Global determinism of *-bands

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Abstract

A class of algebras is said to be globally determined if any two members of that class having isomorphic power algebras are isomorphic. In 1984, Y. Kobayashi proved that semilattices are globally determined. By using his methods, we prove that the class of all *-bands has the same property[1].

Let $\langle A, \mathcal{F} \rangle$ be a universal algebra. Clearly, the operations from $\mathcal{F}$ can be extended to the collection of all nonempty subsets of A by defining

$$f(A_1, \ldots, A_n) = \{f(a_1, \ldots, a_n) : \ a_i \in A_i, \ 1 \leq i \leq n\}$$

for all n-ary operations $f \in \mathcal{F}$ and $A_1, \ldots, A_n \subseteq A$. In the way just described, we obtain *the power algebra*, or *the global* of the algebra A, denoted by $\mathbf{P}(A)$. In particular, if S is a semigroup, then $\mathbf{P}(S)$ is a semigroup too, with the operation given by

$$XY = \{xy : \ x \in X, \ y \in Y\},$$

for all nonempty $X, Y \subseteq S$.

A class $\mathcal{C}$ of algebras of the same similarity type is said to be *globally determined* if for each $A, B \in \mathcal{C}$ the following implication holds:

$$\mathbf{P}(A) \cong \mathbf{P}(B) \implies A \cong B.$$

The problem of global determinism was formulated for the first time in the sixties by B. M. Schein, and also in the pioneering paper of Tamura and Shafer [15], where it was proved that the class of all groups possesses this property. Further research in this field was mainly concentrated on various classes of semigroups and related associative systems. Although the class of all semigroups is not globally determined, as proved by Mogiljanskaja [9], a number of relevant globally determined classes of semigroups were found: rectangular groups [12], finite semilattices of torsion groups and finite simple semigroups [5], completely regular periodic monoids (in particular, all bands with an identity element) and their orthogonal sums [6], completely simple and completely 0-simple semigroups [13, 14], and so on.

Recall that an *involution semigroup* is a semigroup equipped with an involutary antiautomorphism. In other words, we are concerned with algebras of

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the form $\langle S, \cdot, ^* \rangle$ such that $\langle S, \cdot \rangle$ is a semigroup and the following identities are satisfied:

$$\begin{aligned}(xy)^* &= y^* x^*,\\ (x^*)^* &= x.\end{aligned}$$

If, in addition, x^* is an inverse of x for each $x \in S$, i.e. if the identity $xx^*x = x$ holds in S, then S is a **-semigroup* [10]. Finally, a **-band* is an idempotent *-semigroup.

There is a number of papers on *-bands in the literature describing their nice and interesting properties, which all show how well-behaved this class actually is. For example, the lattice of its subvarieties is completely described in [1], valuable information on projectives and injectives is provided by [11], and the structure of free *-bands is well understood [16, 4]. In the present note, we add one more entry to the list of 'good' features of *-bands: we prove that they are globally determined. We recall, however, that the latter property is not shared by the class of all involution semigroups [2].

The present note is inspired by the paper of Kobayashi [8], who proved the the class of all semilattices is globally determined by analyzing some order-theoretic properties of semilattices. But since any band can be made into a partially ordered set by defining

$$a \leq b \iff ab = ba = a,$$

it was tempting to see to what extent can be the method of Kobayashi applied in some more general situations. It turned out that *-bands provide conditions under which such a generalization is completely successful. Note that a *-semilattice is necessarily a semilattice with trivial involution (identity mapping), yielding the original framework discussed in [8].

For any involution semigroup S involved in the sequel, we denote by $\mathsf{Su}(S)$ the set of all involution subsemigroups of S. We start with the following fact.

Lemma 1 *For any involution band B, $\mathsf{Su}(B)$ coincides with the set of all projections (idempotent fixed points of the involution) of $\mathbf{P}(B)$. Therefore, if B_1, B_2 are involution bands, any isomorphism $\varphi : \mathbf{P}(B_1) \to \mathbf{P}(B_2)$ induces (by restriction) a bijection $\mathsf{Su}(B_1) \to \mathsf{Su}(B_2)$.*

Proof. First of all, note that for all $X \subseteq B$ we have $(X^*)^* = X$ and, by the idempotency, $X \subseteq X^2$. Thus, $X \in \mathsf{Su}(B)$ if and only if $X^2 \subseteq X$ and $X^* \subseteq X$, which is, in turn, equivalent to $X = X^2 = X^*$. ∎

Further, we point out one distinguished subfamily of $\mathsf{Su}(B)$:

$$\mathsf{Ch}(B) = \{X \in \mathsf{Su}(B) : \ X = Y^2 \Rightarrow X = Y\}.$$

Clearly, as in the above lemma, any isomorphism of $\mathbf{P}(B_1)$ and $\mathbf{P}(B_2)$ defines a bijection between $\mathsf{Ch}(B_1)$ and $\mathsf{Ch}(B_2)$. We characterize the members of $\mathsf{Ch}(B)$, as follows.

Lemma 2 *Let B be a *-band. Then $X \in \mathsf{Ch}(B)$ if and only if X is a chain of projections.*

Proof. Assume that $xy \notin \{x, y\}$ for some $x, y \in X$. Then $x, y \in X \setminus \{xy\}$ and so $xy \in (X \setminus \{xy\})^2$. Since $X \setminus \{xy\} \subseteq (X \setminus \{xy\})^2$, it follows $(X \setminus \{xy\})^2 = X$, i.e. $X \notin \mathsf{Ch}(B)$. Now for each $x \in X$ we have $xx^* \in \{x, x^*\}$, implying that x must be a projection. Finally, since $xy \in \{x, y\}$ for all $x, y \in X$ and x, y are projections, xy must be a projection too, and therefore $xy = yx$. Thus, X is a chain.

Conversely, let X be a chain of projections and $X = Y^2$ for a subset $Y \subseteq B$. Then however, $Y \subseteq X$ and since every subset of a chain is its subsemigroup, we obtain $Y = Y^2 = X$. ∎

We have already mentioned the partial order relation which can be defined on an arbitrary band B. In an analogous way, one defines a partial order on $\mathsf{Su}(B)$ as the restriction of the natural order on the set of idempotents of $\mathbf{P}(B)$:

$$X \leq Y \iff XY = YX = X.$$

The fact that y covers x in B (i.e. if $x < y$ and there is no $z \in B$ such that $x < z < y$) we write $x \to y$. However, when elements of $\mathsf{Su}(B)$ are involved, we are going to use two kinds of arrows. If Y covers X in $\mathsf{Su}(B)$, then we write $X \Rightarrow Y$. On the other hand, the weaker assertion that $X < Y$ and that there is no element of $\mathsf{Ch}(B)$ between X and Y we denote by $X \to Y$.

Following Kobayashi's terminology [8], we call a sequence $Y_1, \ldots, Y_n$ of elements of $\mathsf{Ch}(B)$ a *hair of X* (of length n) if

$$X \Rightarrow Y_1 \Rightarrow \ldots \Rightarrow Y_n.$$

The above hair is *maximal* if it cannot be prolonged to a hair of length $n+1$. Finally, a quadruple X, Y, Z, T of distinct elements of $\mathsf{Ch}(B)$ we call a *topknot* of X provided that the following relations hold:

$$\begin{array}{ccccc} Y & \Rightarrow & T & \Leftarrow & Z \\ & \nwarrow & & \nearrow & \\ & & X & & \end{array}$$

The main characterization theorem of [8] that allowed to prove the global determinism of semilattices is now the following one.

Proposition 3 (Kobayashi, [8]) *Let S be a semilattice and let X be a subchain of S. Then $|X| = 1$ if and only if X has neither topknots, nor maximal hairs of length 1.*

Our aim is here to prove that an analogous statement holds for *-bands.

It is well known (e.g. from [1, 3]) that if Σ is the greatest semilattice image of a *-band B, then each of the rectangular band $\mathcal{D}$-classes of B is closed for the * operation, yielding that these classes must be square. In other words, B is a semilattice of rectangular *-bands. Let $\sigma : B \to \Sigma$ denote the corresponding surjective homomorphism. The following simple observation enables us to use Lemma 1 from [8] for *-bands.

Lemma 4 *Let B be a *-band. If $X, Y \in \mathsf{Ch}(B)$ are such that $X < Y$ and that $\sigma(X) \Rightarrow \sigma(Y)$ ($\sigma(X) \to \sigma(Y)$) holds in $\mathbf{P}(\Sigma)$, then $X \Rightarrow Y$ ($X \to Y$).*

Proof. Assume there exists $Z \in \mathsf{Su}(B)$ ($Z \in \mathsf{Ch}(B)$) such that $X < Z < Y$. Then, clearly, $\sigma(X) < \sigma(Z) < \sigma(Y)$. The lemma now easily follows by noting that $\sigma(Z)$ is a subsemilattice (subchain) of Σ. ∎

This immediately yields

Lemma 5 *Let $X \in \mathsf{Ch}(B)$, where B is a *-band. If $x \in X$ is not a maximal element of X, then $X \Rightarrow X \setminus \{x\}$.*

Proof. Since the restriction of σ to a given chain of projections (in this case, X) is an isomorphism of X and $\sigma(X)$, and since by Lemma 1 of [8] we have $\sigma(X) < \sigma(X) \setminus \{\sigma(x)\} = \sigma(X \setminus \{x\})$, it follows that $X < X \setminus \{x\}$. Now the conclusion follows by the above lemma and Lemma 1 of [8]. ∎

We need two more lemmata, which are the analogues (but, it is important to stress, *not* corollaries) of Lemma 2 and Lemma 3 from [8].

Lemma 6 *For a *-band B, let $X \in \mathsf{Ch}(B)$ and assume that X has a greatest element x'. If, moreover, there exists a projection $y \in B$ such that $x' \to y$, then $X \to X \cup \{y\}$.*

Proof. Since $X(X \cup \{y\}) = (X \cup \{y\})X = X$, we have $X < X \cup \{y\}$. Assume that $X \leq Y \leq X \cup \{y\}$ holds for some $Y \in \mathsf{Ch}(B)$. Then $XY = YX = X$ and

$$Y = (X \cup \{y\})Y = XY \cup \{y\}Y = X \cup \{y\}Y,$$

so $X \subseteq Y$. Similarly, $Y = X \cup Y\{y\}$. Now if $X \neq Y$, let $z \in Y \setminus X$. Then $x' < z$, for otherwise $z = x'z \in XY = X$. On the other hand, z belongs to $\{y\}Y \cap Y\{y\}$, i.e. $z = yu = vy$ holds for some $u, v \in Y$, and thus $yz = zy = z$, $z \leq y$. But we have $x' \to y$, hence $y = z$. Therefore, $Y = X \cup \{y\}$. ∎

Lemma 7 *Let B be a *-band and let $x \in B$ be a projection. If $\{x\} \to Y$ for some $Y \in \mathsf{Ch}(B)$ then $Y = \{x, y\}$, with $x \to y$.*

Proof. First of all, we know that $\{x\}Y = Y\{x\} = \{x\}$. In other words, for all $y \in Y$ we have $xy = yx = x$, i.e. $x \leq y$. Therefore, if $Y' = \{x\} \cup Y$, it follows

$$YY' = Y(\{x\} \cup Y) = \{x\} \cup Y = Y',$$

and similarly, $Y'Y = Y$ and $\{x\}Y' = Y'\{x\} = \{x\}$, i.e. $\{x\} < Y' \leq Y$, which means that $Y = Y'$, that is, $x \in Y$.

Now pick $z \in Y \backslash \{x\}$ in an arbitrary way and consider the chain of projections $Z = \{y \in Y : \ y \leq z\}$. Clearly, we have $\{x\}Z = Z\{x\} = \{x\}$ and $YZ = ZY = Z$, so that $\{x\} < Z \leq Y$, implying $Z = Y$. This shows that Y has only two elements, $Y = \{x, y\}$. We must have $x \to y$, for otherwise if $x < u < y$, then obviously $\{x\} < \{x, u\} < Y$, a contradiction. ∎

Now we launch into the main part of the proof of global determinism for *-bands.

Proposition 8 *Let $X \in \mathsf{Ch}(B)$ for a *-band B such that $|X| \geq 3$. Then X has a topknot.*

Proof. Assume that X contains elements $x < y < z$. Then by Lemma 5, X has the topknot

$$\begin{tikzcd} X \setminus \{x\} \arrow[r, Rightarrow] & X \setminus \{x, y\} & X \setminus \{y\} \arrow[l, Rightarrow] \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

(moreover, the simple arrows $\to$ are in fact double ones, $\Rightarrow$), and the proposition is proved. ∎

Proposition 9 *If X is an involution subchain of a *-band B with exactly two elements, then it has either a maximal hair of length 1, or a topknot.*

Proof. Let $X = \{x, x'\}$ with $x < x'$. By Lemma 5, we have $X \Rightarrow X \setminus \{x\} = \{x'\}$. If this hair can be prolonged, then Lemma 7 yields a projection $y \in B$ such that $x' \to y$ and $\{x'\} \Rightarrow \{x', y\}$. Now by Lemma 6 we have $X \to X \cup \{y\} = \{x, x', y\}$, and $\{x, x', y\} \Rightarrow \{x', y\}$ by Lemma 5. Hence, we have just constructed the following topknot:

$$\begin{tikzcd} \{x'\} \arrow[r, Rightarrow] & \{x', y\} & X \cup \{y\} \arrow[l, Rightarrow] \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and the required conclusion follows. ∎

Finally, we prove the main results of the paper.

Theorem 10 *Let $X \in \mathsf{Ch}(B)$, where B is a *-band. Then $|X| = 1$ if and only if X has neither maximal hairs of length 1, nor topknots.*

Proof. ($\Rightarrow$) First of all, it is clear that if $X = \{x\}$ has a hair of length 1, $\{x\} \Rightarrow Y$, then by Lemma 7 it must be of the form $\{x\} \Rightarrow \{x, y\}$, with $x \to y$. Obviously, it can be prolonged, because by Lemma 5, $\{x, y\} \Rightarrow \{y\}$.

Assume that $\{x\}$ has a topknot of the form

$$\begin{array}{ccccc} Y & \Rightarrow & T & \Leftarrow & Z \\ & \nwarrow & & \nearrow & \\ & & \{x\} & & \end{array}$$

Then $Y = \{x, y\}$ and $Z = \{x, z\}$ (by Lemma 7), and we have $x \to y$, $x \to z$, $y \neq z$, so that $yz = zy = x$. Now the argument from the bottom of p.220 of [8] applies verbatim in order to show that $Y, Z < \{x, y, z\} < T$ holds, a conclusion which prevents the existence of the above topknot.

($\Leftarrow$) This follows immediately from Propositions 8 and 9. ∎

Corollary 11 *The variety of all* **-bands is a globally determined class.*

Proof. By the above theorem and the remarks following the definition of $\mathsf{Ch}(B)$, if B_1 and B_2 are *-bands, then any isomorphism $\phi : \mathbf{P}(B_1) \to \mathbf{P}(B_2)$ defines a bijection between singleton subsets of B_1 and B_2 containing projections. Note that in any *-band) each element x is a product of two projections, namely of xx^* and x^*x, since $(xx^*)(x^*x) = xx^*x = x$. Therefore, if $x \in B_1$, then $x = pq$ for some projections $p, q \in B_1$, and so

$$\phi(\{x\}) = \phi(\{pq\}) = \phi(\{p\}\{q\}) = \phi(\{p\})\phi(\{q\}) = \{p'\}\{q'\} = \{p'q'\},$$

which implies that ϕ defines an injection of the family of singleton subsets of B_1 into the corresponding family of singletons of B_2. Of course, this conclusion holds for the inverse mapping ϕ^{-1}, so that the singleton subsets of B_1 and B_2 are in a bijective correspondence under ϕ. It is now straightforward to verify that the mapping $\psi : B_1 \to B_2$ defined by

$$\psi(x) = x' \text{ if and only if } \phi(\{x\}) = \{x'\}$$

is indeed an isomorphism. Thus, $B_1 \cong B_2$. ∎

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