-1} - (t-s)^{\alpha-\beta_i-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\beta_i-1}(1-s)^{\alpha-\beta_0-1}, & 0 \le t \le s \le 1, \end{cases} \end{aligned} \tag{6}$$

for all $(t,s) \in [0,1] \times [0,1]$, $i = 1, \dots, m$.

Here the functions g_{2i} may have negative values, because we did not impose a relation between β_0 and β_i, $i = 1, \dots, m$. In [1], the authors used the condition $\beta_m \le \beta_0$ which implies that g_{2i}, $i = 1, \dots, m$ are nonnegative functions (see Lemma 2.3 from [1], or Lemma 2.3 from [12]).

Based on some properties of the function g_1 given by (6) (see [12]), we obtain the following lemma.

Lemma 2.2. *We suppose that* $\Delta \ne 0$ *and* $F(s) := \dfrac{1}{\Delta} \sum_{i=1}^{m} \int_0^1 a_i(\tau) g_{2i}(\tau,s)\, dH_i(\tau) \ge 0$, *for all* $s \in [0,1]$. *Then the Green function* G *given by (5) is a continuous function on* $[0,1] \times [0,1]$ *and satisfies the inequalities:*

a) $G(t,s) \le J(s)$ *for all* t, $s \in [0,1]$, *where* $J(s) = h(s) + F(s)$, $s \in [0,1]$, *and* $h(s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta_0-1}(1-(1-s)^{\beta_0})$, $s \in [0,1]$;

b) $G(t,s) \ge t^{\alpha-1} J(s)$ *for all* t, $s \in [0,1]$;

c) $G(t,s) \le t^{\alpha-1} K(s)$ *for all* t, $s \in [0,1]$, *where* $K(s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta_0-1} + F(s)$, $s \in [0,1]$.

In a similar manner as the authors obtained Lemma 2.5 from [12], we deduce here the following result.

Lemma 2.3. *We suppose that* $\Delta \ne 0$, $F(s) \ge 0$ *for all* $s \in [0,1]$, $y \in C(0,1) \cap L^1(0,1)$ *and* $y(t) \ge 0$ *for all* $t \in (0,1)$. *Then the solution* x *of problem (3),(2) given by (4) satisfies the inequality* $x(t) \ge t^{\alpha-1}\|x\|$ *for all* $t \in [0,1]$, *where* $\|x\| = \sup_{t\in[0,1]} |x(t)|$, *and so* $x(t) \ge 0$ *for all* $t \in [0,1]$.

Now we recall some theorems concerning the fixed point index theory. Let $\mathcal{X}$ be a real Banach space with the norm $\|\cdot\|$, $Q \subset \mathcal{X}$ a cone, "$\le$" the partial ordering defined by Q and θ the zero element in $\mathcal{X}$. For $\rho > 0$, let $B_\rho = \{u \in \mathcal{X},\ \|u\| < \rho\}$ be the open ball of radius ρ centered at θ, its closure $\overline{B}_\rho = \{u \in \mathcal{X},\ \|u\| \le \rho\}$ and its boundary $\partial B_\rho = \{u \in \mathcal{X},\ \|u\| = \rho\}$. The proofs of our main theorems are based on the following fixed point index theorems.

Theorem 2.4. *(see [7]) Let* $\mathcal{A} : \overline{B}_\rho \cap Q \to Q$ *be a completely continuous operator. If there exists* $u_0 \in Q \setminus \{\theta\}$ *such that* $u - \mathcal{A}u \ne \lambda u_0$ *for all* $\lambda \ge 0$ *and* $u \in \partial B_\rho \cap Q$, *then* $i(\mathcal{A}, B_\rho \cap Q, Q) = 0$.

Theorem 2.5. *(see [7]) Let* $\mathcal{A} : \overline{B}_\rho \cap Q \to Q$ *be a completely continuous operator. If* $\mathcal{A}u \ne \mu u$ *for all* $u \in \partial B_\rho \cap Q$ *and* $\mu \ge 1$, *then* $i(\mathcal{A}, B_\rho \cap Q, Q) = 1$.

Let the space $C[0,1]$ and the cone $P = \{u \in C[0,1],\ \ u(t) \ge 0,\ \ \forall\, t \in [0,1]\}$. We present next an application of the Krein-Rutman theorem in the space $C[0,1]$.

Theorem 2.6. *(see [17], [31]) Suppose that* $A : C[0,1] \to C[0,1]$ *is a completely continuous linear operator, and* $A(P) \subset P$. *If there exist* $v \in C[0,1] \setminus (-P)$ *and a constant* $c > 0$ *such that* $cAv \ge v$, *then the spectral radius* $r(A) \ne 0$ *and* A *has an eigenvector* $u_0 \in P \setminus \{\theta\}$ *corresponding to its principal characteristic value* $\lambda_1 = (r(A))^{-1}$, *that is* $\lambda_1 A u_0 = u_0$ *or* $Au_0 = r(A)u_0$, *and so* $r(A) > 0$.

3. Main results

We give in this section some theorems for the existence of at least one or two positive solutions for problem (1),(2). We present firstly the assumptions that we will use in the sequel.

($I1$) $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, n, $m \in \mathbb{N}$, $n \geq 3$, $\beta_i \in \mathbb{R}$ for all $i = 0, \dots, m$, $0 \leq \beta_1 < \beta_2 < \dots < \beta_m < \alpha - 1$, $1 \leq \beta_0 < \alpha - 1$.

($I2$) $a_i \in C(0,1) \cap L^1(0,1)$ for all $i = 1, \dots, m$, and $H_i : [0,1] \to \mathbb{R}$, $i = 1, \dots, m$ are functions of bounded variation.

($I3$) $\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_0)} - \sum_{i=1}^m \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)} \int_0^1 s^{\alpha-\beta_i-1} a_i(s)\, dH_i(s) \neq 0$, and

$F(s) = \dfrac{1}{\Delta} \displaystyle\sum_{i=1}^m \int_0^1 a_i(\tau) g_{2i}(\tau, s)\, dH_i(\tau) \geq 0$ for all $s \in [0,1]$.

($I4$) The function $f : (0,1) \times (0,\infty) \to [0,\infty)$ is continuous. Besides for any $0 < r < R$ there exists $\phi_{r,R} \in C((0,1),[0,\infty)) \cap L^1(0,1)$ such that $f(t,u) \leq \phi_{r,R}(t)$ for all $t \in (0,1)$ and $u \in [rt^{\alpha-1}, R]$.

($I5$) There exist $R_1 > 0$ and a function $p_1 \in C((0,1);[0,\infty)) \cap L^1(0,1)$ with $\int_0^1 p_1(t)\, dt > 0$ such that $f(t,u) \geq p_1(t)u$ for all $(t,u) \in (0,1) \times (0, R_1]$.

($I6$) There exist $R_2 > 0$ and a function $p_2 \in C((0,1);[0,\infty)) \cap L^1(0,1)$ with $\int_0^1 p_2(t)\, dt > 0$ such that $f(t,u) \leq p_2(t)u$ for all $(t,u) \in (0,1) \times [R_2, \infty)$.

($I7$) There exist $R_3 > 0$ and a function $p_3 \in C((0,1);[0,\infty)) \cap L^1(0,1)$ with $\int_0^1 p_3(t)\, dt > 0$ such that $f(t,u) \leq p_3(t)u$ for all $(t,u) \in (0,1) \times (0, R_3]$.

($I8$) There exist $R_4 > 0$ and a function $p_4 \in C((0,1);[0,\infty)) \cap L^1(0,1)$ with $\int_0^1 p_4(t)\, dt > 0$ such that $f(t,u) \geq p_4(t)u$ for all $(t,u) \in (0,1) \times [R_4, \infty)$.

We consider $X = C[0,1]$ the space of continuous functions defined on $[0,1]$ with the supremum norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$, and the cone $P = \{x \in X,\ \ x(t) \geq t^{\alpha-1}\|x\|,\ \forall\, t \in [0,1]\}$.

We define the operators

$$\mathcal{A}x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds,$$
$$\mathcal{T}_i x(t) = \int_0^1 G(t,s) p_i(s) x(s)\, ds, \quad i = 1, \dots, 4.$$

Lemma 3.1. *Assume that* ($I1$) – ($I4$) *hold. Then for any* $r > 0$*, the operator* $\mathcal{A} : P \setminus B_r \to P$ *is completely continuous.*

Proof. For any $x \in P \setminus B_r$, we have $rt^{\alpha-1} \leq x(t) \leq \|x\|$. Let $R = \|x\|$. By ($I4$) it follows that there exists a function $\phi_{r,R} \in C((0,1),[0,\infty)) \cap L^1(0,1)$ such that $f(t, x(t)) \leq \phi_{r,R}(t)$ for all $t \in (0,1)$. Then by using Lemma 2.2, we find

$$\begin{aligned} \mathcal{A}x(t) &= \int_0^1 G(t,s) f(s,x(s))\, ds \leq \int_0^1 J(s) f(s,x(s))\, ds \\ &\leq \int_0^1 J(s)\phi_{r,R}(s)\, ds \leq J_0 \int_0^1 \phi_{r,R}(s)\, ds < \infty, \quad \forall\, t \in [0,1], \end{aligned}$$

where $J_0 = \max_{t\in[0,1]} J(t)$, and then $\mathcal{A}x$ is well defined.

On the other hand, we obtain

$$\mathcal{A}x(t) = \int_0^1 G(t,s) f(s,x(s))\, ds \geq t^{\alpha-1} \int_0^1 J(s) f(s,x(s))\, ds \geq t^{\alpha-1} \mathcal{A}x(t_1),$$

for all $t, t_1 \in [0,1]$, and so $\mathcal{A}x(t) \geq t^{\alpha-1}\|\mathcal{A}x\|$ for all $t \in [0,1]$. Therefore $\mathcal{A}(P \setminus B_r) \subset P$.

We will prove next that $\mathcal{A}$ is completely continuous. Firstly we show that $\mathcal{A}$ is continuous. Let $\{x_n\}_{n\geq 1} \subset P \setminus B_r$ and $\|x_n - x_0\| \to 0$ as $n \to \infty$, with $x_0 \in P \setminus B_r$. Then there exists $R > r$ such that $r \leq \|x_n\| \leq R$

for all $n = 0, 1, \ldots$. By $(I4)$, for the above r, R we deduce (by the absolute continuity of the integral) that for any $\varepsilon > 0$ there exists $\theta \in (0, 1/2)$ such that $\int_0^\theta \phi_{r,R}(s)\,ds < \varepsilon/(6J_0)$ and $\int_{1-\theta}^1 \phi_{r,R}(s)\,ds < \varepsilon/(6J_0)$. Because $f(t,x)$ is uniformly continuous on $[\theta, 1-\theta] \times [\theta^{\alpha-1} r, R]$, then there exists $N > 0$ such that for any $n > N$, we have

$$|f(t, x_n(t)) - f(t, x_0(t))| < \frac{\varepsilon}{3\int_0^1 J(s)\,ds}, \ \ \forall\, t \in [\theta, 1-\theta].$$

Therefore for any $n > N$, we find

$$\begin{aligned}
&\|\mathcal{A}x_n - \mathcal{A}x_0\| \le \max_{t\in[0,1]} \int_0^1 G(t,s)|f(s, x_n(s)) - f(s, x_0(s))|\,ds \\
&\le \int_0^1 J(s)|f(s, x_n(s)) - f(s, x_0(s))|\,ds \\
&\le 2\int_0^\theta J(s)\phi_{r,R}(s)\,ds + \int_\theta^{1-\theta} J(s)|f(s, x_n(s)) - f(s, x_0(s))|\,ds + 2\int_{1-\theta}^1 J(s)\phi_{r,R}(s)\,ds \\
&< 2J_0 \int_0^\theta \phi_{r,R}(s)\,ds + \int_\theta^{1-\theta} J(s)|f(s, x_n(s)) - f(s, x_0(s))|\,ds + 2J_0\int_{1-\theta}^1 \phi_{r,R}(s)\,ds < \varepsilon.
\end{aligned}$$

Hence $\|\mathcal{A}x_n - \mathcal{A}x_0\| \to 0$ as $n \to \infty$, and so $\mathcal{A}$ is a continuous operator.

Next we will show that $\mathcal{A}$ is a compact operator, that is, it maps bounded sets into relatively compact sets. For this, let $E \subset P \setminus B_r$ be a bounded set. Then there exists $R_1 > r$ such that $r \le \|x\| \le R_1$ for all $x \in E$. By the above proof we obtain

$$\mathcal{A}x(t) \le \int_0^1 J(s)\phi_{r,R_1}(s)\,ds \le J_0 \int_0^1 \phi_{r,R_1}(s)\,ds, \ \ \forall\, t \in [0,1], \ x \in E,$$

which implies that $\mathcal{A}(E)$ is uniformly bounded.

The function $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$. So for any $\varepsilon > 0$ there exists $\zeta_1 > 0$ such that for any t_1, $t_2 \in [0,1]$ with $|t_1 - t_2| < \zeta_1$, and for any $s \in [0,1]$ we have

$$|G(t_1, s) - G(t_2, s)| < \frac{\varepsilon}{2\max\{\int_0^1 \phi_{r,R_1}(s)\,ds, 1\}}.$$

Therefore, for any $x \in E$, we deduce

$$\begin{aligned}
&|\mathcal{A}x(t_1) - \mathcal{A}x(t_2)| \le \int_0^1 |G(t_1,s) - G(t_2,s)| f(s, x(s))\,ds \\
&\le \int_0^1 |G(t_1,s) - G(t_2,s)|\phi_{r,R_1}(s)\,ds \le \int_0^1 \frac{\varepsilon}{2\max\{\int_0^1 \phi_{r,R_1}(s)\,ds, 1\}}\phi_{r,R_1}(s)\,ds < \varepsilon.
\end{aligned}$$

This implies that $\mathcal{A}(E)$ is equicontinuous. By using Arzela-Ascoli theorem, we conclude that $\mathcal{A}(E)$ is relatively compact, and then $\mathcal{A} : P \setminus B_r \to P$ is a compact operator. □

Under assumptions $(I1) - (I4)$, by the extension theorem, for any $r > 0$, the operator $\mathcal{A}$ has a completely continuous extension (also denoted by $\mathcal{A}$) from P to P.

By using similar arguments as those used in the proof of Lemma 3.2 from [24], based on Theorem 2.6, we obtain the following result.

Lemma 3.2. *Assume that $(I1) - (I3)$ hold, and $p_i \in C((0,1); [0,\infty)) \cap L^1(0,1)$ with $\int_0^1 p_i(t)\,dt > 0$, $i = 1, \ldots, 4$. Then the operators $\mathcal{T}_i : P \to P$ are linear and completely continuous. Besides, the spectral radius $r(\mathcal{T}_i) > 0$ and $\mathcal{T}_i$ has an eigenfunction $\psi_i \in P \setminus \{\theta\}$ corresponding to the eigenvalue $r(\mathcal{T}_i)$, that is $\mathcal{T}_i\psi_i = r(\mathcal{T}_i)\psi_i$, $i = 1, \ldots, 4$.*

Theorem 3.3. *We assume that* $(I1) - (I4)$ *hold, and there exist* $R_2 > R_1 > 0$ *such that* $(I5)$ *and* $(I6)$ *are satisfied. Besides, we suppose that* $r(\mathcal{T}_1) \geq 1 > r(\mathcal{T}_2) > 0$. *Then the boundary value problem (1),(2) has at least one positive solution.*

Proof. By $(I5)$, for any $x \in \partial B_{R_1} \cap P$, we obtain

$$\mathcal{A}x(t) = \int_0^1 G(t,s)f(s,x(s))\,ds \geq \int_0^1 G(t,s)p_1(s)x(s)\,ds = \mathcal{T}_1 x(t), \quad \forall\, t \in [0,1].$$

We assume that $\mathcal{A}$ has no fixed points on $\partial B_{R_1} \cap P$ (otherwise the theorem is proved). We will show that

$$x - \mathcal{A}x \neq \mu\psi_1, \ \forall\, x \in \partial B_{R_1} \cap P, \ \mu \geq 0, \tag{7}$$

where ψ_1 is given in Lemma 3.2. In fact, if not, there exist $x_1 \in \partial B_{R_1} \cap P$ and $\mu_1 \geq 0$ such that $x_1 - \mathcal{A}x_1 = \mu_1\psi_1$. Then $\mu_1 > 0$ and $x_1 = \mathcal{A}x_1 + \mu_1\psi_1 \geq \mu_1\psi_1$. We denote by $\mu^0 = \sup\{\mu,\ x_1 \geq \mu\psi_1\}$. Then $\mu^0 \geq \mu_1$, $x_1 \geq \mu^0\psi_1$ and

$$\mathcal{A}x_1 \geq \mathcal{T}_1 x_1 \geq \mu^0 \mathcal{T}_1 \psi_1 = \mu^0 r(\mathcal{T}_1)\psi_1 \geq \mu^0\psi_1.$$

Hence $x_1 = \mathcal{A}x_1 + \mu_1\psi_1 \geq \mu^0\psi_1 + \mu_1\psi_1 = (\mu^0 + \mu_1)\psi_1$, which contradicts the definition of μ^0. We deduce that relation (7) holds, and by Theorem 2.4 we deduce

$$i(\mathcal{A}, B_{R_1} \cap P, P) = 0. \tag{8}$$

Now we consider the set

$$V = \{x \in P \setminus B_{R_1},\ \mathcal{A}x = \mu x \ \text{ with } \ \mu \geq 1\}.$$

We will prove next that the set V is bounded. For any $x \in V$, we find

$$\begin{aligned} x(t) &\leq \mu x(t) = \mathcal{A}x(t) = \int_0^1 G(t,s)f(s,x(s))\,ds \\ &= \int_{D_1} G(t,s)f(s,x(s))\,ds + \int_{D_2} G(t,s)f(s,x(s))\,ds \\ &\leq \int_{D_1} G(t,s)p_2(s)x(s)\,ds + \int_0^1 G(t,s)f(s,\widetilde{x}(s))\,ds \\ &\leq \int_0^1 G(t,s)p_2(s)x(s)\,ds + \int_0^1 G(t,s)f(s,\widetilde{x}(s))\,ds \\ &= \mathcal{T}_2 x(t) + \mathcal{A}\widetilde{x}(t) \leq \mathcal{T}_2 x(t) + J_0 M, \end{aligned}$$

where $D_1 = \{s,\ x(s) \geq R_2\}$, $D_2 = \{s,\ x(s) < R_2\}$, $\widetilde{x}(s) = \min\{x(s), R_2\}$, $M = \int_0^1 \phi_{R_1,R_2}(s)\,ds < \infty$ (by $(I4)$). Then we obtain $(I - \mathcal{T}_2)x(t) \leq M$ for all $t \in [0,1]$. Because $r(\mathcal{T}_2) < 1$, we deduce that the inverse operator of $(I - \mathcal{T}_2)$ exists and $(I - \mathcal{T}_2)^{-1} = \sum_{i=1}^{\infty} \mathcal{T}_2^i$. Therefore we find $x(t) \leq (I - \mathcal{T}_2)^{-1}(M)$ and $x(t) \leq M\|(I - \mathcal{T}_2)^{-1}\|$ for all $t \in [0,1]$, which means that V is bounded. We choose $\widetilde{R}_2 > \max\{R_2, \sup\{\|x\|,\ x \in V\}\}$. Then $\mathcal{A}x \neq \mu x$ for all $\mu \geq 1$, $x \in \partial B_{\widetilde{R}_2} \cap P$, and by Theorem 2.5, we obtain

$$i(\mathcal{A}, B_{\widetilde{R}_2} \cap P, P) = 1. \tag{9}$$

By (8) and (9) we conclude

$$i(\mathcal{A}, (B_{\widetilde{R}_2} \setminus \overline{B}_{R_1}) \cap P, P) = i(\mathcal{A}, B_{\widetilde{R}_2} \cap P, P) - i(\mathcal{A}, B_{R_1} \cap P, P) = 1.$$

Then we deduce that $\mathcal{A}$ has at least one fixed point $\widetilde{x}$ on $(B_{\widetilde{R}_2} \setminus \overline{B}_{R_1}) \cap P$, which is a positive solution of problem (1),(2). Taking into account the remark from the beginning of the proof (that $\mathcal{A}$ may have fixed points on $\partial B_{R_1} \cap P$), we conclude that the solution $\widetilde{x}$ of problem (1),(2) satisfies $R_1 \leq \|\widetilde{x}\| < \widetilde{R}_2$. $\square$

Theorem 3.4. *We assume that* $(I1) - (I4)$ *hold and there exist* $R_4 > R_3 > 0$ *such that* $(I7)$ *and* $(I8)$ *are satisfied. Besides, we suppose that* $r(\mathcal{T}_4) > 1 \geq r(\mathcal{T}_3) > 0$. *Then the boundary value problem (1),(2) has at least one positive solution.*

Proof. We suppose that $\mathcal{A}$ has no fixed points on $\partial B_{R_3} \cap P$ (otherwise the theorem is proved). We will show that

$$\mathcal{A}x \neq \mu x, \quad \forall\, x \in \partial B_{R_3} \cap P, \quad \mu > 1. \tag{10}$$

If not, there exist $x_1 \in \partial B_{R_3} \cap P$ and $\mu_1 > 1$ such that $\mathcal{A}x_1 = \mu_1 x_1$. By using $(I7)$ we obtain

$$\begin{aligned} \mu_1 x_1(t) = \mathcal{A}x_1(t) &= \int_0^1 G(t,s) f(s, x_1(s))\, ds \\ &\leq \int_0^1 G(t,s) p_3(s) x_1(s)\, ds = \mathcal{T}_3 x_1(t), \quad \forall\, t \in [0,1]. \end{aligned}$$

Because $\mathcal{T}_3$ is a nondecreasing operator, we deduce

$$\mu_1^2 x_1(t) \leq \mu_1 \mathcal{T}_3 x_1(t) = \mathcal{T}_3(\mu_1 x_1)(t) \leq \mathcal{T}_3(\mathcal{T}_3 x_1(t)) = \mathcal{T}_3^2 x_1(t), \quad \forall\, t \in [0,1].$$

Repeating the process, we find

$$\mu_1^n x_1(t) \leq \mathcal{T}_3^n x_1(t), \quad \forall\, t \in [0,1], \quad n \geq 1,$$

and so

$$\mu_1^n \|x_1\| = \|\mu_1^n x_1\| \leq \|\mathcal{T}_3^n x_1\| \leq \|\mathcal{T}_3^n\| \|x_1\|, \quad \forall\, n \geq 1.$$

We conclude that $\|\mathcal{T}_3^n\| \geq \mu_1^n$ for all $n \geq 1$, and then $r(\mathcal{T}_3) = \lim_{n\to\infty} \sqrt[n]{\|\mathcal{T}_3^n\|} \geq \mu_1 > 1$, which is a contradiction, because $r(\mathcal{T}_3) \leq 1$. Therefore the relation (10) is satisfied, and by Theorem 2.5 we deduce that

$$i(\mathcal{A}, B_{R_3} \cap P, P) = 1. \tag{11}$$

Now we consider a decreasing sequence $(c_n)_{n\geq 1}$, with $0 < c_n < 1$, for all $n \geq 1$, convergent to 0, and we define the operators

$$\mathcal{F}_n x(t) = \int_{c_n}^1 G(t,s) p_4(s) x(s)\, ds.$$

By Theorem 3.7 from [29], the sequence of spectral radii $(r(\mathcal{F}_n))_n$ is increasing and converges to $r(\mathcal{T}_4)$. Then we can choose n_0 sufficiently large such that $r(\mathcal{F}_{n_0}) > 1$. We define $R_{n_0} = R_4 c_{n_0}^{1-\alpha}$. Then for any $x \in \partial B_{R_{n_0}} \cap P$, we have

$$x(t) \geq t^{\alpha-1}\|x\| = t^{\alpha-1} R_{n_0} \geq c_{n_0}^{\alpha-1} R_{n_0} = R_4, \quad \forall\, t \in [c_{n_0}, 1]. \tag{12}$$

In a similar manner as we obtained Lemma 3.2 (see [24]), we deduce that $\mathcal{F}_{n_0}$ has an eigenfunction $\psi_0 \in P\setminus\{\theta\}$ corresponding to the eigenvalue $r(\mathcal{F}_{n_0})$, that is $\mathcal{F}_{n_0}\psi_0 = r(\mathcal{F}_{n_0})\psi_0$. Let $x \in \partial B_{R_{n_0}} \cap P$. By $(I8)$ and (12), we find

$$\begin{aligned} \mathcal{A}x(t) &= \int_0^1 G(t,s) f(s, x(s))\, ds \geq \int_{c_{n_0}}^1 G(t,s) f(s, x(s))\, ds \\ &\geq \int_{c_{n_0}}^1 G(t,s) p_4(s) x(s)\, ds = \mathcal{F}_{n_0} x(t), \quad \forall\, t \in [0,1]. \end{aligned}$$

We assume that $\mathcal{A}$ has no fixed points on $\partial B_{R_{n_0}} \cap P$ (otherwise the theorem is proved). We will show that

$$x - \mathcal{A}x \neq \mu \psi_0, \quad \forall\, x \in \partial B_{R_{n_0}} \cap P, \quad \mu > 0. \tag{13}$$

In fact, if not, there exist $x_2 \in \partial B_{R_{n_0}} \cap P$ and $\mu_2 > 0$ such that $x_2 - \mathcal{A}x_2 = \mu_2\psi_0$. We denote by $\mu_0 = \sup\{\mu,\ x_2 \geq \mu\psi_0\}$. Then $\mu_0 \geq \mu_2$ and $x_2 \geq \mu_0\psi_0$. In addition we have

$$\begin{aligned} x_2(t) &= \mathcal{A}x_2(t) + \mu_2\psi_0(t) \geq \mathcal{F}_{n_0}x_2(t) + \mu_2\psi_0(t) \geq \mu_0\mathcal{F}_{n_0}\psi_0(t) + \mu_2\psi_0(t) \\ &= \mu_0 r(\mathcal{F}_{n_0})\psi_0(t) + \mu_2\psi_0(t) \geq \mu_0\psi_0(t) + \mu_2\psi_0(t) = (\mu_0 + \mu_2)\psi_0(t),\ \ \forall\, t \in [0,1]. \end{aligned}$$

The obtained inequality contradicts the definition of μ_0. Therefore the relation (13) is satisfied, and by Theorem 2.4, we deduce

$$i(\mathcal{A}, B_{R_{n_0}} \cap P, P) = 0. \tag{14}$$

Then by (11) and (14) we conclude that

$$i(\mathcal{A}, (B_{R_{n_0}} \setminus \overline{B}_{R_3}) \cap P, P) = i(\mathcal{A}, B_{R_{n_0}} \cap P, P) - i(\mathcal{A}, B_{R_3} \cap P, P) = -1.$$

This means that $\mathcal{A}$ has at least one fixed point $\widetilde{x}$ on $(B_{R_{n_0}} \setminus \overline{B}_{R_3}) \cap P$, which is a positive solution of problem (1),(2). Taking into account that $\mathcal{A}$ may have fixed points on $(\partial B_{R_{n_0}} \cup \partial B_{R_3}) \cap P$, then the solution $\widetilde{x}$ satisfies $R_3 \leq \|\widetilde{x}\| \leq R_{n_0}$. □

We can also obtain existence results for multiple positive solutions by imposing various conditions similar to $(I5) - (I8)$.

Theorem 3.5. *We assume that* $(I1) - (I4)$ *hold and there exist* $R_4 > R_5 > R_1 > 0$ *such that* $(I5)$, $(I8)$ *and*

$(I9)$ *There exists a function* $p_5 \in C((0,1);[0,\infty)) \cap L^1(0,1)$ *with* $\int_0^1 p_5(t)\,dt > 0$ *such that* $f(t,u) \leq p_5(t)R_5$ *for all* $u \in [R_1 t^{\alpha-1}, R_5]$ *and* $t \in (0,1)$,

hold. Besides, we suppose that $r(\mathcal{T}_1) \geq 1$, $r(\mathcal{T}_4) > 1$ *and* $\|\mathcal{T}_5\| < 1$, *where* $\mathcal{T}_5 x(t) = \int_0^1 G(t,s)p_5(s)x(s)\,ds$ *for* $t \in [0,1]$ *and* $x \in P$. *Then the boundary value problem (1),(2) has at least two positive solutions* x_1 *and* x_2 *with* $R_1 \leq \|x_1\| < R_5 < \|x_2\|$.

Proof. We will show that for any $x \in \partial B_{R_5} \cap P$, we have $\mathcal{A}x \neq \lambda x$ for all $\lambda \geq 1$. If not, there exist $x_0 \in \partial B_{R_5} \cap P$ and $\lambda_0 \geq 1$ such that $\mathcal{A}x_0 = \lambda_0 x_0$. Then we obtain

$$\begin{aligned} x_0(t) &\leq \lambda_0 x_0(t) = \mathcal{A}x_0(t) = \int_0^1 G(t,s)f(s,x_0(s))\,ds \\ &\leq \int_0^1 G(t,s)p_5(s)R_5\,ds = R_5(\mathcal{T}_5 I_d)(t) \leq R_5\|\mathcal{T}_5\| < R_5,\ \ \forall\, t \in [0,1], \end{aligned}$$

where $I_d(t) = t$ for all $t \in [0,1]$. So we deduce $\|x_0\| < R_5$, which is a contradiction, because $x_0 \in \partial B_{R_5} \cap P$. Therefore, by Theorem 2.5 we conclude

$$i(\mathcal{A}, B_{R_5} \cap P, P) = 1. \tag{15}$$

By the proof of Theorem 3.3 we obtain that

$$i(\mathcal{A}, B_{R_1} \cap P, P) = 0, \tag{16}$$

or $\mathcal{A}$ has a fixed point on $\partial B_{R_1} \cap P$.

By the proof of Theorem 3.4, we deduce that there exists $R_{n_0} > R_4$ such that

$$i(\mathcal{A}, B_{R_{n_0}} \cap P, P) = 0, \tag{17}$$

or $\mathcal{A}$ has a fixed point on $\partial B_{R_{n_0}} \cap P$.

If $\mathcal{A}$ has no fixed points on $(\partial B_{R_1} \cup \partial B_{R_{n_0}}) \cap P$, then by the relations (15)-(17) we conclude

$$i(\mathcal{A}, (B_{R_{n_0}} \setminus \overline{B}_{R_5}) \cap P, P) = -1, \ \text{ and } \ i(\mathcal{A}, (B_{R_5} \setminus \overline{B}_{R_1}) \cap P, P) = 1.$$

Therefore the operator $\mathcal{A}$ has at least two fixed points $x_1 \in (B_{R_{n_0}} \setminus \overline{B}_{R_5}) \cap P$ and $x_2 \in (B_{R_5} \setminus \overline{B}_{R_1}) \cap P$, which are positive solutions for problem (1),(2). Because $\mathcal{A}$ may have fixed points on $(\partial B_{R_{n_0}} \cup \partial B_{R_1}) \cap P$, we deduce that the solutions x_1, x_2 of problem (1),(2) satisfy $R_1 \leq \|x_1\| < R_5 < \|x_2\| \leq R_{n_0}$. □

Theorem 3.6. *We assume that* $(I1)-(I4)$ *hold and there exist* $R_2 > R_6 > R_3 > 0$ *such that* $(I6)$, $(I7)$ *and*

$(I10)$ *There exist* $c \in (0,1)$ *and a function* $p_6 \in C((0,1);[0,\infty)) \cap L^1(0,1)$ *with* $\int_0^1 p_6(t)\,dt > 0$ *such that* $f(t,u) \geq p_6(t)R_6$ *for all* $t \in [c,1)$ *and* $u \in [c^{\alpha-1}R_6, R_6]$,

hold. Besides, we suppose that $r(\mathcal{T}_2) < 1$, $r(\mathcal{T}_3) \leq 1$ *and* $\int_c^1 J(s)p_6(s)\,ds > 1$. *Then the boundary value problem (1),(2) has at least two positive solutions* x_1 *and* x_2 *with* $R_3 \leq \|x_1\| < R_6 < \|x_2\|$.

Proof. For any $x \in \partial B_{R_6} \cap P$, we have $x(t) \geq t^{\alpha-1}\|x\| = t^{\alpha-1}R_6 \geq c^{\alpha-1}R_6$ for all $t \in [c,1]$. Then we deduce

$$\begin{aligned}
\|\mathcal{A}x\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)f(s,x(s))\,ds \geq \max_{t\in[0,1]} \int_0^1 t^{\alpha-1}J(s)f(s,x(s))\,ds \\
&\geq \max_{t\in[0,1]} t^{\alpha-1} \int_c^1 J(s)f(s,x(s))\,ds = \int_c^1 J(s)f(s,x(s))\,ds \\
&\geq \int_c^1 J(s)p_6(s)R_6\,ds > R_6 = \|x\|.
\end{aligned}$$

Hence $\|\mathcal{A}x\| > \|x\|$ for all $x \in \partial B_{R_6} \cap P$. This last inequality implies that $\mathcal{A}x \not\leq x$ for all $x \in \partial B_{R_6} \cap P$, and then we obtain (see [7])

$$i(\mathcal{A}, B_{R_6} \cap P, P) = 0. \tag{18}$$

By the proof of Theorem 3.3 we deduce that there exists $\widetilde{R_2} > R_2$ such that

$$i(\mathcal{A}, B_{\widetilde{R_2}} \cap P, P) = 1. \tag{19}$$

By the proof of Theorem 3.4 we conclude that

$$i(\mathcal{A}, B_{R_3} \cap P, P) = 1, \tag{20}$$

or $\mathcal{A}$ has a fixed point on $\partial B_{R_3} \cap P$.

If $\mathcal{A}$ has no fixed points on $\partial B_{R_3} \cap P$, then by the relations (18)-(20) we obtain

$$i(\mathcal{A}, (B_{\widetilde{R_2}} \setminus \overline{B}_{R_6}) \cap P, P) = 1, \text{ and } i(\mathcal{A}, (B_{R_6} \setminus \overline{B}_{R_3}) \cap P, P) = -1.$$

Hence the operator $\mathcal{A}$ has at least two fixed points $x_1 \in (B_{\widetilde{R_2}} \setminus \overline{B}_{R_6}) \cap P$ and $x_2 \in (B_{R_6} \setminus \overline{B}_{R_3}) \cap P$, which are positive solutions for problem (1),(2). Because $\mathcal{A}$ may have fixed points on $\partial B_{R_3} \cap P$, we deduce that the solutions x_1, x_2 of problem (1),(2) satisfy $R_3 \leq \|x_1\| < R_6 < \|x_2\| < \widetilde{R_2}$. □

4. An example

Let $\alpha = 5/2$ $(n = 3)$, $m = 2$, $\beta_0 = 5/4$, $\beta_1 = 1/2$, $\beta_2 = 4/3$, $H_1(t) = \{0,\ t \in [0,1/3);\ 1,\ t \in [1/3,1]\}$, $H_2(t) = t$ for all $t \in [0,1]$, $a_1 = 1$, $a_2 = 1/2$.

We consider the fractional differential equation

$$D_{0+}^{5/2}x(t) + f(t,x(t)) = 0, \quad t \in (0,1), \tag{21}$$

with the boundary conditions

$$x(0) = x'(0) = 0, \quad D_{0+}^{5/4}x(1) = D_{0+}^{1/2}x\left(\frac{1}{3}\right) + \frac{1}{2}\int_0^1 D_{0+}^{4/3}x(t)\,dt. \tag{22}$$

We obtain $\Delta \approx 0.40939289 \neq 0$. We also deduce

$$
\begin{aligned}
&g_1(t,s) = \frac{1}{\Gamma(5/2)} \begin{cases} t^{3/2}(1-s)^{1/4} - (t-s)^{3/2}, & 0 \le s \le t \le 1, \\ t^{3/2}(1-s)^{1/4}, & 0 \le t \le s \le 1, \end{cases} \\
&g_{21}(t,s) = \begin{cases} t(1-s)^{1/4} - (t-s), & 0 \le s \le t \le 1, \\ t(1-s)^{1/4}, & 0 \le t \le s \le 1, \end{cases} \\
&g_{22}(t,s) = \frac{1}{\Gamma(7/6)} \begin{cases} t^{1/6}(1-s)^{1/4} - (t-s)^{1/6}, & 0 \le s \le t \le 1, \\ t^{1/6}(1-s)^{1/4}, & 0 \le t \le s \le 1, \end{cases} \\
&F(s) = \frac{1}{\Delta} \begin{cases} \frac{1}{3}(1-s)^{1/4} - \left(\frac{1}{3} - s\right) + \frac{1}{2\Gamma(13/6)}[(1-s)^{1/4} - (1-s)^{7/6}], & 0 \le s < \frac{1}{3}, \\ \frac{1}{3}(1-s)^{1/4} + \frac{1}{2\Gamma(13/6)}[(1-s)^{1/4} - (1-s)^{7/6}], & \frac{1}{3} \le s \le 1, \end{cases} \\
&G(t,s) = g_1(t,s) + t^{3/2}F(s), \quad t,\ s \in [0,1], \\
&J(s) = \tfrac{1}{\Gamma(5/2)}(1-s)^{1/4}(1-(1-s)^{5/4}) + F(s), \quad s \in [0,1].
\end{aligned}
$$

We have $F(s) \ge 0$, for all $s \in [0,1]$. We mention here that the function g_{22} has negative values in the vicinity of $t = 1$.

We consider the function

$$
f(t,u) = \begin{cases} 10(t-\frac{1}{4})^2 t^{-1/3} u^{-1/5} + (t-\frac{1}{3})^2(1-t)^{-1/4}u, & (t,u) \in (0,1) \times (0,1], \\ \left[10(t-\frac{1}{4})^2 t^{-1/3} + (t-\frac{1}{3})^2(1-t)^{-1/4}\right]\cos^2\frac{\pi(u-1)}{7}, & (t,u) \in (0,1) \times (1,64], \\ \frac{5}{2}(t-\frac{1}{4})^2 t^{-1/3} u^{1/3} + \frac{1}{8}(t-\frac{1}{3})^2(1-t)^{-1/4}u^{1/2}, & (t,u) \in (0,1) \times (64,\infty). \end{cases}
$$

For any $0 < r < R$, we obtain the inequality $f(t,u) \le \phi_{r,R}(t)$ for all $(t,u) \in (0,1) \times [rt^{3/2}, R]$, where $\phi_{r,R}(t),\ t \in (0,1)$ is defined by

$$
\phi_{r,R}(t) = \begin{cases} 10(t-\frac{1}{4})^2 t^{-1/3}(rt^{3/2})^{-1/5} + (t-\frac{1}{3})^2(1-t)^{-1/4}R, & 0 < R \le 1, \\ \max\Big\{10(t-\frac{1}{4})^2 t^{-1/3} + (t-\frac{1}{3})^2(1-t)^{-1/4}; & \\ \quad 10(t-\frac{1}{4})^2 t^{-1/3}(rt^{3/2})^{-1/5} + (t-\frac{1}{3})^2(1-t)^{-1/4}\Big\}, & 1 < R \le 64, \\ \max\Big\{\frac{5}{2}(t-\frac{1}{4})^2 t^{-1/3}R^{1/3} + \frac{1}{8}(t-\frac{1}{3})^2(1-t)^{-1/4}R^{1/2}, & \\ \quad 10(t-\frac{1}{4})^2 t^{-1/3}(rt^{3/2})^{-1/5} + (t-\frac{1}{3})^2(1-t)^{-1/4}\Big\}, & R > 64. \end{cases}
$$

The above function $\phi_{r,R} \in C((0,1);[0,\infty)) \cap L^1(0,1)$, and so the assumptions $(I1) - (I4)$ are satisfied.

We choose $R_1 = 1$, $R_2 = 64$ and the functions

$$
\begin{aligned}
&p_1(t) = 10(t-\tfrac{1}{4})^2 t^{-1/3} + (t-\tfrac{1}{3})^2(1-t)^{-1/4}, \quad t \in (0,1), \\
&p_2(t) = \tfrac{p_1(t)}{3}, \quad t \in (0,1).
\end{aligned}
$$

We have the inequalities $f(t,u) \ge p_1(t)u$ for all $(t,u) \in (0,1) \times (0,1]$, and $f(t,u) \le p_2(t)u$ for all $(t,u) \in (0,1) \times [64,\infty)$. Evidently, $p_1,\ p_2 \in C((0,1);[0,\infty)) \cap L^1(0,1)$, $\int_0^1 p_1(t)\,dt \approx 1.88182 > 0$, $\int_0^1 p_2(t)\,dt \approx 0.62727 > 0$, and so assumptions $(I5)$ and $(I6)$ are also satisfied.

We define the linear operators $\mathcal{T}_1,\ \mathcal{T}_2 : P \to P$, where $P = \{x \in C[0,1],\ x(t) \ge t^{3/2}\|x\|,\ \forall\, t \in [0,1]\}$, by

$$
\begin{aligned}
&\mathcal{T}_1 x(t) = \int_0^1 G(t,s)p_1(s)x(s)\,ds, \\
&\mathcal{T}_2 x(t) = \int_0^1 G(t,s)p_2(s)x(s)\,ds = \frac{1}{3}\mathcal{T}_1 x(t), \quad t \in [0,1], \ x \in P.
\end{aligned}
$$

We will show that $r(\mathcal{T}_1) \ge 1$ and $r(\mathcal{T}_2) < 1$. We denote by $I_d(t) = t,\ \forall\, t \in [0,1]$, and $\zeta(t) = t^{3/2},\ \forall\, t \in [0,1]$. Then we find

$$
\begin{aligned}
\mathcal{T}_1\zeta(t) &= \int_0^1 G(t,s)p_1(s)\zeta(s)\,ds = \int_0^1 G(t,s)p_1(s)s^{3/2}\,ds \\
&\ge \int_0^1 t^{3/2}J(s)p_1(s)s^{3/2}\,ds = \left(\int_0^1 J(s)p_1(s)s^{3/2}\,ds\right)\zeta(t),
\end{aligned}
$$

and therefore

$$\mathcal{T}_1^n\zeta(t) = \mathcal{T}_1(\mathcal{T}_1^{n-1}\zeta)(t) \geq \left(\int_0^1 J(s)p_1(s)s^{3/2}\,ds\right)^n \zeta(t).$$

The last inequality gives us

$$r(\mathcal{T}_1) = \lim_{n\to\infty} \sqrt[n]{\|\mathcal{T}_1^n\|} \geq \lim_{n\to\infty} \sqrt[n]{\left(\int_0^1 J(s)p_1(s)s^{3/2}\,ds\right)^n \max_{t\in[0,1]} \zeta(t)}$$
$$= \int_0^1 J(s)p_1(s)s^{3/2}\,ds \approx 1.70534,$$

and then $r(\mathcal{T}_1) > 1$.

On the other hand, we obtain

$$(\mathcal{T}_1 I_d)(t) = \int_0^1 G(t,s)p_1(s)\,ds \leq \int_0^1 J(s)p_1(s)\,ds \approx 2.41727, \quad \forall\, t \in [0,1],$$

and so $\|\mathcal{T}_1 I_d\| < 2.418$. Therefore we find

$$r(\mathcal{T}_2) = \frac{1}{3}r(\mathcal{T}_1) \leq \frac{1}{3}\|\mathcal{T}_1\| = \frac{1}{3}\|\mathcal{T}_1 I_d\| < 1.$$

Then $0 < \frac{1}{3} < r(\mathcal{T}_2) < 1 < r(\mathcal{T}_1)$. By Theorem 3.3 we conclude that problem (21),(22) has at least one positive solution $x(t)$, $t \in [0,1]$, which satisfies the inequality $x(t) \geq t^{3/2}\|x\|$ for all $t \in [0,1]$.

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