Antipodal graphs of small diameter

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Abstract

A proper metric space $X = (X, d)$ is called *antipodal* if—with $[x, y] = \{z \in X : d(x, y) = d(x, z) + d(z, y)\}$—for every $x \in X$ there exists some $y \in X$ such that $[x, y] = X$. A connected undirected finite graph G is called *antipodal* if its associated graph metric is antipodal.

Here we characterize antipodal graphs of diameter 3 and show that almost every graph is an induced subgraph of some antipodal graph of diameter 3[1].

1 Introduction

A proper metric space $X = (X, d)$ is called *antipodal* if—with $[x, y] = \{z \in X : d(x, y) = d(x, z) + d(z, y)\}$—for every $x \in X$ there exists some $y \in X$ such that $[x, y] = X$. A connected undirected finite graph G is called *antipodal* if its associated graph metric is antipodal.

The only antipodal graphs of diameter 2 are Cocktail-party graphs. Namely, if G is an antipodal graph of diameter 2 with a mapping $* : V(G) \mapsto V(G)$ such that $[u, u^*] = V(G)$ for each $u \in V(G)$, then each vertex of G is adjacent to both u and u^*, while u and u^* are not adjacent. Since $(u^*)^* = u$, the vertices of G are partitioned into pairs (u, u^*) such that the vertices from different pairs are adjacent, while the vertices from the same pair are not adjacent. This shows that G is a Cocktail-party graph.

In [1] it is shown that a locally connected antipodal graph of diameter 3 is either the Johnson graph $J(6, 3)$, the halved 6-cube or the Gosset graph.

Our goal here is to give a full characterization of antipodal graphs of dimaeter 3. In order to do this, we first devise a new unary graph operation $P(G)$ and then show that each antipodal graph of diameter 3 is isomorphic to $P(G)$ for some graph G. Since this operation is defined in such a way that G is an induced subgraph of $P(G)$, we will then show that almost every graph is an induced subgraph of an antipodal graph of diameter 3.

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2 Antipodal graphs of diameter 3

For an arbitrary vertex u of a graph G, the set of vertices of G adjacent to u is called the *open neighborhood* of u and denoted $N_G(u)$. The set $N_G[u] = \{u\} \cup N_G(u)$ is the *closed neighborhood* of u in G. If there are no conflicts, we omit the index G and write $N(u)$ for $N_G(u)$ and $N[u]$ for $N_G[u]$. The graph G is *semi-cannonical* if for each vertex u of G it holds that $N[u] \neq V(G)$ and for each pair of vertices u and v of G it holds that $N[u] \neq N[v]$.

We first prove a handy lemma.

Lemma 1 *Let G be an antipodal graph with the mapping $*\colon V(G) \mapsto V(G)$ such that $[u, u^*] = V(G)$ for each $u \in V(G)$. The mapping $*$ is the isomorphism of G.*

Proof. We first prove that $*$ is the bijection of G. Suppose first that for some $u \in V(G)$ there exist vertices u' and u'' such that $[u, u'] = [u, u''] = V(G)$. Then we have that

$$\begin{aligned} d(u, u') &= d(u, u'') + d(u'', u'), \\ d(u, u'') &= d(u, u') + d(u', u''), \end{aligned}$$

from which follows that $d(u', u'') = 0$, showing that $u' = u''$. Therefore, the mapping $*$ is one-to-one. Next, it must be onto, since $(u^*)^* = u$, and we conclude that $*$ is the bijection of G.

Let D be the diameter of G, and let $v, w \in V(G)$ be such that $d(v, w) = D$. Then from $D \geq d(v, v^*) = d(v, w) + d(w, v^*)$, we conclude that $w = v^*$. Next, for an arbitrary vertex $u \in V(G)$ we have that

$$\begin{aligned} 2d(u, u^*) &= (d(u, v) + d(v, u^*)) + (d(u, v^*) + d(v^*, u^*)) \\ &= (d(v, u) + d(u, v^*)) + (d(v, u^*) + d(u^*, v^*)) = 2d(v, v^*) = 2D. \end{aligned}$$

Therefore, for each $u \in V(G)$ the distance between u and u^* is equal to D.

It remains only to prove that the mapping $*$ preserves adjacency of vertices. For arbitrary vertices u, v of G, from $u \in [v, v^*]$ and $v^* \in [u, u^*]$, it follows that

$$d(v, u) + d(u, v^*) = D = d(u, v^*) + d(v^*, u^*).$$

From here we see that $d(u^*, v^*) = D - d(u, v^*) = d(u, v)$. Therefore, vertices u and v are adjacent if and only if vertices u^* and v^* are adjacent. ■

Next we describe new unary graph operation $P(G)$. Let G be an arbitrary graph and let G^* be isomorphic to G. Further for each $u \in V(G)$ let u^* be the corresponding vertex of G^* under some fixed isomorphism between G and G^*, and let a and a^* be two new vertices which do not belong to either G or G^*. The graph $P(G)$ is then constructed in the following way:

$$\begin{aligned} V(P(G)) &= V(G) \cup V(G^*) \cup \{a, a^*\}, \\ E(P(G)) &= E(G) \cup E(G^*) \\ &\quad \cup \{(a, u) \mid u \in V(G)\} \cup \{(a^*, u^*) \mid u^* \in V(G^*)\} \\ &\quad \cup \{(u, v^*) \mid u \in V(G),\ v^* \in V(G^*),\ (u, v) \notin E(G),\ u \neq v\}. \end{aligned}$$

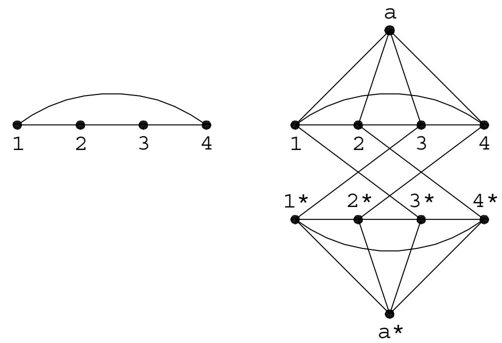

Figure 1: The construction of $P(C_4)$ for cycle C_4.

We illustrate this construction in Fig. 1 on cycle C_4.

Our main goal is to prove the following theorem.

Theorem 2 *The graph H is antipodal with diameter 3 if and only if there exists semi-cannonical graph G such that $H \cong P(G)$.*

The proof of this theorem directly follows from the following two theorems.

Theorem 3 *If G is semi-cannonical, then $P(G)$ is an antipodal graph with diameter 3.*

Proof. First, we prove that $[a, a^*] = V(P(G))$. The distance between a and a^* is at least 3, since they are not adjacent and have no common neighbor.

Since G is semi-cannonical, for each vertex u of G there exists vertex v of G that is not adjacent to u in G. Then u and v^* are adjacent in $P(G)$, and a, u, v^*, a^* is the path of length 3 in $P(G)$. Also u^* and v are adjacent in $P(G)$, and a, v, u^*, a^* is another path of length 3 in $P(G)$. This shows that $u, u^* \in [a, a^*]$ and we conclude that $[a, a^*] = V(P(G))$.

Now we show that for each $u \in V(G)$, it holds that $[u, u^*] = V(P(G))$. The distance between u and u^* is at least 3, since they are not adjacent and have no common neighbor.

Let v be an arbitrary vertex of G. If u and v are not adjacent in G, then u, a, v, u^* and u, v^*, a^*, u^* are paths of length 3 in $P(G)$ showing that $v, v^* \in [u, u^*]$. If u and v are adjacent in G, then $N[u] \neq N[v]$. If $N[u] \setminus N[v] \neq \emptyset$, then there exists vertex w of G adjacent to u and not adjacent to v. The paths u, v, w^*, u^* and u, w, v^*, u^* have length 3 in $P(G)$. If $N[v] \setminus N[u] \neq \emptyset$, then there exists vertex w of G adjacent to v and not adjacent to u. In this case, the paths u, v, w, u^* and u, w^*, v^*, u^* have length 3 in $P(G)$. In any case, we see that $v, v^* \in [u, u^*]$.

From above we see that the distance between a and a^*, and between u and u^* for each vertex u of G is equal to 3, thus showing that $P(G)$ is an antipodal graph with diameter 3. ■

Theorem 4 *Let H be an antipodal graph with diameter 3. Then there exists semi-cannonical graph G such that $H \cong P(G)$.*

Proof. Let $*: V(H) \mapsto V(H)$ be the mapping such that for each $u \in V(H)$ it holds that $[u, u^*] = V(H)$. Fix the vertex a of H and let $N(a)$ denote also a subgraph of H induced by an open neighborhood of a in H. It will be obvious from the context whether $N(a)$ denotes the open neighborhood or the graph induced by it. We show that $H \cong P(N(a))$.

Since H has diameter 3, each vertex u of H is adjacent either to a or a^*.

Let u and v be arbitrary vertices adjacent to a. Then u^* and v^* are adjacent to a^*. Since the mapping $*$ is isomorphism of H, vertices u and v are adjacent if and only if u^* and v^* are adjacent. This shows that $N(a)$ and $N(a^*)$ are isomorphic graphs.

Further, u is adjacent to exactly one of v and v^*, which finally shows that $H \cong P(N(a))$.

It remains to show that $N(a)$ is semi-cannonical. Let u be an arbitrary vertex of $N(a)$. Since H is antipodal with diameter 3, we have that $u \in [a, a^*]$ and there exists a vertex w in H such that a, u, w, a^* is a path of length 3 in H. The vertex w is adjacent to u and not adjacent to a. Then w^* is not adjacent to u and it is adjacent to a, showing that $N_{N(a)}[u] \neq N(a)$.

Let u, v be a pair of adjacent vertices in $N(a)$ (and also in H). Since H is antipodal with diameter 3, we have that $v \in [u, u^*]$ and there exists a vertex w in H such that u, v, w, u^* is a path of length 3 in H. The vertex w is not equal to either a or a^*. If w belongs to $N(a)$, then $w \in N_{N(a)}[v]$ and $w \notin N_{N(a)}[u]$. If w belongs to $N(a^*)$, then $w^* \in N_{N(a)}[u]$ and $w^* \notin N_{N(a)}[v]$. In any case, $N_{N(a)}[u] \neq N_{N(a)}[v]$, showing that $N(a)$ is semi-cannonical. ■

3 Induced subgraphs of antipodal graphs

In this section we show that almost every graph is semi-cannonical. Since every graph G is an induced subgraph of $P(G)$, in view of Theorem 2 this means that almost every graph G is an induced subgraph of an antipodal graph of diameter 3—namely, of $P(G)$.

Theorem 5 *Almost every graph is semi-cannonical.*

Proof. Let U_n denote the number of all nonisomorphic unlabelled graphs on n vertices, and let S_n denote the number of all nonisomorphic unlabelled semi-cannonical graphs on n vertices. The number of all nonisomorphic unlabelled graphs that are not semi-cannonical is then equal to $U_n - S_n$.

Let G be an arbitrary graph on $n-1$ vertices and construct n graphs G' and G^u for each $u \in V(G)$ in the following way:

- the graph G' is obtained by adding a new vertex adjacent to all vertices of G;
- for each $u \in V(G)$ the graph G^u is obtained by adding a new vertex adjacent to vertices of $N_G[u]$.

If a graph H on n vertices is not semi-cannonical, then either there exists a vertex u such that $N_H[u] = V(H)$ (and then $H \cong (H-u)'$) or there exist two vertices u and v such that $N_H[u] = N_H[v]$ (and then $H \cong (H-v)^u$). Since some graphs on n vertices that are not semi-cannonical may be obtained more than once in this way from a graph with $n-1$ vertices, we have that $U_n - S_n \leq nU_{n-1}$.

Therefore we have that

$$\frac{S_n}{U_n} \geq 1 - \frac{nU_{n-1}}{U_n},$$

and since it holds that (e.g. see [2])

$$U_n = (1+o(n))\frac{2^{\binom{n}{2}}}{n!}, \tag{1}$$

we easily get that

$$\lim_{n\to\infty} \frac{S_n}{U_n} = 1.$$

■

Corollary 6 *Almost every graph is an induced subgraph of an antipodal graph of diameter* 3.

Proof. Almost every graph G is semi-cannonical, and for such G graph $P(G)$ is antipodal graph of diameter 3 and contains G as an induced subgraph (as a matter of fact, $P(G)$ contains two copies of G as induced subgraphs). ■

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