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Ricci Curvature on Warped Product Submanifolds of Sasakian-Space-Forms

Pradip Mandal[a], Tanumoy Pal[a], Shyamal Kumar Hui[a]

[a]*Department of Mathematics, The University of Burdwan, Burdwan, 713104, West Bengal, India.*

Abstract. The paper deals with the study of Ricci curvature on warped product pointwise bi-slant submanifolds of Sasakian-space-form. We obtained some inequalities for such submanifold involving intrinsic invariant, namely the Ricci curvature invariant and extrinsic invariant, namely the squared mean curvature invariant. Some relations of Hamiltonian, Lagrangian and Hessian tensor of warping function are studied here.

1. Introduction

As a generalization of Riemannian product manifold Bishop and O'Neill [5] defined and studied warped product manifolds and in the context of submanifold, such warped product initiated by Chen ([10], [11]). Then many authors studied warped product submanifolds of different ambient manifold ([1], [2], [13], [15], [21]).
On the other hand, Cabrerizo et al. [7] studied bi-slant and semi-slant submanifolds of an almost contact metric manifold and in particular, of Sasakian manifold. Then Khan and Khan [18] studied contact version of pseudo-slant submanifolds. Also then Mihai and Uddin studied warped product pointwise semi-slant submanifolds of Sasakian manifolds [20]. Recently, Chen and Uddin [14] studied warped product pointwise bi-slant submanifolds of Kaehler manifolds. Then Hui et al. [17] studied such submanifolds of Kenmotsu manifolds.
In 1999, Chen [9] established a new relation between intrinsic invariant (the Ricci curvature invariant) and extrinsic invariant (the squared mean curvature invariant) on submanifolds of real space forms. Thereafter, Tripathi [23] studied this inequality on Kaehlerian slant and C-totally real submanifolds in $2m$-dimensional complex space forms and Sasakian-space-forms respectively. Many author studied Ricci curvature on submanifolds in different space forms such as Kenmotsu-space-form [4], cosymplectic-space-form [24], Sasakian-space-form [19] and generalized Sasakian-space-form [16].
Recently Ali et al. [3] studied Ricci curvature on warped product pointwise bi-slant submanifolds of complex space forms and obtain several inequalities between intrinsic invariant and extrinsic invariant. Motivated by the above studies, here we have studied on warped product pointwise bi-slant submanifolds of Sasakian-space-form. The paper is organized as follows. Section 2 is concerned with some preliminaries.

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Corresponding author: Shyamal Kumar Hui
Email addresses: pm2621994@gmail.com (Pradip Mandal), tanumoypalmath@gmail.com (Tanumoy Pal), skhui@math.buruniv.ac.in (Shyamal Kumar Hui)

In Section 3 we have obtained some interesting inequalities between intrinsic and extrinsic invariant on warped product pointwise bi-slant submanifolds of Sasakian-space-form. In section 4, we have obtained some relations on Lagrangian, Hamiltonian and Hessian tensor of warping function with Ricci curvature on pointwise bi-slant submanifolds of Sasakian-space forms.

2. Preliminaries

In an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$, we have [6]

$$\phi^2(X) = -X + \eta(X)\xi, \;\; \phi\xi = 0, \tag{1}$$

$$\eta(\xi) = 1, \;\; g(X, \xi) = \eta(X), \;\; \eta(\phi X) = 0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

$$g(\phi X, Y) = -g(X, \phi Y). \tag{4}$$

$\bar{M}(\phi, \xi, \eta, g)$ is said to be a Sasakian manifolds if it satisfies

$$(\bar{\nabla}_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{5}$$

$$\bar{\nabla}_X\xi = -\phi X. \tag{6}$$

A Sasakian manifold with constant ϕ-sectional curvature c is called Sasakian-space-form, it is denoted by $\bar{M}^{2n+1}(c)$. In $\bar{M}^{2n+1}(c)$, we have

$$\begin{aligned}\bar{R}(X, Y)Z = &\frac{c+3}{4}\Big\{g(Y, Z)X - g(X, Z)Y\Big\} + \frac{c-1}{4}\Big\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\Big\} \\ &+ \frac{c-1}{4}\Big\{\eta(Z)\eta(X)Y - \eta(Z)\eta(Y)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\Big\}.\end{aligned} \tag{7}$$

Let M^{m+1} be a submanifold of $\bar{M}^{2n+1}(c)$, where m is even. If ∇ and $\nabla^{\perp}$ are the induced connections on $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ respectively, then the Gauss and Weingarten formulas are given by [25]

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \; \bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{8}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where σ and A_V are second fundamental form and shape operator respectively such that [25] $g(\sigma(X, Y), V) = g(A_V X, Y)$.
From (8) we have

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)) \tag{9}$$

where $X, Y, Z, W \in \chi(M)$.
For any $X \in \Gamma(TM)$ we can write

$$\phi X = PX + QX \tag{10}$$

where PX and QX are tangential and normal component of ϕX. M is said to be totally geodesic, totally umbilical and minimal according as $\sigma(X, Y) = 0$, $\sigma(X, Y) = g(X, Y)H$ and $H = 0$ where $X, Y \in \Gamma(TM)$ and H is mean curvature. Here the related null space of σ is defined by

$$\mathcal{N}_p = \{X \in T_pM| \;\; \sigma(X, Y) = 0, \text{ for all } Y \in T_pM\}. \tag{11}$$

Definition 2.1. *[22] An immersion $\phi : M \to \bar{M}^{2n+1}(c)$ is called pointwise slant if, any point $p \in M$, the Wirtinger angle $\theta(X)$ between ϕX and T_pM is independent of the choice of non-zero tangent vector $X \in T_pM$ which is linearly independent of ξ. The function θ on M is called the slant function. A pointwise slant submanifold is called pointwise proper slant if it contains no points $p \in M$ such that $\cos\theta = 0$ at p.*

In a pointwise slant submanifolds we have the following relations [22]

$$P^2 = \cos^2\theta[-I + \eta \otimes \xi] \tag{12}$$

$$g(PU, PV) = \cos^2\theta\, g(U, V) \tag{13}$$

$$g(FU, FV) = \sin^2\theta\, g(U, V) \tag{14}$$

for $U, V \in \chi(M)$.

Definition 2.2. *[14] A submanifold M of $\bar{M}^{2n+1}(c)$ is said to be pointwise bi-slant if there exist two distribution $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$, at any point $p \in M$ satisfying the following conditions:*

(i) $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are orthogonal,

(ii) $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$,

(iii) $\phi\mathcal{D}^{\theta_1} \perp \mathcal{D}^{\theta_2}$ and $D^{\theta_1} \perp \phi D^{\theta_2}$,

(iv) The distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are pointwise slant with slant functions θ_1 and θ_2 from TM to $\mathbb{R}$.

Remark 2.3. *(a) If θ_1 and θ_2 from TM to $\mathbb{R}$ are constant functions, then M is called bi-slant submanifold,*

(b) If one of θ_1 and θ_2 is $\frac{\pi}{2}$, then M is called pointwise pseudo-slant submanifold,

(c) If one of θ_1 and θ_2 is 0, then M is called pointwise semi-slant submanifold,

(d) If $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ or $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = 0$, then M is called CR submanifold.

If M is a pointwise bi-slant submanifold of $\bar{M}^{2n+1}(c)$, then for any $X \in \Gamma(TM)$ we have

$$X = T_1X + T_2X \tag{15}$$

where T_1, T_2 are projections from TM onto $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ respectively. Using $P_1 = T_1 \circ P$ and $P_2 = T_2 \circ P$ in (10) we get

$$\phi X = P_1X + P_2X + QX \tag{16}$$

$\forall\ X \in \Gamma(TM)$. From (12) we have

$$P_i^2 X = \cos^2\theta_i\Big(-X + \eta(X)\xi\Big),\ X \in \Gamma(TM),\ i = 1, 2. \tag{17}$$

Let $p \in M$ and $\{e_1, \cdots, e_m, e_{m+1} = \xi\}$ be an orthonormal basis of T_pM. Then the mean curvature vector $H(p)$ is define as

$$H(p) = \frac{1}{m+1}\sum_{i=1}^{m+1}\sigma(e_i, e_i).$$

Also we define

$$||\sigma||^2 = \sum_{i,j=1}^{m+1} g(\sigma(e_i, e_j), \sigma(e_i, e_j)). \tag{18}$$

The gradient of a smooth function f on M is denoted by ∇f and defined by

$$g(\nabla f, X) = Xf. \tag{19}$$

Also we get

$$\|\nabla f\|^2 = \sum_{j=1}^{m+1} (e_j(f))^2. \tag{20}$$

Thus, from the above equation, the Hamiltonian in a local orthonormal frame is defined by

$$H(df, p) = \frac{1}{2}\sum_{i=1}^{m+1} df(e_i)^2 = \frac{1}{2}\sum_{i=1}^{m+1} e_i(f)^2 = \frac{1}{2}\|\nabla f\|^2. \tag{21}$$

Also the Laplacian Δf of f is given by

$$\begin{aligned} \Delta f &= \sum_{i=1}^{m+1} \{(\nabla_{e_i} e_i) f - e_i(e_i(f))\} \\ &= \sum_{i=1}^{m+1} g(\nabla_{e_i} grad\ f, e_i). \end{aligned} \tag{22}$$

Similarly the Hessian tensor of f is defined by

$$\Delta f = -Trace\ H^f = -\sum_{i=1}^{m+1} H^f(e_i, e_i), \tag{23}$$

where H^f is the Hessian function of f. Moreover the Lagrangian for f is defined as

$$L_f = \frac{1}{2}\|\nabla f\|^2. \tag{24}$$

The Euler-Lagrange equation for the Lagrangian (24) is given by [8]

$$\Delta f = 0. \tag{25}$$

The warped product of two smooth manifolds N_1 and N_2 with Riemannian metric g_1, g_2 is denoted by $N_1 \times_f N_2$ and defined by another Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where $g = g_1 + f^2 g_2$ and $f : N_1 \to \mathbb{R}^+$ is a smooth function. If f is constant then the warped product is said to be trivial warped product.

In $M = N_1 \times_f N_2$, we have [5]

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z, \tag{26}$$

for any $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$.

Let L be a k-plane section of T_pM and X is a unit vector in L. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_k\}$ of L such that $X = e_A \in \{e_1, e_2, \cdots, e_k\}$. The Ricci curvature, denoted by $Ric_L(X)$ is defined by

$$Ric_L(X) = \sum_{\substack{i=1 \\ i \neq A}}^{k} K_{iA}, \tag{27}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by $\{e_i, e_j\}$. Such a curvature is called k-Ricci curvature.

The scalar curvature of the k-plane section L is given by

$$\tau(L) = \sum_{1 \le i < j \le k} K_{ij}. \tag{28}$$

Let $M = M_{\theta_1} \times_f M_{\theta_2}$ be a warped product pointwise bi-slant submanifold of $\bar{M}^{2n+1}(c)$ such that $\xi \in \Gamma(D^{\theta_1})$. K_{ij} and $\bar{K}_{ij}$ are the sectional curvatures of the plane sections in M^{m+1} and $\bar{M}^{2n+1}(c)$ respectively. Let dim $M_{\theta_1} = 2n_1 + 1$ and dim $M_{\theta_2} = 2n_2$ such that dim $M = m + 1 = 2n_1 + 2n_2 + 1$. Then from Gauss equation we get

$$\sum_{1\le i<j\le m+1} K_{ij} = \sum_{1\le i<j\le m+1} \bar{K}_{ij} + \sum_{r=m+2}^{2n+1} \sum_{1\le i<j\le m+1} \left\{\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right\}. \tag{29}$$

From (28) and (29), we have

$$\tau(T_pM_{\theta_1}) = \bar{\tau}(T_pM_{\theta_1}) + \sum_{r=m+2}^{2n+1} \sum_{1\le i<j\le 2n_1+1} \left\{\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right\}, \tag{30}$$

$$\tau(T_pM_{\theta_2}) = \bar{\tau}(T_pM_{\theta_2}) + \sum_{r=m+2}^{2n+1} \sum_{2n_1+2\le i<j\le m+1} \left\{\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right\}. \tag{31}$$

Since M^{m+1} is a warped product submanifold we have [12]

$$\sum_{1\le i<j\le m+1} K_{ij} = 2n_2(\Delta(\ln f) - \|\nabla \ln f\|^2) = \frac{2n_2\Delta(f)}{f}, \tag{32}$$

where $\Delta(f)$ is the Laplacian of the warping function f.

3. Main Results

Theorem 3.1. *Let $\psi : M^{m+1} = M_{\theta_1} \times_f M_{\theta_2} \to \bar{M}^{2n+1}(c)$ be a $\mathcal{D}^{\theta_1}$-minimal isometric immersion of a $(m+1)$-dimensional warped product pointwise bi-slant submanifold M^{m+1} into $\bar{M}^{2n+1}(c)$, where $M_{\theta_1}, M_{\theta_2}$ are pointwise slant submanifolds with distinct slant functions θ_1, θ_2 respectively and $\xi \in \Gamma(\mathcal{D}^{\theta_1})$. Then we have*

$$\begin{aligned} Ric(X) + 2n_2(\Delta \ln f) \quad &\le \quad \frac{(m+1)^2}{4}\|H\|^2 + 2n_2\|\nabla(\ln f)\|^2 + \frac{(c+3)}{4}(3n_1 + 5n_2 + 4n_1n_2) \\ &+ \quad \frac{(c-1)}{4}(3\cos^2\theta_1 - 2), \end{aligned} \tag{33}$$

if $X \in \Gamma(\mathcal{D}^{\theta_1})$,

$$\begin{aligned} Ric(X) + 2n_2(\Delta \ln f) \quad &\le \quad \frac{(m+1)^2}{4}\|H\|^2 + 2n_2\|\nabla(\ln f)\|^2 + \frac{(c+3)}{4}(5n_1 + 3n_2 + 4n_1n_2) \\ &+ \quad \frac{(c-1)}{4}(3\cos^2\theta_2 - 2), \end{aligned} \tag{34}$$

if $X \in \Gamma(\mathcal{D}^{\theta_2})$.

Proof. Let $X \in T_pM$ be a unit tangent vector at $p \in M$. Consider a local orthonormal frame $\{e_1, e_2, \cdots, e_{2n_1}, e_{2n_1+1} = \xi, e_{2n_1+2}, \cdots, e_{2n_1+2n_2+1} = e_{m+1}, e_{m+2}, \cdots, e_{2n+1}\}$ of $\bar{M}^{2n+1}(c)$ such that $\{e_1, e_2, \cdots, e_{2n_1}, e_{2n_1+1} = \xi, e_{2n_1+2}, \cdots, e_{2n_1+2n_2+1} = e_{m+1}\}$ is tangent to M.
Assume that $\{e_1, e_2 = \sec\theta_1 P_1 e_1, e_3, e_4 = \sec\theta_1 P_1 e_3, \cdots, e_{2n_1-1}, e_{2n_1} = \sec\theta_1 P_1 e_{2n_1-1}, e_{2n_1+1} = \xi\}$ is an orthogonal frame of $\mathcal{D}^{\theta_1}$, the distribution corresponding to M_{θ_1} and $\{e_{2n_1+2}, e_{2n_1+3} = \sec\theta_2 P_2 e_{2n_1+2}, e_{2n_1+4}, e_{2n_1+5} = \sec\theta_2 P_2 e_{2n_1+4}, \cdots, e_{2n_1+2n_2}, e_{2n_1+2n_2+1} = \sec\theta_2 P_2 e_{2n_1+2n_2}\}$ is an orthonormal frame of $\mathcal{D}^{\theta_2}$, the distribution corresponding to M_{θ_2}.

Set $X = e_A \in \{e_1, e_2, \cdots, e_{2n_1}, e_{2n_1+1} = \xi, e_{2n_1+2}, \cdots, e_{2n_1+2n_2+1} = e_{m+1}\}$.
Now from Gauss equation we have

$$(m+1)^2\|H\|^2 = 2\tau(T_pM) + \|\sigma\|^2 - 2\bar{\tau}(T_pM). \tag{35}$$

We expand (35) for our constructed frame as follows

$$\begin{aligned}(m+1)^2\|H\|^2 &= 2\tau(T_pM) + \sum_{r=m+2}^{2n+1}\left\{(\sigma^r_{11} + \cdots + \sigma^r_{m+1\ m+1} - \sigma^r_{AA})^2 + (\sigma^r_{AA})^2\right\} \\ &- 2\sum_{r=m+2}^{2n+1}\sum_{\substack{1\le i<j\le m+1,\\ i,j\neq A}}\sigma^r_{ii}\sigma^r_{jj} - 2\bar{\tau}(T_pM) + 2\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2.\end{aligned} \tag{36}$$

From (36) we get

$$\begin{aligned}(m+1)^2\|H\|^2 &= 2\tau(T_pM) + \frac{1}{2}\sum_{r=m+2}^{2n+1}\left[(\sigma^r_{11} + \cdots + \sigma^r_{m+1\ m+1})^2 + \left\{2\sigma^r_{AA} - (\sigma^r_{11} + \cdots + \sigma^r_{m+1\ m+1})\right\}^2\right] \\ &+ 2\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2 - 2\sum_{r=m+2}^{2n+1}\sum_{\substack{1\le i<j\le m+1,\\ i,j\neq A}}\sigma^r_{ii}\sigma^r_{jj} - 2\bar{\tau}(T_pM).\end{aligned} \tag{37}$$

Using the $\mathcal{D}^{\theta_1}$-minimality of M^{m+1} in (37) we get

$$\begin{aligned}(m+1)^2\|H\|^2 &= 2\tau(T_pM) + \frac{1}{2}(m+1)^2\|H\|^2 + \frac{1}{2}\sum_{r=m+2}^{2n+1}\left[\left\{2\sigma^r_{AA} - (\sigma^r_{2n_1+2\ 2n_1+2} + \cdots + \sigma^r_{m+1\ m+1})\right\}^2\right] \\ &+ \sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2 - \sum_{\substack{1\le i<j\le m+1,\\ i,j\neq A}}\sigma^r_{ii}\sigma^r_{jj} + \sum_{\substack{i=1\\ i\neq A}}^{m+1}(\sigma^r_{iA})^2 + \sum_{\substack{1\le i<j\le m+1\\ i,j\neq A}}(\sigma^r_{ij})^2 - \sum_{\substack{1\le i<j\le m+1,\\ i,j\neq A}}\sigma^r_{ii}\sigma^r_{jj}\Big] - 2\bar{\tau}(T_pM).\end{aligned} \tag{38}$$

By virtue of (29) in (38) we get

$$\begin{aligned}\frac{1}{2}(m+1)^2\|H\|^2 &= 2\tau(T_pM) - 2\bar{\tau}(T_pM) + \frac{1}{2}\sum_{r=m+2}^{2n+1}\Big[\{(2\sigma^r_{AA} - (\sigma^r_{2n_1+2\ 2n_1+2} + \cdots + \sigma^r_{m+1\ m+1})\}^2 \\ &+ \sum_{\substack{1\le i<j\le m+1,\\ i,j\neq A}}\bar{K}_{ij} - \sum_{\substack{1\le i<j\le m+1,\\ i,j\neq A}}K_{ij} + \sum_{\substack{i=1,\\ i\neq A}}^{m+1}(\sigma^r_{iA})^2 + \sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2 - \sum_{\substack{1\le i<j\le m+1,\\ i,j\neq A}}\sigma^r_{ii}\sigma^r_{jj}\Big].\end{aligned} \tag{39}$$

Using (30), (31) and (32) in (39), we have

$$\begin{aligned}\frac{1}{2}(m+1)^2\|H\|^2 &= Ric(X) + \bar{\tau}(T_pM_{\theta_1}) + \bar{\tau}(T_pM_{\theta_2}) - 2\bar{\tau}(T_pM) + 2n_2\frac{\Delta f}{f} \\ &+ \sum_{\substack{1\le i<j\le m+1\\ i,j\neq A}}\bar{K}_{ij} + \sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le 2n_1+1}\{\sigma^r_{ii}\sigma^r_{jj} - (\sigma_{ij})^2\} \\ &+ \sum_{2n_1+2\le i<j\le m+1}\{\sigma^r_{ii}\sigma^r_{jj} - (\sigma_{ij})^2\} + \sum_{\substack{i=1,\\ i\neq A}}^{m+1}(\sigma^r_{iA})^2 + \frac{1}{2}\sum_{r=m+2}^{2n+1}\Big\{(2\sigma^r_{AA} \\ &-(\sigma^r_{2n_1+2\ 2n_1+2} + \cdots + \sigma^r_{m+1\ m+1})\Big\}^2 + \sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2 - \sum_{\substack{1\le i<j\le m+1,\\ i,j\neq A}}\sigma^r_{ii}\sigma^r_{jj}\Big].\end{aligned} \tag{40}$$

Case-1:
e_A is tangent to M_{θ_1}, i.e., $e_A \in \{e_1, e_2, \cdots, e_{2n_1+1}\}$. We choose $e_A = e_1 = X$. Then from (40), we get

$$\begin{aligned}
\frac{1}{2}(m+1)^2\|H\|^2 &= Ric(X) + \bar{\tau}(T_pM_{\theta_1}) + \bar{\tau}(T_pM_{\theta_2}) - 2\bar{\tau}(T_pM) + 2n_2\frac{\Delta f}{f} + \sum_{2\le i<j\le m+1} \bar{K}_{ij} \\
&+ \sum_{i=2}^{m+1}(\sigma^r_{1i})^2 + \sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le 2n_1+1}\sigma^r_{ii}\sigma^r_{jj} + \sum_{2n_1+2\le i<j\le m+1}\sigma^r_{ii}\sigma^r_{jj} - \sum_{2\le i<j\le m+1}\sigma^r_{ii}\sigma^r_{jj}\Big] \\
&- \sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le 2n_1+1}(\sigma^r_{ij})^2 + \sum_{2n_1+2\le i<j\le m+1}(\sigma^r_{ij})^2 - \sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2\Big] \\
&+ \frac{1}{2}\sum_{r=m+2}^{2n+1}\Big\{2\sigma^r_{11} - (\sigma^r_{2n_1+2\ 2n_1+2} + \cdots + \sigma^r_{m+1\ m+1})\Big\}^2.
\end{aligned} \tag{41}$$

Now, from (7), we get

$$\sum_{1\le i<j\le m+1}\bar{R}(e_i, e_j, e_j, e_i) = \frac{c+3}{4}m(m+1) + \frac{c-1}{4}\Big\{-2m + 3\sum_{1\le i<j\le m+1} g^2(e_i, \phi e_j)\Big\}. \tag{42}$$

For our constructed frame field, we obtained

$$g^2(e_i, \phi e_j) = \begin{cases}\cos^2\theta_1, & \text{for } 1 \le i, j \le 2n_1, \\ \cos^2\theta_2, & \text{for } 1 \le i, j \le 2n_2.\end{cases} \tag{43}$$

Therefore

$$\sum_{1\le i<j\le m+1} g^2(e_i, \phi e_j) = 2(n_1\cos^2\theta_1 + n_2\cos^2\theta_2).$$

Thus we obtain

$$\begin{aligned}
&\bar{\tau}(T_pM_{\theta_1}) + \bar{\tau}(T_pM_{\theta_2}) + \sum_{2\le i<j\le m+1}\bar{K}_{ij} - 2\bar{\tau}(T_pM) \\
&= -\frac{c+3}{4}(3n_1 + 5n_2 + 4n_1n_2) + \frac{c-1}{4}(2 - 3\cos^2\theta_1).
\end{aligned} \tag{44}$$

Also, we calculate

$$\begin{aligned}
&\sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le 2n_1+1}\sigma^r_{ii}\sigma^r_{jj} + \sum_{2n_1+2\le i<j\le m+1}\sigma^r_{ii}\sigma^r_{jj} - \sum_{2\le i<j\le m+1}\sigma^r_{ii}\sigma^r_{jj}\Big] \\
&= \sum_{r=m+2}^{2n+1}\Big[\sum_{j=2}^{2n_1+1}\sigma^r_{11}\sigma^r_{jj} - \sum_{i=2}^{2n_1+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{ii}\sigma^r_{jj}\Big].
\end{aligned} \tag{45}$$

Since M is D^{θ_1}-minimal, we have

$$\sum_{j=2}^{2n_1+1}\sigma^r_{jj} = -\sigma^r_{11}. \tag{46}$$

Therefore, from (45) and (46), we get

$$\sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le 2n_1+1}\sigma^r_{ii}\sigma^r_{jj}+\sum_{2n_1+2\le i<j\le m+1}\sigma^r_{ii}\sigma^r_{jj}-\sum_{2\le i<j\le m+1}\sigma^r_{ii}\sigma^r_{jj}\Big] \tag{47}$$
$$=-\sum_{r=m+2}^{2n+1}(\sigma^r_{11})^2+\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{11}\sigma^r_{jj}.$$

Again we find

$$\sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le 2n_1+1}(\sigma^r_{ij})^2+\sum_{2n_1+2\le i<j\le m+1}(\sigma^r_{ij})^2-\sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2\Big] \tag{48}$$
$$=-\sum_{r=m+2}^{2n+1}\sum_{i=1}^{2n_1+1}\sum_{j=2n_1+2}^{m+1}(\sigma^r_{ij})^2.$$

Also,

$$\sum_{r=m+2}^{2n+1}\{2\sigma^r_{11}-(\sigma^r_{2n_1+2\ 2n_1+2}+\cdots+\sigma^r_{m+1\ m+1})\}^2 \tag{49}$$
$$=4\sum_{r=m+2}^{2n+1}(\sigma^r_{11})^2-4\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{11}\sigma^r_{jj}+(m+1)^2\|H\|^2.$$

Thus, using (44), (47), (48) and (49) in (41) we get

$$\begin{aligned}\frac{1}{2}(m+1)^2\|H\|^2 &= Ric(X)+2n_2\frac{\Delta f}{f}-\frac{(c+3)}{4}(3n_1+5n_2+4n_1n_2)+\frac{c-1}{4}(2-3\cos^2\theta_1)\\ &+\sum_{i=2}^{m+1}(\sigma^r_{1i})^2+\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{11}\sigma^r_{jj}+\sum_{r=m+2}^{2n+1}\sum_{i=1}^{2n_1+1}\sum_{j=2n_1+2}^{m+1}(\sigma^r_{ij})^2+\sum_{r=m+2}^{2n+1}(\sigma^r_{11})^2\\ &-2\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{11}\sigma^r_{jj}+\frac{1}{2}(m+1)^2\|H\|^2.\end{aligned} \tag{50}$$

Neglecting the positive terms $\sum_{i=2}^{m+1}(\sigma^r_{1i})^2$ and $\sum_{r=m+2}^{2n+1}\sum_{i=1}^{2n_1+1}\sum_{j=2n_1+2}^{m+1}(\sigma^r_{ij})^2$ from the right hand side of (50), we get

$$\begin{aligned}Ric(X)+2n_2\frac{\Delta f}{f} &\le \frac{(c+3)}{4}(3n_1+5n_2+4n_1n_2)+\frac{(c-1)}{4}(3\cos^2\theta_1-2)\\ &-\sum_{r=m+2}^{2n+1}(\sigma^r_{11})^2+\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{11}\sigma^r_{jj}.\end{aligned} \tag{51}$$

Using (32) in (51) we get

$$\begin{aligned}Ric(X)+2n_2\Delta(\ln f) &\le 2n_2\|\nabla\ln f\|^2+\frac{(c+3)}{4}(3n_1+5n_2+4n_1n_2)+\frac{(c-1)}{4}(3\cos^2\theta_1-2)\\ &-\sum_{r=m+2}^{2n+1}\{\sigma^r_{11}-\frac{1}{2}\sum_{j=2n_1+2}^{m+1}\sigma^r_{jj}\}^2+\frac{(m+1)^2}{4}\|H\|^2.\end{aligned} \tag{52}$$

Neglecting the term $\sum_{r=m+2}^{2n+1}\{\sigma^r_{11}-\frac{1}{2}\sum_{j=2n_1+2}^{m+1}\sigma^r_{jj}\}^2$ from (52), we get the required equation (33).

Case-2.
e_A is tangent to M_{θ_2} i.e., $e_A \in \{e_{2n_1+2},\cdots,e_{m+1}\}$. We choose $e_A = e_{m+1} = X$. Then from (40), we get

$$\begin{aligned}\frac{1}{2}(m+1)^2\|H\|^2 &= Ric(X)+\bar{\tau}(T_pM_{\theta_1})+\bar{\tau}(T_pM_{\theta_2})-2\bar{\tau}(T_pM)+\sum_{1\le i<j\le m}\bar{K}_{ij}+\sum_{i=1}^{m}(\sigma^r_{i\,m+1})^2 \\ &+ 2n_2\frac{\Delta f}{f}+\sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le 2n_1+1}\sigma^r_{ii}\sigma^r_{jj}+\sum_{2n_1+2\le i<j\le m+1}\sigma^r_{ii}\sigma^r_{jj}-\sum_{1\le i<j\le m}\sigma^r_{ii}\sigma^r_{jj}\Big] \\ &+ \frac{1}{2}\sum_{r=m+2}^{2n+1}\{(2\sigma^r_{m+1\,m+1}-(\sigma^r_{2n_1+2\,2n_1+2}+\cdots+\sigma^r_{m+1\,m+1})\}^2 \\ &- \sum_{r=m+2}^{2n+1}\Big[\sum_{1\le i<j\le 2n_1+1}(\sigma^r_{ij})^2+\sum_{2n_1+2\le i<j\le m+1}(\sigma^r_{ij})^2-\sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2\Big].\end{aligned} \tag{53}$$

Now we calculate

$$\begin{aligned}&\bar{\tau}(T_pM_{\theta_1})+\bar{\tau}(T_pM_{\theta_2})-2\bar{\tau}(T_pM)+\sum_{1\le i<j\le m}\bar{K}_{ij} \\ &= -\frac{(c+3)}{4}(5n_1+3n_2+4n_1n_2)-\frac{(c-1)}{4}(3\cos^2\theta_2-2).\end{aligned} \tag{54}$$

Again we find

$$\begin{aligned}&\sum_{r=m+2}^{2n+1}\Big\{\sum_{1\le i<j\le 2n_1+1}\sigma^r_{ii}\sigma^r_{jj}+\sum_{2n_1+2\le i<j\le m+1}\sigma^r_{ii}\sigma^r_{jj}-\sum_{1\le i<j\le m}\sigma^r_{ii}\sigma^r_{jj}\Big\} \\ &= -\sum_{r=m+2}^{2n+1}(\sigma^r_{m+1\,m+1})^2+\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{m+1\,m+1}\sigma^r_{jj}.\end{aligned} \tag{55}$$

Also we obtain

$$\begin{aligned}&\sum_{r=m+2}^{2n+1}\{2\sigma^r_{m+1\,m+1}-(\sigma^r_{2n_1+2\,2n_1+2}+\cdots+\sigma^r_{m+1\,m+1})\}^2 \\ &= 4\sum_{r=m+2}^{2n+1}\Big[(\sigma^r_{m+1\,m+1})^2-\sum_{2n_1+2\le i<j\le m+1}\sigma^r_{m+1\,m+1}\sigma^r_{jj}\Big]+(m+1)^2\|H\|^2.\end{aligned} \tag{56}$$

Using (48), (54), (55) and (56) in (53) we get

$$\begin{aligned}\frac{1}{2}(m+1)^2\|H\|^2 &= Ric(X)+\frac{2n_2\Delta f}{f}-\frac{(c+3)}{4}(5n_1+3n_2+4n_1n_2)-\frac{(c-1)}{4}(3\cos^2\theta_2-2) \\ &+ \sum_{r=m+2}^{2n+1}\sum_{i=1}^{m}(\sigma^r_{i\,m+1})^2-\sum_{r=m+2}^{2n+1}(\sigma^r_{m+1\,m+1})^2+\sum_{r=m+2}^{2n+1}\Big[\sum_{j=2n_1+2}^{m+1}\sigma^r_{m+1\,m+1}\sigma^r_{jj}+\sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2 \\ &+ 2(\sigma^r_{m+1\,m+1})^2-2\sum_{j=2n_1+2}^{m+1}\sigma^r_{m+1\,m+1}\sigma^r_{jj}\Big]+\frac{1}{2}(m+1)^2\|H\|^2.\end{aligned} \tag{57}$$

Neglecting the positive terms $\sum_{r=m+2}^{2n+1}\sum_{i=1}^{m}(\sigma^r_{i\,m+1})^2$ and $\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m+1}(\sigma^r_{ij})^2$ from the right hand side of (57), we get

$$\begin{aligned} Ric(X)+\frac{2n_2\Delta f}{f} &\le \frac{(c+3)}{4}(5n_1+3n_2+4n_1n_2)+\frac{(c-1)}{4}(3\cos^2\theta_2-2) \\ &- \sum_{r=m+2}^{2n+1}(\sigma^r_{m+1m+1})^2+\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}\sigma^r_{m+1\,m+1}\sigma^r_{jj}. \end{aligned} \tag{58}$$

Using (32) in (58) we get

$$\begin{aligned} Ric(X)+2n_2\Delta(\ln f) &\le 2n_2\|\nabla\ln f\|^2+\frac{(c+3)}{4}(5n_1+3n_2+4n_1n_2)+\frac{(c-1)}{4}(3\cos^2\theta_2-2) \\ &- \sum_{r=m+2}^{2n+1}\{\sigma^r_{m+1m+1}-\frac{1}{2}\sum_{j=2n_1+2}^{m+1}\sigma^r_{jj}\}^2+\frac{(m+1)^2}{4}\|H\|^2. \end{aligned} \tag{59}$$

Neglecting the term $\sum_{r=m+2}^{2n+1}\{\sigma^r_{m+1m+1}-\frac{1}{2}\sum_{j=2n_1+2}^{m+1}\sigma^r_{jj}\}^2$ from (59), we get (34).

For equality case:

(1) If $H(p)=0$, then a unit vector $X\in T_pM$ orthogonal to ξ satisfies the equality case of (33) or (34) if and only if $X\in\mathcal{N}_p$, the relative null space at p.

(2) The equality case of (33) holds identically for all unit vectors tangent to M_{θ_1} and orthogonal to ξ at p if and only if p is a mixed totally geodesic and $\mathcal{D}^{\theta_1}$-totally geodesic point at p.

(3) The equality case of (34) holds identically for all unit vectors tangent to M_{θ_2} at p if and only if p is a mixed totally geodesic point and either $\mathcal{D}^{\theta_2}$-totally geodesic or $\mathcal{D}^{\theta_2}$-totally umbilical point with $n_2=1$.

For (1) we assume that $H(p)=0$. Then for any unit tangent vector $e_A\in\{e_1,e_2,\cdots,e_{2n_1+1},e_{2n_1+2},\cdots,e_{m+1}\}$, the equality of (33) and (34) holds if and only if the following conditions hold

(a) $\sum_{i=1}^{2n+1}\sum_{j=2n_1+2}^{m+1}(\sigma^r_{ij})^2=0$, (b) $\sum_{i=1,\, i\ne A}^{2n_1+1}(\sigma^r_{iA})^2=0$,

(c) $2\sigma^r_{AA}=\sum_{j=2n_1+2}^{m+1}\sigma_{jj}$ such that $r\in\{m+2,\cdots,2n+1\}$.

The condition (a) implies that p is a mixed totally geodesic point. Thus using $\mathcal{D}^{\theta_1}$-minimality of M^{m+1} and combining the conditions of (b) and (c) it is clear that $X=e_A\in\mathcal{N}_p$, the relative null space at p. The converse is trivial.

For (2) the equality condition of (33) holds if and only if the following conditions hold

(a) $\sum_{i=1}^{2n+1}\sum_{j=2n_1+2}^{m+1}(\sigma^r_{ij})^2=0$, (b) $\sum_{i=1,\, i\ne\alpha}^{2n_1+1}(\sigma^r_{\alpha i})^2=0$,

(c) $2\sigma^r_{\alpha\alpha}=\sum_{j=2n_1+2}^{m+1}\sigma_{jj}$, $\alpha\in\{1,\cdots,2n_1\}$, $r\in\{m+2,\cdots,2n+1\}$.

By virtue of $\mathcal{D}^{\theta_1}$-minimality of M^{m+1} and condition (c), we get $\sigma_{\alpha\alpha}=0$, $\alpha\in\{1,\cdots,2n_1+1\}$ and then combining conditions (a) and (b), we get $\sigma^r_{ij}=0\ \forall\ i\ne j,\ i,j\in\{1,2,\cdots,2n_1+1\}$. From which we get the desired result.

For (3), the equality case of (34) holds if and only if

(a) $\sum_{r=m+2}^{2n+1}\sum_{j=2n_1+2}^{m+1}(\sigma^r_{ij})^2 = 0$, (b) $\sum_{i=1, i\neq\alpha}^{2n_1+1}(\sigma^r_{\alpha i})^2 = 0$,

(c) $2\sigma^r_{AA} = \sum_{j=2n_1+2}^{m+1}\sigma_{jj}$, $A \in \{2n_1+2, \cdots, m+1\}$, $r \in \{m+2, \cdots, 2n+1\}$.

Using $\mathcal{D}^{\theta_1}$-minimality of M^{m+1} with the condition (c) we get $\sigma^r_{mm} = \sigma^r_{m+1\,m+1}$ for $n_2 = 1$ and $\sigma^r_{AA} = 0\ \forall\ A \in \{2n_1+2, \cdots, m+1\}$. Combining the conditions (a) and (b), we get $\sigma^r_{ij} = 0\ \forall\ i \neq j,\ i, j \in \{2n_1+2, \cdots, m+1\}$. Thus we get the desired results. □

Remark 3.2. *The above theorem also holds good if we take $\mathcal{D}^{\theta_2}$-minimal isometric immersion of $M^{m+1} = M_{\theta_1} \times_f M_{\theta_2}$ into $\bar{M}^{2n+1}(c)$.*

Corollary 3.3. *Let $\psi : M^{m+1} = M_\theta \times_f M_\perp \to \bar{M}^{2n+1}(c)$ be a $\mathcal{D}^\theta$-minimal isometric immersion of a $(m+1)$-dimensional warped product pointwise pseudo-slant submanifold M^{m+1} into $\bar{M}^{2n+1}(c)$, where M_θ and $M_\perp$ are pointwise slant submanifold with slant function θ and an anti-invariant submanifold respectively and $\xi \in \Gamma(\mathcal{D}^\theta)$. Then we have*

$$\begin{aligned} Ric(X) + n_2(\Delta \ln f) &\leq \frac{(m+1)^2}{4}\|H\|^2 + n_2\|\nabla(\ln f)\|^2 + \frac{(c+3)}{8}(6n_1 + 5n_2 + 4n_1n_2) \\ &+ \frac{(c-1)}{4}(3\cos^2\theta - 2), \quad if\ X \in \Gamma(\mathcal{D}^\theta) \end{aligned}$$

$$\begin{aligned} Ric(X) + n_2(\Delta \ln f) &\leq \frac{(m+1)^2}{4}\|H\|^2 + n_2\|\nabla(\ln f)\|^2 + \frac{(c+3)}{8}(10n_1 + 3n_2 + 4n_1n_2) \\ &- \frac{(c-1)}{2}, \quad if\ X \in \Gamma(\mathcal{D}^\perp) \end{aligned}$$

where $dim(M_\theta) = 2n_1 + 1$ and $dim(M_\perp) = n_2$.

Proof. Substituting $\theta_1 = \theta$ for simplification and $\theta_2 = \frac{\pi}{2}$ in (33) and (34) we get the result. □

Corollary 3.4. *Let $\psi : M^{m+1} = M_T \times_f M_\theta \to \bar{M}^{2n+1}(c)$ be a $\mathcal{D}^T$-minimal isometric immersion of a $(m+1)$-dimensional warped product pointwise semi-slant submanifold M^{m+1} into $\bar{M}^{2n+1}(c)$, where M_T and M_θ are invariant submanifold and a pointwise slant submanifold with slant function θ respectively and $\xi \in \Gamma(\mathcal{D}^T)$. Then we have*

$$\begin{aligned} Ric(X) + 2n_2(\Delta \ln f) &\leq \frac{(m+1)^2}{4}\|H\|^2 + 2n_2\|\nabla(\ln f)\|^2 + \frac{(c+3)}{4}(3n_1 + 5n_2 + 4n_1n_2) \\ &+ \frac{(c-1)}{4} \quad if X \in \Gamma(\mathcal{D}^T) \end{aligned}$$

$$\begin{aligned} Ric(X) + 2n_2(\Delta \ln f) &\leq \frac{(m+1)^2}{4}\|H\|^2 + 2n_2\|\nabla(\ln f)\|^2 + \frac{(c+3)}{4}(5n_1 + 3n_2 + 4n_1n_2) \\ &+ \frac{(c-1)}{4}(3\cos^2\theta - 2), \quad if\ X \in \Gamma(\mathcal{D}^\theta) \end{aligned}$$

where $dim(M_T) = 2n_1 + 1$ and $dim(M_\theta) = 2n_2$.

Proof. Substituting $\theta_1 = 0$ and $\theta_2 = \theta$ for simplification in (33) and (34) we get the result. □

Corollary 3.5. *Let $\psi : M^{m+1} = M_T \times_f M_\perp \rightarrow \bar{M}^{2n+1}(c)$ be a $\mathcal{D}^T$-minimal isometric immersion of a $(m+1)$-dimensional warped product CR submanifold M^{m+1} into $\bar{M}^{2n+1}(c)$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds respectively and $\xi \in \Gamma(\mathcal{D}^T)$. Then we have*

$$\begin{aligned} Ric(X) + n_2(\Delta \ln f) &\leq \frac{(m+1)^2}{4}\|H\|^2 + n_2\|\nabla(\ln f)\|^2 + \frac{(c+3)}{8}(6n_1 + 5n_2 + 4n_1n_2) \\ &+ \frac{(c-1)}{4} \quad if\ X \in \Gamma(\mathcal{D}^T) \end{aligned}$$

$$\begin{aligned} Ric(X) + n_2(\Delta \ln f) &\leq \frac{(m+1)^2}{4}\|H\|^2 + n_2\|\nabla(\ln f)\|^2 + \frac{(c+3)}{8}(10n_1 + 3n_2 + 4n_1n_2) \\ &- \frac{(c-1)}{2}, \quad if\ X \in \Gamma(\mathcal{D}^\perp) \end{aligned}$$

where $dim(M_T) = 2n_1 + 1$ and $dim(M_\perp) = n_2$.

Proof. Substituting $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ in (33) and (34) we get the result. □

4. Some Applications

Theorem 4.1. *Suppose $\psi : M^{m+1} = M_{\theta_1} \times_f M_{\theta_2} \rightarrow \bar{M}^{2n+1}(c)$ be a $\mathcal{D}^{\theta_1}$-minimal isometric immersion of a $(m+1)$-dimensional warped product pointwise bi-slant submanifold M^{m+1} into $\bar{M}^{2n+1}(c)$, where $M_{\theta_1}, M_{\theta_2}$ are pointwise slant submanifolds with distinct slant functions θ_1, θ_2 respectively and $\xi \in \Gamma(\mathcal{D}^{\theta_1})$. Then we have the following*

$$L_{\ln f} \geq \frac{1}{4n_2}\Big[Ric(X) - \frac{(m+1)^2}{4}\|H\|^2 - \frac{c+3}{4}(3n_1 + 5n_2 + 4n_1n_2) - \frac{(c-1)}{4}(3\cos^2\theta_1 - 2)\Big] \ on\ M_{\theta_1}. \quad (60)$$

and

$$L_{\ln f} \geq \frac{1}{4n_2}\Big[Ric(X) - \frac{(m+1)^2}{4}\|H\|^2 - \frac{c+3}{4}(5n_1 + 3n_2 + 4n_1n_2) - \frac{(c-1)}{4}(3\cos^2\theta_2 - 2)\Big] \ on\ M_{\theta_2}. \quad (61)$$

Proof. The theorem follows from (33), (34), (24) and (25). □

Theorem 4.2. *Suppose $\psi : M^{m+1} = M_{\theta_1} \times_f M_{\theta_2} \rightarrow \bar{M}^{2n+1}(c)$ be a $\mathcal{D}^{\theta_1}$-minimal isometric immersion of $(m+1)$-dimensional warped product pointwise bi-slant submanifold M^{m+1} into $\bar{M}^{2n+1}(c)$, where $M_{\theta_1}, M_{\theta_2}$ are pointwise slant submanifolds with distinct slant functions θ_1, θ_2 respectively and $\xi \in \Gamma(\mathcal{D}^{\theta_1})$. Then we have the following*

$$\begin{aligned} H(d\ln f, p) &\geq \frac{1}{4n_2}\Big[Ric(X) + 2n_2\Delta(\ln f) - \frac{(m+1)^2}{4}\|H\|^2 - \frac{c+3}{4}(3n_1 + 5n_2 + 4n_1n_2) \\ &- \frac{(c-1)}{4}(3\cos^2\theta_1 - 2)\Big] \ on\ M_{\theta_1} \end{aligned} \quad (62)$$

and

$$\begin{aligned} H(d\ln f, p) &\geq \frac{1}{4n_2}\Big[Ric(X) + 2n_2\Delta(\ln f) - \frac{(m+1)^2}{4}\|H\|^2 - \frac{c+3}{4}(5n_1 + 3n_2 + 4n_1n_2) \\ &- \frac{(c-1)}{4}(3\cos^2\theta_2 - 2)\Big] \ on\ M_{\theta_2}. \end{aligned} \quad (63)$$

Proof. The theorem follows from (33), (34) and (21). □

Theorem 4.3. *Suppose* $\psi : M^{m+1} = M_{\theta_1} \times_f M_{\theta_2} \to \bar{M}^{2n+1}(c)$ *be a* $\mathcal{D}^{\theta_1}$*-minimal isometric immersion of a* $(m+1)$*-dimensional warped product pointwise bi-slant submanifold* M^{m+1} *into* $\bar{M}^{2n+1}(c)$*, where* $M_{\theta_1}, M_{\theta_2}$ *are pointwise slant submanifolds with distinct slant functions* θ_1, θ_2 *respectively and* $\xi \in \Gamma(\mathcal{D}^{\theta_1})$*. Then we have the following*

$$Ric(X) \leq \frac{(m+1)^2}{4}||H||^2 + n_2\frac{Tr.\mathcal{H}^{\ln f}}{f} + \frac{c+3}{4}(3n_1 + 5n_2 + 4n_1n_2) + \frac{(c-1)}{4}(3\cos^2\theta_1 - 2), \quad (64)$$

for any $X \in \Gamma(\mathcal{D}^{\theta_1})$ *and*

$$Ric(X) \leq \frac{(m+1)^2}{4}||H||^2 + n_2\frac{trace\mathcal{H}^{\ln f}}{f} + \frac{c+3}{4}(5n_1 + 3n_2 + 4n_1n_2) + \frac{(c-1)}{4}(3\cos^2\theta_2 - 2), \quad (65)$$

for any $X \in \Gamma(\mathcal{D}^{\theta_2})$.

Proof. The theorem follows from (33), (34) and (23). □

Theorem 4.4. *Suppose* $\psi : M^{m+1} = M_{\theta_1} \times_f M_{\theta_2} \to \bar{M}^{2n+1}(c)$ *be a* $\mathcal{D}^{\theta_1}$*-minimal isometric immersion of a* $(m+1)$*-dimensional warped product pointwise bi-slant submanifold* M^{m+1} *into* $\bar{M}^{2n+1}(c)$*, where* $M_{\theta_1}, M_{\theta_2}$ *are pointwise slant submanifolds with distinct slant functions* θ_1, θ_2 *respectively and* $\xi \in \Gamma(\mathcal{D}^{\theta_1})$*. If the warping function* f *be harmonic, then we have the following*

$$Ric(X) \leq \frac{(m+1)^2}{4}||H||^2 + \frac{c+3}{4}(3n_1 + 5n_2 + 4n_1n_2) + \frac{(c-1)}{4}(3\cos^2\theta_1 - 2), \quad (66)$$

for any $X \in \Gamma(\mathcal{D}^{\theta_1})$ *and*

$$Ric(X) \leq \frac{(m+1)^2}{4}||H||^2 + \frac{c+3}{4}(5n_1 + 3n_2 + 4n_1n_2) + \frac{(c-1)}{4}(3\cos^2\theta_2 - 2), \quad (67)$$

for any $X \in \Gamma(\mathcal{D}^{\theta_2})$.

Proof. Since f is said to harmonic, then $\Delta f = 0$. Using this in (33) and (34) we get the result. □

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