Note on near-P-polyagroups

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Abstract

Among the results of the paper[1] we have the following proposition. Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, (Q, A) is an near-P-polyagroup (briefly: NP-polyagroup) of the type $(s, n-1)$ *[*:[11],1.3*]* iff the following statements hold: (i) (Q, A) is an $< 1, n > -$ and $< 1, s+1 > -$associative $n-$groupoid *[***or** $< 1, n > -$ and $< (k-1) \cdot s + 1,\ k \cdot s + 1 > -$ associative $n-$groupoid*]*; and (ii) for every $a_1^n \in Q$ there is **at least one** $x \in Q$ and **at least one** $y \in Q$ such that the following equalities hold $A(a_1^{n-1}, x) = a_n$ and $A(y, a_1^{n-1}) = a_n$.

1 Preliminaries

Definition 1.1 *[11] Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, we say that (Q, A) is an $Ps-$***associative** $n-$**groupoid** *iff for every $i, j \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}, i < j$, the following law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})$$

[:$< i, j > -$associative law].

Remark: For $s = 1$ (Q, A) is a $(k+1)-$semigroup; $k > 1$.
A notion of an $s-$**associative** $n-$**groupoid** was introduced by F.M. Sokhatsky (for example [6]).

Definition 1.2 *[11] $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, we say that (Q, A) is an $P-$***polyagroup of the type** $(s, n-1)$ *iff is an $Ps-$associative $n-$groupoid and a $n-$quasigroup.*
A notion of an **polyagroup** *was introduced by F.M. Sokhatsky (for example [7]).*

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Definition 1.3 *[11] $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $Ps-$associative $n-$groupoid. Then, we say that (Q, A) is an* **near-P–polyagroup (briefly: NP-polyagroup) of the type** *$(s, n-1)$ iff for every $i \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$ and for all $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality*

$A(x_1^{i-1}, x_i, a_i^{n-1}) = a_n$
holds.

Remark: Every P-polyagroup of the type $(s, n-1)$ is an NP–polyagroup of the type $(s, n-1)$.

2 Auxiliary propositions

Proposition 2.1 *[8] Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. Further on, let the $< 1, n > -$associative law holds in (Q, A), and let for every $a_1^n \in Q$ there is* **at least one** *$x \in Q$ and* **at least one** *$y \in Q$ such that the following equalities $A(a_1^{n-1}, x) = a_n$ and $A(y, a_1^{n-1}) = a_n$ hold. Then, there are mappings $\mathbf{e}$ and $^{-1}$ respectively of the sets Q^{n-2} an Q^{n-1} into the set Q such that the following laws*

$$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x,\ A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x,$$

$$A((a_1^{n-2}, x)^{-1}, a_1^{n-2}, x) = \mathbf{e}(a_1^{n-2}) \text{ and } A(x, a_1^{n-2}, (a_1^{n-2}, x)^{-1}) = \mathbf{e}(a_1^{n-2})$$

hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$.

Proposition 2.2 *[11] Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Also let*

(i) the $< 1, s+1 > -$associative [$< (k-1) \cdot s + 1, k \cdot s + 1 > -$associative] law holds in the (Q, A); and

(ii) for every $x, y, a_1^{n-1} \in Q$ the following implication holds

$$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y.$$
$$[A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y\ ].$$

Then (Q, A) is an $Ps-$associative $n-$groupoid.
(See, also [8].)

Proposition 2.3 *[11] Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then the following statements are equivalnt: (i) (Q, A) is an NP–polyagroup of the type $(s, n-1)$; (ii) there are mappings $^{-1}$ and $\mathbf{e}$ respectively* *of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]*

(a) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1})$,

(b) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

(c) $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) there are mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ [of type $< n, n-1, n-2 >$]

($\overline{a}$) $A(x_1^{(k-1)\cdot s}, A(x_{(k-1)\cdot s+1}^{(k-1)\cdot s+n}), x_{(k-1)\cdot s+n+1}^{2n-1}) = A(x_1^{k\cdot s}, A(x_{k\cdot s+1}^{2n-1}))$,

($\overline{b}$) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

($\overline{c}$) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$.

(See, also [8].)

3 Results

Theorem 3.1 *Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, (Q, A) is an* **NP–polyagroup of the type** *$(s, n-1)$ iff following statements hold:*

(i) (Q, A) is an $< 1, n > -$associative $n-$groupoid;

(ii) (Q, A) is an $< 1, s+1 > -$associative $n-$groupoid **or** *$< (k-1) \cdot s + 1,\ k \cdot s + 1 > -$associative $n-$groupoid; and*

(iii) for every $a_1^n \in Q$ there is **at least one** *$x \in Q$ and* **at least one** *$y \in Q$ such that the following equalities hold*

$$A(a_1^{n-1}, x) = a_n \text{ and } A(y, a_1^{n-1}) = a_n.$$

Proof. *a)* $\Leftarrow$: Considering (i) and (iii), by Proposition 2.1, we conclude that there is mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the laws $(b),\ (c),\ (\overline{b})$ and $(\overline{c})$ from 2.3 hold in algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$. Whence considering (ii), by Proposition 2.3, we conclude that (Q, A) is an NP–polyagroup of the type $(s, n-1)$.

b) $\Rightarrow$: Considering 1.3, we conclude that the statements $(i) - (iii)$ hold. ■

Remark: Group as a semigroup and a quasigroup was characterized by Weber H. in 1896 (cf. [4], pp. 19–20). A notion of an $n-$group was introduced by Dörnte W. in [1] as a generalization of the Weber's characterization of a group. Group as a semigroup $(Q, \cdot)$ in which the following formula holds

$$(\forall a \in Q)(\forall b \in Q)(\exists x \in Q)(\exists y \in Q)(a \cdot x = b \wedge y \cdot a = b)$$

was characterized by Hungtington E.V. in 1902 (cf. [4], p. 20). Note that following proposition has been proved in [3]. An $n-$**semigroup** (Q, A) is an $n-$**group** iff for each $a_1^n \in Q$ there exists **at least one** $x \in Q$ and **at least one** $y \in Q$ such that the following equalities hold $A(a_1^{n-1}, x) = a_n$ and $A(y, a_1^{n-1}) = a_n$. This assertion has been already formulated in [2], but the proof is missing there. (See, also [9].) Note that the following proposition has been proved in [10]: Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. Then, (Q, A) is an $n-$group iff following statements hold: (i) (Q, A) is an $< 1, n > -$ and $< 1, 2 >$ $-$**associative** $n-$**groupoid** *[* **or** $< 1, n > -$and $< n-1, n > -$**associative** $n-$groupoid *]*; and (ii) for every $a_1^n \in Q$ there is **at least one** $x \in Q$ and **at least one** $y \in Q$ such that the following equalities hold $A(a_1^{n-1}, x) = a_n$ and $A(y, a_1^{n-1}) = a_n$. This proposition for $n \geq 3$ appears as a special case of Theorem 3.1 (for $s = 1$). (See also [12].)

Theorem 3.2 *Let* $k > 1, s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then,* (Q, A) *is an* **NP–polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

(I) (Q, A) *is an* $< 1, s+1 > -$*associative* $n-$*groupoid;*

(II) *For every* $a_1^n \in Q$ *there is* **exactly one** $x \in Q$ *such that the following equality holds*

$$A(x, a_1^{n-1}) = a_n; \textit{ and}$$

(III) *For every* $a_1^n \in Q$ *there is* **at least one** $y \in Q$ *such that the following equality holds*

$$A(a_1^{n-1}, y) = a_n.$$

Proof. $a)$ $\Leftarrow$: Considering (I) and (II), by Proposition 2.2, we conclude that (Q, A) is an $Ps-$associative $n-$groupoid. Whence, considering (II) and (III), by Theorem 3.1, we conclude that (Q, A) is an NP–polyagroup of the type $(s, n-1)$.

$b)$ $\Rightarrow$: Considering 1.3, we conclude that the statements (I)–(III) hold. ■

Similarly, it is possible to prove that the following proposition holds:

Theorem 3.3 *Let* $k > 1, s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then,* (Q, A) *is an* **NP–polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

(1) (Q, A) *is a* $< (k-1) \cdot s + 1, k \cdot s + 1 > -$*associative* $n-$*groupoid;*

(2) *For every* $a_1^n \in Q$ *there is* **exactly one** $x \in Q$ *such that the following equality holds*

$$A(a_1^{n-1}, x) = a_n; \text{ and}$$

(3) *For every* $a_1^n \in Q$ *there is* **at least one** $y \in Q$ *such that the following equality holds*

$$A(y, a_1^{n-1}) = a_n.$$

Remark: For $s = 1$ Theorem 3.3 and Theorem 3.4 is proved in [5].

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