# Potential surfaces and their graphical representations

**Eberhard Malkowsky and Vesna Veličković**

**Abstract**

Potential surfaces arise in physics. Here we use our own software [6, 1, 2, 3, 4] for their graphical representation. We also consider their fundamental coefficients and their Gaussian and mean curvatures[1].

## 1 Introduction

In this paper, we deal with the graphical representation of *potential surfaces* and their *Gaussian* and *mean curvatures*. We use our own software for geometry and differential geometry [6, 1, 2, 3, 4]; no commercial software package for computer graphics is needed.

Many interesting functions in physics, such as surface energy functions, are real–valued functions depending on the direction in three dimensional space. They can be represented as surfaces over the unit sphere, so–called *potential surfaces*.

## 2 Potential surfaces

Throughout, we assume that $D \subset \mathbb{R}^2$ is a domain, and surfaces are given by a parametric representation

$$\vec{x} = \vec{x}(u^i) = (x^1(u^i), x^2(u^i), x^3(u^i)) \quad ((u^1, u^2) \in D) \tag{1}$$

where $x^j \in C^r(D)$ for $j = 1, 2, 3$, that is the component functions $x^j$ have continuous partial derivatives of order $r \geq 1$ on $D$, and the vectors $\vec{x}_k = \partial\vec{x}/\partial u^k$ $(k = 1, 2)$ satisfy the condition

$$\vec{x}_1(u^i) \times \vec{x}_2(u^i) \neq \vec{0} \quad \text{for all } (u^1, u^2) \in D \text{ where } \vec{x}_k = \frac{\partial \vec{x}}{\partial u^k} \ (k = 1, 2); \tag{2}$$

[1]Presented at the IMC "Filomat 2001", Niš, August 26–30, 2001

2000 Mathematics Subject Classification: 53A05, 68N05

Keywords: Computer Graphics, Differential Geometry, Potential Surfaces, Mean and Gaussian curvature

(2) ensures the existence of the tangent plane to a surface at each of its points. If we denote the *surface unit normal vectors*, the *first* and *second fundamental coefficients of a surface* $S$ given by (1) by

$$\vec{N}(u^i) = \frac{\vec{x}_1(u^i) \times \vec{x}_2(u^i)}{\|\vec{x}_1(u^i) \times \vec{x}_2(u^i)\|}, \; g_{jk}(u^i) = \vec{x}_j(u^i) \bullet \vec{x}_k(u^i),$$
$$L_{jk}(u^i) = \vec{N}(u^i) \bullet \vec{x}_{jk}(u^i) \text{ where } \vec{x}_{jk}(u^i) = \frac{\partial^2 \vec{x}(u^i)}{\partial u^j \partial u^k} \text{ for } j,k = 1,2,$$

respectively, and write $g(u^i) = \det(g_{jk}(u^i))$ and $L(u^i) = \det(L_{jk}(u^i))$ then the functions $K : D \to \mathbb{R}$ and $H : D \to \mathbb{R}$ with

$$K = \frac{L}{g} \text{ and } H = \frac{1}{2g}(L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11})$$

are the *Gaussian curvature* and the *mean curvature* of $S$.

Let $D$ be a subset of the rectangle $R = (-\pi/2, \pi/2) \times (0, 2\pi)$. Then the part of the unit sphere $S^2_D$ in $\mathbb{R}^3$ corresponding to $D$ has a parametric representation

$$\vec{y}(u^i) = (\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1) \quad ((u^1, u^2) \in D)$$

where $u^1$ and $u^2$ are the spherical coordinates of the point $Y(u^1, u^2)$ with position vector $\vec{y}(u^1, u^2)$. Each vector $\vec{y}(u^1, u^2)$ represents a direction in $\mathbb{R}^3$, given by $u^1$ and $u^2$. Let $f : D \to \mathbb{R}$ be a function depending on the direction. If, at each point $Y(u^i) \in S^2_D$, we move the distance $f(u^i)$ in the direction of the vector $-\vec{N}(u^i)$, then we obtain a *potential surface* $PS_f$ generated by $f$; thus potential surfaces are natural representations over the unit sphere of real–valued functions of two variables from a subset of $R$. We put $h = f + 1$; then a parametric representation for a potential surface $PS_f$ is given by

$$\vec{x}(u^i) = h(u^i)(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1) \quad ((u^1, u^2) \in D). \tag{3}$$

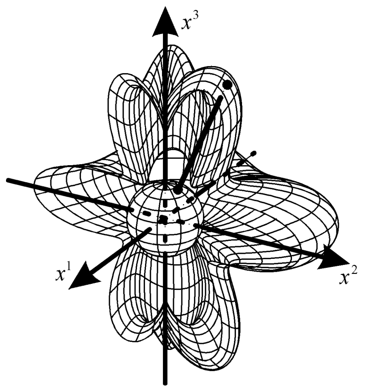


Figure 1: Generation of a potential surface

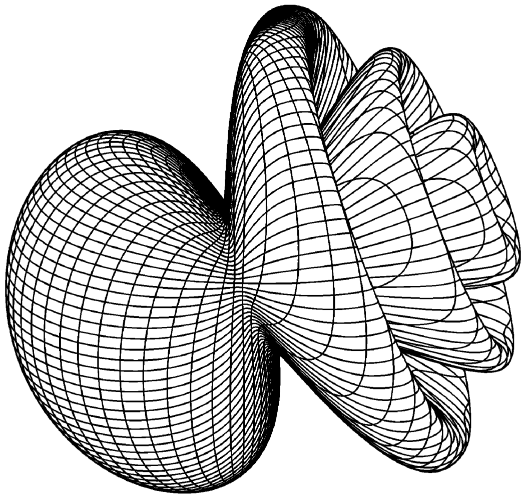

Figure 2: A potential surface

As an example we consider the growth of crystals.

**Example 1** *According to Wulff's principle [8, 1], the shape of a crystal that grows under ideal laboratory conditions is uniquely defined by it surface energy function, a real–valued function that depends on the direction in* $\mathbb{R}^3$. *It was shown in [7] that if the surface energy function is a norm then the shape of the corresponding crystal is a sphere with respect to the dual norm.*

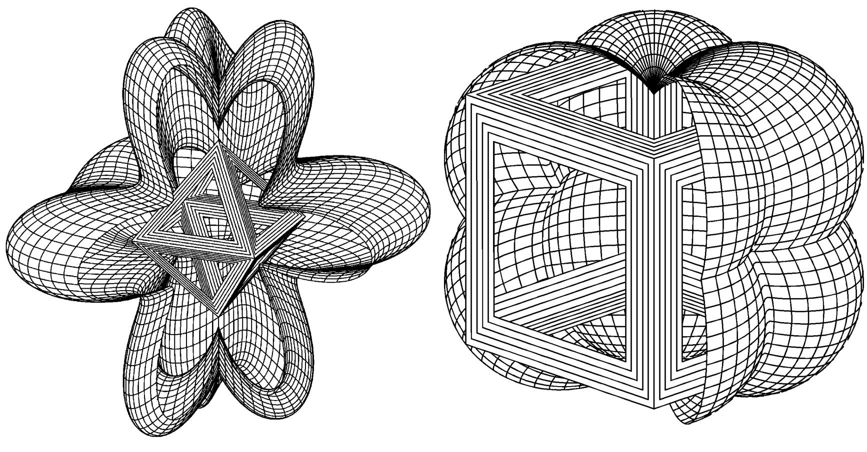

Figure 3: Wulff's crystals corresponding to the $\ell_\infty$ and $\ell_1$ norms

# 3 The first and second fundamental coefficients and the Gaussian and mean curvature of potential surfaces

In this section we compute the first and second fundamental coefficients and the Gaussian and mean curvature of potential surfaces given by a parametric representation (3). Furthermore, we represent some of these quantities as potential surfaces by using the corresponding functions as $f$ in $h = f + 1$.

Omitting $u^1$ and $u^2$ in the notations, from (3), that is, from $\vec{x} = h\vec{y}$, we obtain

$$\vec{x}_1 = h_1\vec{y} + h\vec{y}_1 \text{ and } \vec{x}_2 = h_2\vec{y} + h\vec{y}_2$$

where $\vec{y}_1 = (-\sin u^1 \cos u^2, -\sin u^1 \sin u^2, \cos u^1)$, $\vec{y}_2 = (-\cos u^1 \sin u^2, \cos u^1 \cos u^2, 0)$, $\vec{y}^{\,2} = \vec{y}_1^{\,2} = 1$, $\vec{y}_2^{\,2} = \cos^2 u^1$, $\vec{y}_1 \bullet \vec{y}_2 = 0$ and $\vec{y} \bullet \vec{y}_k = \dfrac{1}{2}\dfrac{\partial \vec{y}^{\,2}}{\partial u^k} = 0 \quad (k = 1, 2)$. Thus the first fundamental coefficients are given by

$$g_{11} = h_1^2 + h^2,\ g_{12} = h_1 h_2,\ g_{22} = h_2^2 h^2 \cos^2 u^1$$

$$\text{and } g = h^2\left((h^2 + h_1^2)\cos^2 u^1 + h_2^2\right).$$

Furthermore

$$\vec{y}_{11} = (-\cos u^1 \cos u^2, -\cos u^1 \sin u^2, -\sin u^1) = -\vec{y},$$

$$\vec{y}_{12} = (\sin u^1 \sin u^2, -\sin u^1 \cos u^2, 0),$$

$$\vec{y}_{22} = (-\cos u^1 \cos u^2, -\cos u^1 \cos u^2, 0),$$

$$\vec{x}_{11} = h_{11}\vec{y} + 2h_1\vec{y}_1 + h\vec{y}_{11} = (h_{11} - h)\vec{y} + 2h_1\vec{y}_1,$$

$$\vec{x}_{12} = h_{12}\vec{y} + h_1\vec{y}_2 + h_2\vec{y}_2 + h\vec{y}_{12},$$

$$\vec{x}_{22} = h_{22}\vec{y} + 2h_2\vec{y}_2 + h\vec{y}_{22} \text{ and}$$

$$\vec{x}_1 \times \vec{x}_2 = hh_1(\vec{y} \times y_2) + hh_2(\vec{y}_1 \times \vec{y}) + h^2(\vec{y}_1 \times \vec{y}_2),$$

hence

$$\begin{aligned} L_{11}\sqrt{g} &= \vec{x}_{11} \bullet (\vec{x}_1 \times \vec{x}_2) = (h_{11} - h)h^2\vec{y} \bullet (\vec{y}_1 \times \vec{y}_2) + 2hh_1^2\vec{y}_1 \bullet (\vec{y} \times \vec{y}_2) \\ &= ((h_{11} - h)h^2 - 2hh_1h_1^2)\vec{y} \bullet (\vec{y}_1 \times \vec{y}_2) \end{aligned}$$

and $\vec{y} \bullet (\vec{y}_1 \times \vec{y}_2) = -\cos u^1$ imply

$$L_{11}\sqrt{g} = h\cos u^1(h^2 - hh_{11} + 2h_1^2).$$

Next

$$\begin{aligned} L_{12}\sqrt{g} = \vec{x}_{12} \bullet (\vec{x}_1 \times \vec{x}_2) = h^2 h_{12}\vec{y} \bullet (\vec{y}_1 \times \vec{y}_2) + hh_1h_2\vec{y}_2 \bullet (\vec{y}_1 \times \vec{y}) + \\ + hh_1h_2\vec{y}_1 \bullet (\vec{y} \times \vec{y}_2) + h\vec{y}_{12} \bullet \left(hh_1(\vec{y} \times \vec{y}_2) + hh_2(\vec{y}_1 \times \vec{y}) + h^2(\vec{y}_1 \times \vec{y}_2)\right) \end{aligned}$$

and $\vec{y}_{12} \bullet (\vec{y} \times \vec{y}_2) = 0$, $\vec{y}_{12} \bullet (\vec{y}_1 \times \vec{y}) = -\sin u^1$ and $\vec{y}_{12} \bullet (\vec{y}_1 \times \vec{y}_2) = 0$ imply

$$L_{12}\sqrt{g} = h\left(\cos u^1(2h_1h_2 - hh_{12}) - hh_2 \sin u^1\right).$$

Finally,

$$\begin{aligned} L_{22}\sqrt{g} = \vec{x}_{22} \bullet (\vec{x}_1 \times \vec{x}_2) = h^2h_{22}\vec{y} \bullet (\vec{y}_1 \times \vec{y}_2) + 2hh_2^2\vec{y}_2 \bullet (\vec{y}_1 \times \vec{y}) + \\ +h\left(hh_1\vec{y}_{22} \bullet (\vec{y} \times \vec{y}_2) + hh_2\vec{y}_{22} \bullet (\vec{y}_1 \times \vec{y}) + h^2\vec{y}_{22} \bullet (\vec{y}_1 \times \vec{y}_2)\right) \end{aligned}$$

and $\vec{y}_{22} \bullet (\vec{y} \times \vec{y}_2) = \cos^2 u^1 \sin u^1$, $\vec{y}_{22} \bullet (\vec{y}_1 \times \vec{y}) = 0$ and $\vec{y}_{22} \bullet (\vec{y}_1 \times \vec{y}_2) = \cos^3 u^1$ imply

$$L_{22}\sqrt{g} = h \cos u^1 \left(2h_2^2 - hh_{22} + h^2 \cos^2 u^1 + hh_1 \sin u^1 \cos u^1\right).$$

From this, we obtain the second fundamental coefficients $L_{jk}$ $(j,k = 1,2)$, $L$ and

$$K = \frac{L}{g} \text{ and } H = \frac{1}{2g}(L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}).$$

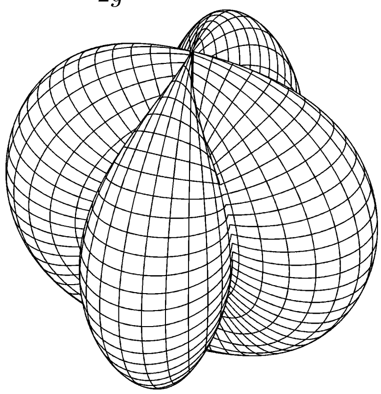

Figure 4: The potential surface of the function $h(u^i) = |\sin 2u^2| + 1$

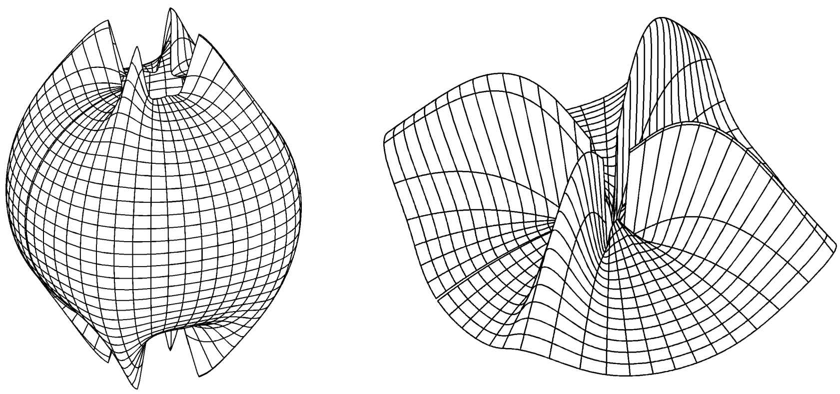

Figure 5: The Gaussian curvature of the potential surface in Figure 4

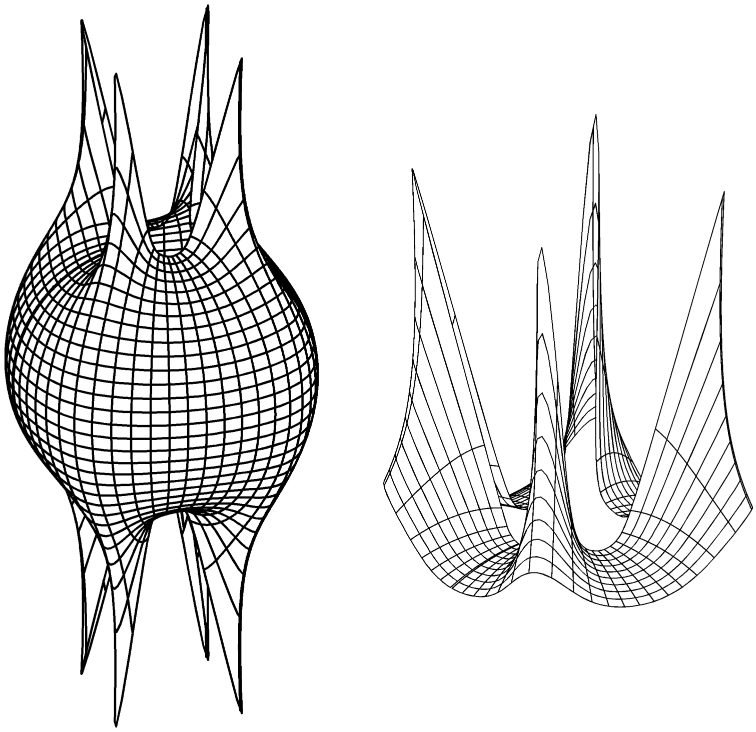

Figure 6: The mean curvature of the potential surface in Figure 4

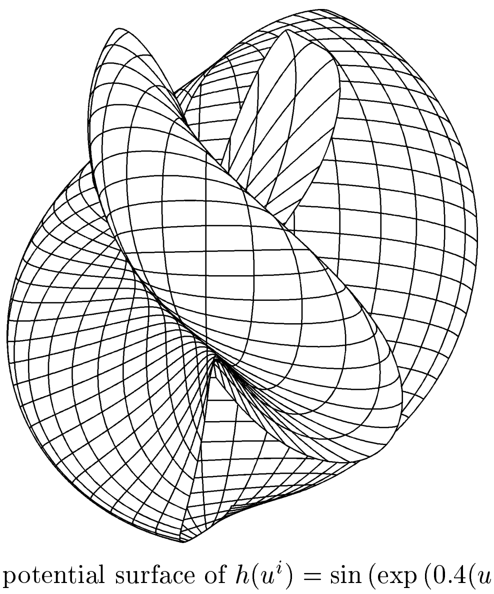

Figure 7: The potential surface of $h(u^i) = \sin(\exp(0.4(u^1 + u^2))) + 2$

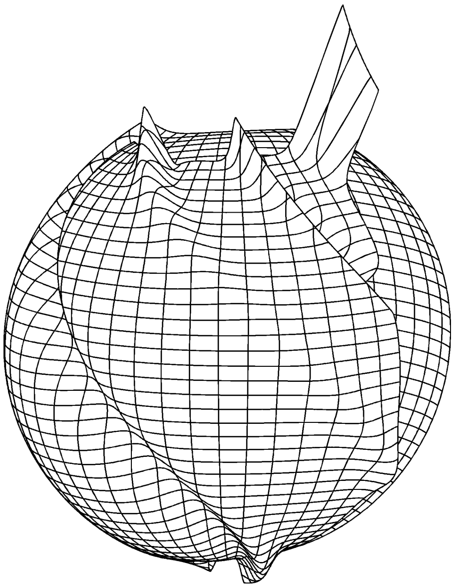

Figure 8: The Gaussian curvature of the potential surface in Figure 7

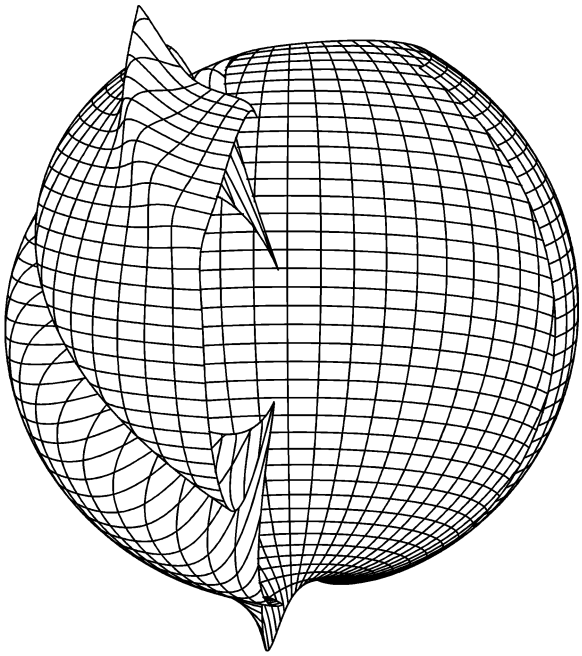

Figure 9: The mean curvature of the potential surface in Figure 7

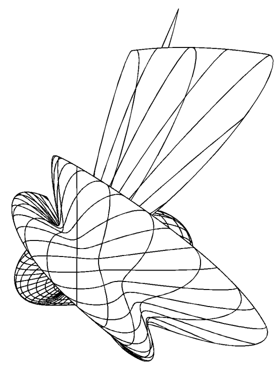

Figure 10: The function $g$ of the potential surface in Figure 7

# References


[1] M. Failing, *Entwicklung numerischer Algorithmen zur computergrafischen Darstellung spezieller Probleme der Differentialgeometrie und Kristallographie*, Ph.D. Thesis, Giessen, 1996.

[2] M. Failing and E. Malkowsky, *Ein effizienter Nullstellenalgorithmus zur computergrafischen Darstellung spezieller Kurven und Flächen*, Mitt. Math. Sem. Giessen **229** (1996), 11–25.

[3] E. Malkowsky, *Computergrafik in der Differentialgeometrie, eine offene Software zur Differentialgeometrie*, Technologie–Vermittlungs–Agentur Berlin, Dokumentationsband zum Hochschul–Computer–Forum , 1993.

[4] E. Malkowsky, *An open software in OOP for computer graphics and some applications in differential geometry*, Proceedings of the 20th South African Symposium on Numerical Mathematics, 1994, 51-80.

[5] E. Malkowsky, *An open sofware in OOP for computer graphics in differential geometry, the basic concepts*, Zeitschrift für Angewandte Mathematik und Mechanik **76, Suppl. 1** (1996), 467–468.

[6] E. Malkowsky and W. Nickel, *Computergrafik in der Differentialgeometrie*, Vieweg–Verlag, Wiesbaden, Braunschweig, 1993.

[7] E. Malkowsky and V. Veličković, *An application of functional analysis in computer graphics and crystallography*, Proceedings of the Conference YUINFO 2000, Kopaonik, Yugoslavia, on CD.

[8] G. Wulff, *Der Curie-Wulff'sche Satz über Combinationsformen von Krystallen*, Zeitschrift für Krystallographie **53** (1990).



Faculty of Science and Mathematics
Višegradska 33, 18000 Niš
Yugoslavia
ema@pmf.pmf.ni.ac.yu
vesna@BankerInter.net