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On the Normal Scalar Curvature Conjecture for Legendrian Submanifolds in Kenmotsu Space Forms

Monia Fouad Naghi[a], Mića S. Stanković[b], Fatimah Alghamdi[a]

[a]*Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[b]*Department of Mathematics, Faculty of Sciences and Mathematics, University of Nis, Serbia*

Abstract. In this paper, we prove DDVV conjecture (the generalized Wintgen inequality) for Legendrian submanifolds in Kenmotsu space forms. Further, we derive an inequality for slant submanifolds in Kenmotsu space forms.

1. Introduction

In differential geometry, one of most fundamental research problem is to discover the relationships for intrinsic and extrinsic invariants. In [19], P. Wintgen found a relationship between Gaussian curvature G (an intrinsic invariant), the squared mean curvature $||H||^2$ (an extrinsic invariant) and the normal curvature $G^\perp$ of any surface M^2 in $\mathbb{E}^4$ always satisfy the inequality

$$G + G^\perp \leq ||H||^2 \tag{1}$$

and the equality holds if and only if the ellipse of curvature of M^2 in $\mathbb{E}^4$ is a circle. The inequality (1) is called Wintgen inequality and the Whitney 2-sphere satisfies the equality case of Wintgen inequality.

Later, the Wintgen inequality was extended for the surfaces M^2 of codimension m in a real space form $\tilde{M}^{m+2}(c)$ in [18] and [11] independently as:

$$G + G^\perp \leq ||H||^2 + c.$$

The equality case was also investigated.

In 1999, De Smet, Dillen, Verstraelen, Vrancken [9] developed the generalized Wintgen inequality named as *DDVV conjecture* for the submanifolds in real space forms as follows:

Conjecture 1.1. *Let $f : M^n \to \tilde{M}^{n+m}(c)$ be an isometric immersion, where $\tilde{M}^{n+m}(c)$ is a real space form of constant sectional curvature c. Then*

$$\rho + \rho^\perp \leq ||H||^2 + c. \tag{2}$$

where ρ is the normalised scalar curvature (intrinsic invariant) and $\rho^\perp$ is the normalised scalar normal curvature (extrinsic invariant).

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Email addresses: mnaghi@kau.edu.sa (Monia Fouad Naghi), stmica@mts.rs (Mića S. Stanković), fmaalghamdi@kau.edu.sa (Fatimah Alghamdi)

If K and $R^\perp$ are the sectional curvature and the normal curvature tensor on M^n, respectively in $\tilde{M}^{n+m}(c)$, then the normalized scalar curvature tensor ρ is given by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} K(e_i \wedge e_j) \tag{3}$$

where τ is the scalar curvature, and the normalized scalar normal curvature $\rho^\perp$ by

$$\rho^\perp = \frac{2\tau^\perp}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1\leq i<j\leq n} \sum_{n+1\leq r<s\leq m+n} \left(R^\perp(e_i, e_j, e_r; e_s)\right)^2} \tag{4}$$

The Conjecture 1.1 was proven in [9] for a submanifold M^n of arbitrary dimension $n \geq 2$ and codimension 2 in the real space form $\tilde{M}^{n+2}(c)$ of constant sectional curvature c. Later, the DDVV conjecture was proved for general case in [12] and in [10] independently.

For a normally flat submanifold, i.e., $R^\perp = 0$, this conjecture was proved by B.-Y. Chen in [6]. Hence, the conjecture is true for the hypersurfaces of real space forms.

Recently, I. Mihai proved DDVV conjecture for Lagrangian submanifolds in complex space forms [14] and for Legendrian submanifolds in Sasakian space forms [15]. In this paper, we derive the generalized Wintgen inequality (DDVV conjecture) for Legendrian submanifolds in Kenmotsu space forms.

2. Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $(\tilde{M}^{2m+1}, g)$ is said to be a Kenmotsu manifold if it admits a $(1,1)$ tensor field φ of its tangent bundle $T\tilde{M}^{2m+1}$, a vector field ξ and a 1-form η, satisfying [4]

$$\begin{aligned} &\varphi^2 = -I + \eta \otimes \xi, \ \ \varphi\xi = 0, \ \ \eta \circ \varphi = 0, \ \ \eta(\xi) = 1, \\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \\ &(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \ \ \tilde{\nabla}_X \xi = X - \eta(X)\xi. \end{aligned}$$

for all vector fields X, Y on $\tilde{M}^{2m+1}$, where $\tilde{\nabla}$ is the Levi-Civita connection of g.

A Kenmotsu manifold with constant φ-sectional curvature c is said to be *Kenmotsu space from* and is denoted by $\tilde{M}^{2m+1}(c)$. Recall that the Riemannian curvature tensor of a Kenmotsu space form $\tilde{M}^{2m+1}(c)$ is given by

$$\begin{aligned} \tilde{R}(X, Y; Z, W) = {} & \frac{(c-3)}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} - \frac{(c+1)}{4}[\eta(Z)\{\eta(Y)g(X, W) - \eta(X)g(Y, W)\} \\ & + \eta(W)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} - g(\varphi X, W)g(\varphi Y, Z) + g(\varphi X, Z)g(\varphi Y, W) \\ & + 2g(\varphi X, Y)g(\varphi Z, W)] \end{aligned} \tag{5}$$

for any vector fields X, Y, Z and W tangent to $\tilde{M}^{2m+1}(c)$. As examples of Kenmotsu space forms we mention $\mathbb{R}^{2m+1}$ and $\mathbb{H}^{2m+1}(-1)$, with usual Kenmotsu structures (for instance, see [4]).

Let M^n be an n-dimensional Riemannian manifold isometrically immersed in a Kenmotsu space from $\tilde{M}^{2m+1}(c)$. We denote by ∇ and h, the Riemannian connection and the second fundamental form of M^n, respectively. Then, the Gauss and Ricci equations are respectively given by

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \tag{6}$$

$$R^\perp(X, Y, N_1, N_2) = \tilde{R}(X, Y, N_1, N_2) - g([A_{N_1}, A_{N_2}]X, Y), \tag{7}$$

for all $X, Y, Z, W \in \Gamma(TM^n)$ and $N_1, N_2 \in \Gamma(T^\perp M^n)$, where $\tilde{R}$ is the curvature tensor of $\tilde{M}^{2m+1}$ and $R^\perp$ is the normal component of $\tilde{R}$, whereas R is the curvature tensor of M^n.

For any orthonormal basis $\{e_1, \cdots, e_n\}$ of the tangent space T_pM^n, the *mean curvature vector* $H(p)$ is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \quad \|H\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} g(h(e_i, e_i), h(e_j, e_j)).$$

A submanifold M^n is *totally geodesic* in $\tilde{M}^{2m+1}$ if $h = 0$, and *minimal* if $H = 0$. If $h(X, Y) = g(X, Y)H$ for all $X, Y \in \Gamma(TM^n)$, then M^n is *totally umbilical* in $\tilde{M}^{2m+1}$.

A submanifold M^n normal to the structure vector field ξ is said to be a *C-totally real* submanifold. In this case, it follows that φ maps any tangent space of M^n into the normal space, that is, $\varphi(T_pM^n) \subset T_p^{\perp}M^n$, for each $p \in M^n$. In particular, if $n = m$, then M^n is called a *Legendrian* submanifold.

For submanifolds tangent to the structure vector field ξ, we mention the following classes of submanifolds.

(i) A submanifold M^n tangent to ξ is said to be an *invariant* submanifold if φ preserves any tangent space of M^n, that is, $\varphi(T_pM^n) \subseteq T_pM^n$, for any $p \in M^n$.

(ii) A submanifold M^n tangent to ξ is called an *anti-invariant* submanifold if φ maps any tangent space of M^n into the normal space, that is, $\varphi(T_pM^n) \subseteq T_p^{\perp}M^n$, for any $p \in M^n$.

(iii) A submanifold M^n tangent to ξ is said to be a *slant* submanifold if for each non-zero vector $X \in T_pM^n$ not proportional to ξ_p, the angle $\theta(X)$ between φX and T_pM^n is constant, which is independent of the choice of $p \in M^n$ and $X \in T_pM^n$. The angle θ is called *slant angle* or *Wirtinger angle* of M^n.

3. DDVV conjecture for Legendrian submanifolds

In this section, we derive the generalized Wintgen inequality for Legendiran submanifolds of Kenmotsu manifolds. An anti invariant submanifold M^n normal to the structure vector field ξ of a Kenmotsu manifold $\tilde{M}^{2m+1}$ is said to be a *Legendrian* submanifold, if $n = m$.

Following [20], we have

$$K_N = \frac{1}{4}\sum_{r,s=1}^{2m+1-n} Trace[A_r, A_s]^2,$$

where $A_r = A_{e_{n+r}},\ r \in \{1, \cdots, 2m + 1 - n\}$, and call it the *scalar normal curvature* of M^n. The *normalized scalar normal curvature* is given by $\rho_N = \frac{2}{n(n-1)}\sqrt{K_N}$. Since $A_\xi = 0$, it follows that

$$K_N = \frac{1}{2}\sum_{1\le r<s\le 2m-n} Trace[A_r, A_s]^2 = \sum_{1\le r<s\le 2m-n}\ \sum_{1\le i<j\le 2m-n} g([A_r, A_s]e_i, e_j)^2. \tag{8}$$

We denote the second fundamental form $h^r_{ij} = g(h(e_i, e_j), e_r),\ i, j \in \{1, \cdots, n\},\ r \in \{n+1, \cdots, 2m+1-n\}$. Then, in terms of the components of the second fundamental from, we write

$$K_N = \sum_{1\le r<s\le 2m-n}\ \sum_{1\le i<j\le n}\left(\sum_{k=1}^{n}(h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk})\right)^2. \tag{9}$$

Lemma 3.1. *Let M^n be an n-dimensional anti-invariant submanifold normal to ξ of a $(2m+1)$-dimensional Kenmotsu space from $\tilde{M}^{2m+1}(c)$. Then,*

$$\rho + \rho_N \le \|H\|^2 + \frac{c-3}{4}$$

with equality holding if and only if, with respect to the suitable orthonormal frames $\{e_1, \cdots, e_n\}$ *and* $\{e_{n+1}, \cdots, e_{2m}, e_{2m+1} = \xi\}$*, the shape operator of* M^n *in* $\tilde{M}^{2m+1}(c)$ *takes the from*

$$A_{e_{n+1}} = \begin{pmatrix} \lambda_1 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix}, \quad A_{e_{n+2}} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix},$$

$$A_{e_{n+3}} = \begin{pmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{pmatrix},$$

where λ_1*,* λ_2*,* λ_3 *and* μ *are real function on* M^n*,*

$$A_{e_{n+4}} = \cdots = A_{e_{2m}} = A_{e_{2m+1}} = 0.$$

Proof. Note that in the proof of this lemma, we use the similar arguments and technique used in [12] (see also [14]). From the definition of H, we have

$$n^2\|H\|^2 = \sum_{r=1}^{2m+1-n} \left(\sum_{i=1}^{n} h_{ii}^r \right)^2 .$$

Since $h_{ii}^{2m+1} = g(A_{e_{2m+1}} e_i, e_i) = 0$, then the above expression will be

$$n^2\|H\|^2 = \frac{1}{n-1} \sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} (h_{ii}^r - h_{jj}^r)^2 + \frac{2n}{n-1} \sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} h_{ii}^r h_{jj}^r. \tag{10}$$

We use the following inequality given in [12],

$$\sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} (h_{ii}^r - h_{jj}^r)^2 + 2n \sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} (h_{ij}^r)^2 \ge 2n \Bigg[\sum_{1 \le r < s \le 2m-n} \sum_{1 \le i < j \le n} \left(\sum_{k=1}^{n} (h_{jk}^r (h_{ik}^s - h_{ik}^r h_{jk}^s) \right)^2 \Bigg]^{\frac{1}{2}}. \tag{11}$$

Then, with the help of above inequality, (10) takes the from

$$n^2\|H\|^2 \ge \frac{2n}{n-1} \Bigg[\sum_{1 \le r < s \le 2m-n} \sum_{1 \le i < j \le n} \left(\sum_{k=1}^{n} (h_{jk}^r (h_{ik}^s - h_{ik}^r h_{jk}^s) \right)^2 \Bigg]^{\frac{1}{2}} + \frac{2n}{n-1} \sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2].$$

Thus, from the definition of normalized scalar curvature and (9), we derive

$$n^2\|H\|^2 \ge n^2 \rho_N + \frac{2n}{n-1} \sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{12}$$

On the other hand, from the Gauss equation we have

$$2\tau = \sum_{1 \le i < j \le n} \tilde{R}(e_i, e_j, e_j, e_i) + 2 \sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2],$$

where for orthonormal vector fields from (5), we derive

$$\sum_{1\le i<j\le n} \tilde{R}(e_i, e_j, e_j, e_i) = \frac{(c-3)}{4} n(n-1).$$

Then, we find

$$\tau = \frac{n(n-1)(c-3)}{8} + \sum_{r=1}^{2m-n} \sum_{1\le i<j\le n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \tag{13}$$

From (3), (12) and (13), we derive

$$\rho + \rho_N \le \|H\|^2 + \frac{c-3}{4},$$

which is the required inequality. The equality case holds identically if and only if the shape operators takes the given form with respect to the suitable frames (we use the similar arguments given in [12] see also [14]). □

We have the following corollary as a consequence of Lemma 3.1.

Corollary 3.2. *Let M^n be an n-dimensional anti-invariant submanifold normal to ξ of $\mathbb{H}^{2m+1}(-1)$. Then, we have*

$$\rho + \rho_N \le \|H\|^2 - 1.$$

Now, we derive the generalized Wintgen inequality (DDVV conjecture) for Legendrian submanifolds in a Kenmotsu space from.

Theorem 3.3. *Let M^n be a Legendrian submanifold of a Kenmotsu space form $\tilde{M}^{2n+1}(c)$. Then, we have*

$$(\rho^\perp)^2 \le \left(\|H\|^2 - \rho + \frac{c-3}{4}\right)^2 + \frac{c+1}{n(n-1)}\left(\rho - \frac{c-3}{4}\right) + \frac{(c+1)^2}{8n(n-1)},$$

Proof. Consider the orthonormal frame fields on M^n as $\{e_1, \cdots, e_n\}$; then $\{e_{n+1} = \varphi e_1, \cdots, e_{2n} = \varphi e_n, e_{2n+1} = \xi\}$ is an orthonormal frame in the normal bundle $T^\perp M^n$. From (5) and (7), we have

$$R^\perp(e_i, e_j, e_{n+r}, e_{n+s}) = -\frac{c+1}{4}\Big[- g(\varphi e_i, e_{n+s})g(\varphi e_j, e_{n+r}) + g(\varphi e_i, e_{n+r})g(\varphi e_j, e_{n+s})\Big] - g([A_r, A_s]e_i, e_j).$$

Using the considered frame field, we derive

$$\begin{aligned} R^\perp(e_i, e_j, e_{n+r}, e_{n+s}) &= \frac{c+1}{4}\Big[g(\varphi e_i, \varphi e_s)\, g(\varphi e_j, \varphi e_r) + g(\varphi e_i, \varphi e_r)\, g(\varphi e_j, \varphi e_s)\Big] - g([A_r, A_s]e_i, e_j) \\ &= \frac{c+1}{4}(\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}) - g([A_r, A_s]e_i, e_j), \end{aligned} \tag{14}$$

for all $i, j \in \{1, \cdots, n\}$, $r, s \in \{1, \cdots, n\}$. Now, we find

$$(\tau^\perp)^2 = \sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \left(R^\perp(e_i, e_j, e_{n+r}, e_{n+s})\right)^2.$$

Then, with the help of (14), the above relation expresses as

$$\begin{aligned} (\tau^\perp)^2 &= \sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \left(\frac{c+1}{4}(\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}) - g([A_r, A_s]e_i, e_j)\right)^2 \\ &= \frac{(c+1)^2}{16} \sum_{1\le r<s\le n} \sum_{1\le i<j\le n} (\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js})^2 + K_N + \frac{c+1}{2} \sum_{1\le r<s\le n} \sum_{1\le i<j\le n} (\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js})\, g([A_r, A_s]e_i, e_j) \\ &= \frac{n(n-1)(c+1)^2}{32} + \frac{n^2(n-1)^2}{4}\rho_N^2 - \frac{c+1}{4}\|h\|^2 + \frac{c+1}{4} n^2\|H\|^2. \end{aligned} \tag{15}$$

On the other hand, by Gauss equation and (5), we have

$$2\tau = n(n-1)\frac{c-3}{4} + n^2\|H\|^2 - \|h\|^2,$$

equivalently,

$$n^2\|H\|^2 - \|h\|^2 = n(n-1)\left(\rho - \frac{c-3}{4}\right).$$

Then, with the help of above relation, (15) takes the form

$$(\rho^\perp)^2 = \rho_N + \frac{c+1}{n(n-1)}\left(\rho - \frac{c-3}{4}\right) + \frac{(c+1)^2}{8n(n-1)}.$$

Then, by Lemma 3.1, we derive

$$(\rho^\perp)^2 \leq \left(\|H\|^2 - \rho + \frac{c-3}{4}\right)^2 + \frac{c+1}{n(n-1)}\left(\rho - \frac{c-3}{4}\right) + \frac{(c+1)^2}{8n(n-1)},$$

which is required inequality. □

4. Another inequality

In this section we derive an inequality for normalized scalar curvature ρ and normalized normal scalar curvature ρ_N for a slant submanifold M in Kenmotsu space forms. We consider the structure vector field ξ tangent to M.

Theorem 4.1. *Let M^n be an n-dimensional slant submanifold of a Kenmotsu space form $\tilde{M}^{2m+1}(c)$. Then, we have*

$$\rho + \rho_N \leq \|H\|^2 + \frac{c-3}{4} + \frac{(3\cos^2\theta - 2)(c+1)}{4n}.$$

Proof. Consider the orthonormal frame field on M^n as follows: $\{e_1, e_2 = \sec\theta\, Te_1, \cdots, e_{n-2}, e_{n-1} = \sec\theta\, Te_{n-2}, e_n = \xi\}$. Then, we have $g(e_1, \varphi e_2) = -g(\varphi e_1, \sec\theta\, Te_1) = -\cos\theta$. Consequently, $g^2(e_i, \varphi e_{i+1}) = \cos^2\theta$. Using this fact in Gauss equation with (2), we derive

$$2\tau = n(n-1)\frac{c-3}{4} - \frac{c+1}{4}(n-1)\left(3\cos^2\theta - 2\right) + 2\sum_{r=1}^{2m+1-n}\sum_{1\leq i<j\leq n}\left[h^r_{ii}h^r_{ii} - (h^r_{ij})^2\right]. \tag{16}$$

On the other hand, by similar argument as in proof of Lemma 3.1 (relation (12)), we have

$$n^2\|H\|^2 \geq n^2\rho_N + \frac{2n}{n-1}\sum_{r=1}^{2m+1-n}\sum_{1\leq i<j\leq n}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \tag{17}$$

Then, from (16) and (17), we derive

$$\rho + \rho_N \leq \|H\|^2 + \frac{c-3}{4} + \frac{(3\cos^2\theta - 2)(c+1)}{4n}.$$

Hence, we achieve the result. □

Corollary 4.2. *Let M^n be a slant submanifold of $\mathbb{H}^{2m+1}$. Then, we have*

$$\rho + \rho_N \leq \|H\|^2 - 1.$$

Notice that the inequality for the normal scalar curvature and normalized normal scalar curvature in terms of mean curvature does not change for the different submanifolds in H^{2m+1}. For example; in Corollary 3.2, the inequality is obtained for anti-invariant submanifolds, while; in Corollary 4.2, it is for slant slant submanifold but in both cases the inequality is same.

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