Some matrix transformations between the difference sequence spaces $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta l_\infty(p)$

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Abstract

For any sequence $x = (x_k)_{k=1}^\infty \in \omega$ and any subset X of ω, we write $\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty$ and $\Delta X = \{x \in \omega : \Delta x \in X\}$. Let $p = (p_k)_{k=1}^\infty$ and $q = (q_k)_{k=1}^\infty$ be bounded sequences of positive reals. We determine the β–duals of the sets $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta \ell_\infty(p)$. Furthermore, we characterize the matrix classes $(\Delta X, Y)$ and $(\Delta X, \Delta Y)$ for $X = c_0(p), c(p), l_\infty(p)$ and $Y = c_0(q), c(q), l_\infty(q)$.[1].

1 Introduction

Let ω be the set of all complex sequences $x = (x_k)_{k=1}^\infty$, and c_0, c, l_∞ and cs be the sets of all null, convergent and bounded sequences and of all convergent series, respectively. Furthermore, let $p = (p_k)_{k=1}^\infty$ and $q = (q_k)_{k=1}^\infty$ be bounded sequences of positive reals throughout, and $c_0(p) = \{x \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\}$, $c(p) = \{x \in \omega : \lim_{k\to\infty} |x_k - l|^{p_k} = 0$ for some $l \in \mathbb{C}\}$ and $l_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$ (cf. [5], [6] and [10]).

Grosse–Erdmann [2] characterized the matrix classes (X, Y) for $X = c_0(p)$, $c(p)$, $l_\infty(p)$ and $Y = c_0(q), c(q), l_\infty(q)$.

Given any sequence $x \in \omega$, we write $\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty$. Furthermore, for any subset X of ω, let $\Delta X = \{x \in \omega : \Delta x \in X\}$. In [1] and [7], the sequence spaces ΔX were introduced and studied for $X = c_0(p), c(p), l_\infty(p)$. If $p_k = const$ for all k then these sets reduce to $c_0(\Delta)$, $c(\Delta)$ and $l_\infty(\Delta)$, respectively (see [3],[8]).

In this paper, we determine the β–duals of the sets $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta \ell_\infty(p)$ and characterize the matrix classes $(\Delta X, Y)$ and $(\Delta X, \Delta Y)$ for $X = c_0(p), c(p)$, $l_\infty(p)$ and $Y = c_0(q), c(q), l_\infty(q)$.

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2 The β–duals of $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta \ell_\infty(p)$

If z is any sequence and Y is any subset of ω then we write $z^{-1} * Y = \{x \in \omega : zx = (z_k x_k)_{k=1}^\infty \in Y\}$. For any subset X of ω, the set $X^\beta = \cap_{x \in X}(x^{-1} * cs)$ is called the β–dual of X. The β–duals of the sets $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta l_\infty(p)$ were studied in [1], [7] and [9]. Boundedness of the sequence p was not assumed. If, however, we assume boundedness of the sequence p a different proof may be applied which considerably improves the results for the β–duals of $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta \ell_\infty(p)$.

Let X and Y be subsets of ω. By (X, Y) we denote the class of all infinite matrices $A = (a_{nk})_{n,k=1}^\infty$ of complex numbers such that $A_n = (a_{nk})_{k=1}^\infty \in X^\beta$ for all n and $A(x) = (A_n(x))_{n=1}^\infty = (\sum_{k=1}^\infty a_{nk}x_k)_{k=1}^\infty \in Y$ for all $x \in X$.

We write e and $e^{(n)}$ $(n = 1, 2, \dots)$ for the sequences with $e_k = 1$ $(k = 1, 2, \dots)$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. If $a \in cs$ we define the sequence R by $R_k = \sum_{j=k+1}^\infty a_j$ for $k = 0, 1, \dots$.

We need the following result.

Lemma 1 ([7, Corollary 1]) *Let d be a non–decreasing sequence of positive reals. Then $a \in d^{-1} * cs$ implies $R \in d^{-1} * c_0$.*

We write $\mathbf{n} = (n)_{n=1}^\infty$, $\mathbf{N}^{1/p} = (N^{1/p_k})_{k=1}^\infty$, $\mathbf{N}^{-1/p} = (N^{-1/p_k})_{k=1}^\infty$, $\mathbf{\Sigma N}^{1/p} = (\sum_{j=1}^{k-1} N^{1/p_j})_{k=1}^\infty$ and $\mathbf{\Sigma N}^{-1/p} = (\sum_{j=1}^{k-1} N^{-1/p_j})_{k=1}^\infty$ for each $N \in I\!N \setminus \{1\}$.

The following result in which the boundedness of the sequence p is not needed is well known.

Lemma 2 (cf. [9, Theorem 2])
We put

$$M_\infty^{(1)}(p) = \bigcap_{N \in I\!N \setminus \{1\}} \left\{ a \in \omega : R \in (\mathbf{N}^{1/p})^{-1} * \ell_1 \right\}$$

and

$$\tilde{M}_\infty^{(2)}(p) = \bigcap_{N \in I\!N \setminus \{1\}} (\mathbf{\Sigma N}^{1/p})^{-1} * cs.$$

Then $(\Delta \ell_\infty(p))^\beta = M_\infty^{(1)}(p) \cap \tilde{M}_\infty^{(2)}(p)$.

Now we give the β–duals of $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta \ell_\infty(p)$ for bounded sequences p.

Theorem 1 *We put*

$$M_0^{(1)}(p) = \bigcup_{N \in I\!N \setminus \{1\}} \left\{ a \in \omega : R \in (\mathbf{N}^{-1/p})^{-1} * \ell_1 \right\},$$

$$M_0^{(2)}(p) = \bigcup_{N \in I\!N \setminus \{1\}} \left\{ a \in \omega : R \in (\mathbf{\Sigma N}^{-1/p})^{-1} * \ell_\infty \right\},$$

$$M(p) = (\mathbf{n})^{-1} * cs$$

and

$$M_{\infty}^{(2)}(p) = \bigcap_{N \in IN \setminus \{1\}} \left\{ a \in \omega : R \in (\mathbf{\Sigma N}^{1/p})^{-1} * c_0 \right\}.$$

Then
(a) $(\Delta c_0(p))^{\beta} = M_0^{(1)}(p) \cap M_0^{(2)}(p)$*, and if* $a \in (\Delta c_0(p))^{\beta}$ *then*

$$\sum_{k=1}^{\infty} a_k y_k = -\sum_{k=1}^{\infty} R_k \Delta y_k + y_1 \sum_{k=1}^{\infty} a_k \text{ for all } y \in \Delta c_0(p); \tag{1}$$

(b) $(\Delta c(p))^{\beta} = (\Delta c_0(p))^{\beta} \cap M(p)$*, and if* $a \in (\Delta c(p))^{\beta}$ *then identity (1) holds for all* $y \in \Delta c(p)$ *;*
(c) $(\Delta \ell_{\infty}(p))^{\beta} = M_{\infty}^{(1)}(p) \cap M_{\infty}^{(2)}(p)$*, and if* $a \in (\Delta \ell_{\infty}(p))^{\beta}$ *then identity (1) holds for all* $y \in \Delta \ell_{\infty}(p)$*.*

Proof. (a) We write $Y = \Delta c_0(p)$ and $X = c_0(p)$.
First we assume $a \in M_0^{(1)}(p) \cap M_0^{(2)}(p)$. Let $y \in Y$ be given. Then $x = \Delta y \in X$. Abel's summation by parts yields

$$\sum_{k=1}^{n} a_k y_k = -\sum_{k=1}^{n-1} R_k x_k - R_n y_n + y_1 R_0 \ (n = 1, 2, \dots). \tag{2}$$

First $a \in M_0^{(1)}(p)$, that is $R \in X^{\beta}$ by [6, Theorem 6] implies $Rx \in cs$. Furthermore

$$R_n(y_n - y_1) = -\sum_{k=1}^{n-1} R_n x_k \text{ for } n = 1, 2, \dots, \tag{3}$$

and we note that $y \in Y$ if and only if $y - y_1 e^{(1)} \in Y$, since Y is a linear space for bounded sequences p. We define the matrix A by

$$a_{nk} = \begin{cases} -R_n & (1 \leq k \leq n-1) \\ 0 & (k > n) \end{cases} \quad (n = 1, 2, \dots). \tag{4}$$

Then $a \in M_0^{(2)}(p)$ implies

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| N^{-1/p_k} = \sup_n |R_n| \sum_{k=1}^{n-1} N^{-1/p_k} < \infty \text{ for some } N \in IN \setminus \{1\}. \tag{5}$$

Furthermore $R_n = \sum_{k=n+1}^{\infty} a_k \to 0 \ (n \to \infty)$, that is

$$\lim_{n \to \infty} a_{nk} = 0 \text{ for each fixed } k. \tag{6}$$

By [4, Corollary 3], conditions (5) and (6) together imply $A \in (X, c)$, that is $R(y - y_1 e^{(1)}) \in c$, and so $Ry \in c$. Finally, by (2), we conclude $ay \in cs$. Thus we have shown

$$M_0^{(1)}(p) \cap M_0^{(2)}(p) \subset Y^{\beta}. \tag{7}$$

Now we assume $a \in Y^\beta$. Then $ay \in cs$ for all $y \in Y$. Since $\Delta e = 0 \in X$, it follows that $e \in Y$, and so $a = ae \in cs$, hence the sequence R is defined. If $y \in Y$ then $x = \Delta y \in X$, and

$$\sum_{k=1}^{n} a_k(y_k - y_1) = -\sum_{k=1}^{n-1} x_k \sum_{j=k+1}^{n} a_j \ (n = 1, 2, \dots).$$

We define the matrix $B = (b_{nk})_{n,k=1}^\infty$ by $b_{nk} = -\sum_{j=k+1}^{n} a_j$ for $1 \leq k \leq n-1$ and $b_{nk} = 0$ for $k \geq n$ $(n = 1, 2, \dots)$. Then $B \in (X, c)$, and again, by [4, Corollary 3], there are $N \in I\!N \setminus \{1\}$ and a constant K such that

$$\sum_{k=1}^{\infty} |b_{nk}| N^{-1/p_k} = \sum_{k=1}^{n-1} \left| \sum_{j=k+1}^{n} a_j \right| N^{-1/p_k} \leq K \text{ for all } n.$$

We fix $m \in \mathbb{N}$. Then

$$\sum_{k=1}^{m-1} \left| \sum_{j=k+1}^{n} a_j \right| N^{-1/p_k} \leq K \text{ for all } n \geq m.$$

Since $R_k = \lim_{n\to\infty} \sum_{j=k+1}^{n} a_j$ exists for each k, this implies $\sum_{k=1}^{m-1} |R_k| N^{-1/p_k} \leq K$, and since $m \in \mathbb{N}$ was arbitrary, we conclude $\sum_{k=1}^{\infty} |R_k| N^{-1/p_k} \leq K$, that is $R \in X^\beta$ by [6, Theorem 6], and so $a \in M_0^{(1)}(p)$. Defining the matrix A as in (4), we have $A \in (X, c)$, and this yields (5) by [4, Corollary 3], hence $a \in M_0^{(2)}(p)$. Thus we have shown $Y^\beta \subset M_0^{(1)}(p) \cap M_0^{(2)}(p)$. This and (7) together yield $Y^\beta = M_0^{(1)}(p) \cap M_0^{(2)}(p)$.
Finally, we assume $a \in Y^\beta$ Then, by what we have just shown, there is $N \in I\!N \setminus \{1\}$ such that, for the matrix A defined in (4),

$$D = \sup_n \sum_{k=1}^{\infty} |a_{nk}| N^{-1/p_k} = \sup_n |R_n| \sum_{k=1}^{n-1} N^{-1/p_k} < \infty,$$

and condition (6) holds. Let $\varepsilon > 0$ be given. We put $P = \sup_k p_k < \infty$ and $M_0 = \max\{N((2D+1)/\varepsilon)^P, N\}$. Then for all $M \geq M_0$ and for all $n \in \mathbb{N}$,

$$\begin{aligned} |R_n| \sum_{k=1}^{n-1} M^{-1/p_k} &\leq |R_n| \sum_{k=1}^{n-1} M_0^{-1/p_k} = |R_n| \sum_{k=1}^{n-1} N^{-1/p_k} \left(\frac{N}{M_0}\right)^{1/p_k} \\ &\leq \left(|R_n| \sum_{k=1}^{n-1} N^{-1/p_k} \right) \left(\frac{N}{M_0}\right)^{1/P} \leq D \left(\frac{N}{M_0}\right)^{1/P} \leq D \frac{N\varepsilon}{(2D+1)N} \leq \varepsilon/2, \end{aligned}$$

and so

$$\lim_{M\to\infty} \sup_n |a_{nk}| M^{-1/p_k} = 0. \tag{8}$$

By [2, Theorem 5.1, 5.], conditions (6) and (8) together imply $A \in (X, c_0)$, hence $Ry \in c_0$, and (1) follows from (2).

(b) First we assume $a \in (c(p))^\beta$. Since $\Delta c_0(p) \subset \Delta c(p)$ implies $(\Delta c(p))^\beta \subset (\Delta c_0(p))^\beta$, we have $a \in (\Delta c_0(p))^\beta$. Furthermore, $\mathbf{n} \in \Delta c(p)$, since $\Delta\mathbf{n} - (-1)e \in c_0(p)$. Thus $a\mathbf{n} \in cs$, that is $a \in M(p)$. Thus we have shown

$$(\Delta c(p))^\beta \subset (\Delta c_0(p))^\beta \cap M(p). \tag{9}$$

Conversely we assume $a \in (\Delta c_0(p))^\beta \cap M(p)$. Let $y \in \Delta c(p)$ be given. Then $x = \Delta y \in c(p)$, hence there is $l \in \mathbb{C}$ such that $x - le \in c_0(p)$. Let $z = y + l\mathbf{n}$. Then $\Delta z = \Delta y + l\Delta\mathbf{n} = x - le \in c_0(p)$, hence $z \in \Delta c_0(p)$, and, as in (2),

$$\begin{aligned}&\sum_{k=1}^{n} a_k y_k = \sum_{k=1}^{n} a_k z_k + l\sum_{k=1}^{n} k a_k = \\ &-\sum_{k=1}^{n-1} R_k \Delta z_k - R_n z_n + z_1 R_0 + l\sum_{k=1}^{n} k a_k \text{ for all } n.\end{aligned} \tag{10}$$

Since $z \in \Delta c_0(p)$, we have $R\Delta z \in cs$ and $Rz \in c$ by Part (a). Furthermore $a\mathbf{n} \in cs$, since $a \in M(p)$. Thus $ay \in cs$, and we have shown $(\Delta c_0(p))^\beta \cap M(p) \subset (\Delta c(p))^\beta$. Together with (9) this yields $(\Delta c(p))^\beta = (\Delta c_0(p))^\beta \cap M(p)$.
Finally, let $a \in (\Delta c(p))^\beta$ and $y \in c(p)$ be given. By (2),

$$\begin{aligned}&\sum_{k=1}^{n} a_k y_k = -\sum_{k=1}^{n-1} R_k \Delta y_k - R_n y_n + y_1 R_0 = \\ &-\sum_{k=1}^{n-1} R_k \Delta y_k - R_n z_n + R_n n + y_1 R_0 \ (n = 0, 1, \dots).\end{aligned}$$

Since $z \in \Delta c_0(p)$ and $a \in M_0^{(2)}(p)$, $Rz \in c_0$ by Part (a). Furthermore $a \in M(p)$ implies $\mathbf{n}R \in c_0$ by Lemma 1. So $ay \in cs$ implies $R\Delta y \in cs$, and (2) holds.

(c) We write $Y = \Delta\ell_\infty(p)$ and $X = \ell_\infty(p)$. By Lemma 1 and Lemma 2,

$$Y^\beta \subset M_\infty^{(1)}(p) \cap M_\infty^{(2)}(p). \tag{11}$$

Conversely we assume $a \in M_\infty^{(1)}(p) \cap M_\infty^{(2)}(p)$. Let $y \in Y$ be given. Then $x = \Delta y \in X$. First $a \in M_0^{(1)}(p)$, that is $R \in X^\beta$ by [5, Theorem 2] implies $Rx \in cs$. We define the matrix A as in (4). Then $a \in M_\infty^{(2)}(p)$ implies

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| N^{1/p_k} = \lim_{n\to\infty} \sum_{k=1}^{n-1} |R_n| \sum_{k=1}^{n-1} M^{1/p_k} = 0 \text{ for all } N \in IN \setminus \{1\},$$

that is $A \in (X, c_0)$ by [2, Theorem 5.1, 7.]. Therefore we conclude from (3) that $R(y - y_1 e^{(1)}) \in c_0$, and so $Ry \in c_0$. By (2), $ay \in cs$. Thus we have shown $M_\infty^{(1)}(p) \cap M_\infty^{(2)}(p) \subset Y^\beta$. This and (11) together yield $Y^\beta = M_\infty^{(1)}(p) \cap M_\infty^{(2)}(p)$. The last part is obvious. ∎

3 Matrix transformations

Let $p = (p_k)_{k=1}^\infty$ and $q = (q_k)_{k=1}^\infty$ be bounded sequences of positive reals throughout.

If $A = (a_{nk})_{n,k=1}^\infty$ is an infinite matrix then we write R^A for the matrix with $r_{nk}^A = \sum_{j=k+1}^\infty a_{nj}$ for all n and k, provided the series converge.

First we reduce the characterizations of the classes $(\Delta X, Y)$ to those of (X, Y) for arbitrary subspaces Y of ω and $X = c_0(p)$, $c(p)$ and $l_\infty(p)$.

Theorem 2 *Let Y be an arbitrary subspace of ω. Then*
(a) $A \in (\Delta c_0(p), Y)$ if and only if

$$R \in (c_0(p), Y), \tag{12}$$

$$A(e) \in Y \tag{13}$$

and

$$\left\{ \begin{array}{c} \text{for each } n \text{ there is } N_n \in I\!N \setminus \{1\} \text{ such that} \\ \sup_k |r_{nk}^A| \sum_{j=1}^{k-1} N^{-1/p_j} < \infty; \end{array} \right. \tag{14}$$

(b) $A \in (\Delta c(p), Y)$ if and only if

$$A \in (\Delta c_0(p), Y) \tag{15}$$

and

$$A(\mathbf{k}) \in Y; \tag{16}$$

(c) $A \in (\Delta l_\infty(p), Y)$ if and only if condition (13) holds and

$$R \in (l_\infty(p), Y) \tag{17}$$

and

$$\left\{ \begin{array}{c} \sup_k |r_{nk}^A| \sum_{j=1}^{k-1} N^{1/p_j} < \infty \\ \text{for all } n = 1, 2, \dots \text{ and for all } N \in I\!N \setminus \{1\}. \end{array} \right. \tag{18}$$

Proof. (a) We put $X = c_0(p)$ and $Z = \Delta X$, and observe that $z \in Z$ if and only if $x = \Delta z \in X$. Furthermore we write $R = R^A$.
First we assume $A \in (Z, Y)$. Condition (13) is obvious, since $e \in Z$. Furthermore $A_n \in Z^\beta$ implies $R_n \in M_0^{(2)}(p)$ for each n by Theorem 1 (a), and so condition (14) holds. Let $x \in X$ be given. We define the sequence z by $z_k = \sum_{j=1}^{k-1} x_j$ $(k = 1, 2, \dots)$. Then $z \in Z$ and

$$R_n(x) = -A_n(z) + z_1 r_{n0} = -A_n(z) \text{ for all } n$$

by identity (1), and $A(z) \in Y$ implies $R(x) \in Y$. Thus condition (12) holds.
Conversely, we assume that conditions (12), (13) and (14) hold. First $R_n \in X^\beta$

for all n and condition (14) together imply $A_n \in Z^\beta$ for all n by Theorem 1 (a). Let $z \in Z$ be given. Then by (1)

$$A_n(z) = -R_n(x) + z_1 r_{n0} = R_n(x) + z_1 A_n(e) \text{ for all } n,$$

and $R(x) \in Y$ and condition (13) together imply $A(z) \in Y$, since Y is a linear space.

(b) First we assume $A \in (\Delta c(p), Y)$. Then obviously $A \in (\Delta c_0(p), Y)$. Furthermore $\mathbf{k} \in \Delta c(p)$ implies condition (16).
Conversely, we assume that conditions (15) and (16) hold. First, condition (16) implies $A_n \in \mathbf{k}^{-1}{*}cs$, that is $A_n \in M(p)$ for all n, and since also $A_n \in (\Delta c_0(p))^\beta$ for all n by condition (15), we conclude $A_n \in (\Delta c(p))^\beta$ by Theorem 1 (b). Let $z \in \Delta c(p)$ be given. Then $\Delta z - le \in c_0(p)$ for some $l \in \mathbb{C}$. We put $x = z + l\mathbf{k}$. Then $x \in \Delta c_0(p)$, and

$$A_n(z) = A_n(x) - lA_n(\mathbf{k}) \text{ for all } n.$$

Now $A(x) \in Y$ and condition (16) together imply $A(z) \in Y$, since Y is a linear space.

(c) Part (c) is proved in the same way as Part (a) by applying Theorem 1 (c) instead of Theorem 1 (a). ■

Remark 1 *Condition (13) in Theorem 2 (a) and (c) may be replaced by*

$$A(e^{(1)}) \in Y. \tag{19}$$

Proof. Let $X = c_0(p)$ or $X = l_\infty(p)$ and $Z = \Delta X$.
First we assume that conditions (12), (13) and (14) or (17), (13) and (18) hold. Then $A \in (Z, Y)$ by Theorem 2 (a) or (c), respectively. Now $e^{(1)} \in Z$ implies $A(e^{(1)}) \in Y$, that is condition (19) holds.
Conversely, we assume that conditions (12), (19) and (14) or (17), (19) and (18) hold. Then $A_n(e) = R_n(e^{(1)}) + A_n(e^{(1)})$ for all n. Since $e^{(1)} \in X$, we have $R(e^{(1)}) \in Y$ by condition (12) or condition (17). This and condition (19) together imply $A(e) \in Y$, that is condition (13) holds. ■

The characterization of $(X, \Delta Y)$ can easily be reduced to that of (X, Y).

Theorem 3 *Let X and Y be arbitrary subsets of ω. Then $A \in (X, \Delta Y)$ if and only if*

$$A_1 \in X^\beta \tag{20}$$

and

$$B \in (X, Y) \text{ where } b_{nk} = a_{nk} - a_{n+1,k} \text{ for all } n \text{ and } k. \tag{21}$$

Proof. First we assume $A \in (X, \Delta Y)$. Then $A_n \in X^\beta$ for all n, in particular, condition (20) holds, and $B_n = \Delta_n A_n = A_n - A_{n+1} \in X^\beta$ for all n. Furthermore, $A(x) \in \Delta Y$, that is $\Delta_n A(x) = B(x) \in Y$ for all $x \in X$, and so (21) holds.
Conversely, we assume that conditions (20) and (21) are satisfied. Then $A_{n+1} = A_n - B_n \in X^\beta$ for all $n \geq 2$ by induction. Furthermore, $B(x) = \Delta_n A(x) \in Y$, that is $A(x) \in \Delta Y$ for all $x \in X$. Thus we have shown $A \in (X, \Delta Y)$. ■

Now we apply Theorem 2 and well–known results from [2] to characterize the classes (X,Y) and $(X,\Delta Y)$ where X is any of the spaces $\Delta l_\infty(p)$, $\Delta c_0(p)$ and $\Delta c(p)$, and Y is any of the spaces $l_\infty(p)$, $c_0(p)$ and $c(p)$.

Theorem 4 *The necessary and sufficient conditions for $A \in (X,Y)$ for $X = \Delta\ell_\infty(p), \Delta c_0(p), \Delta c(p)$ and $Y = \ell_\infty(q), c_0(q), c(q), \Delta\ell_\infty(p), \Delta c_0(q), \Delta c(q)$ can be read from the following table*

To / *From*	$\ell_\infty(q)$	$c_0(q)$	$c(q)$	$\Delta\ell_\infty(q)$	$\Delta c_0(q)$	$\Delta c(q)$
$\Delta\ell_\infty(p)$	*(1.)*	*(2.)*	*(3.)*	*(10.)*	*(11.)*	*(12.)*
$\Delta c_0(p)$	*(4.)*	*(5.)*	*(6.)*	*(13.)*	*(14.)*	*(15.)*
$\Delta c(p)$	*(7.)*	*(8.)*	*(9.)*	*(16.)*	*(17.)*	*(18.)*

where, with $r^A_{nk} = \sum_{j=k+1}^\infty a_{nj}$ *and* $r^B_{nk} = \sum_{j=k+1}^\infty (a_{nj} - a_{n+1,j})$ $(n,k = 1,2,\dots)$,

(1.): *(1.1), (1.2), (1.3) where*

(1.1) $\sup_n \left(\sum_{k=1}^\infty |r^A_{nk}| N^{1/p_k}\right)^{q_n} < \infty$ *for all* $N \in I\!N \setminus \{1\}$

(1.2) $\begin{cases} \sup_k |r^A_{nk}| \sum_{j=1}^{k-1} N^{1/p_j} < \infty \\ \text{for all } N \in I\!N \setminus \{1\} \text{ and for all } n = 1,2,\dots \end{cases}$

(1.3) $\sup_n |a_{n1}|^{q_n} < \infty$

(2.): *(2.1), (1.2), (2.2) where*

(2.1) $\lim_{n\to\infty} \left(\sum_{k=1}^\infty |r^A_{nk}| N^{1/p_k}\right)^{q_n} = 0$ *for all* $N \in I\!N \setminus \{1\}$

(2.2) $\lim_{n\to\infty} |a_{n1}|^{q_n} = 0$

(3.): *(3.1), (3.2), (1.2), (3.3) where*

(3.1) $\sup_n \sum_{k=1}^\infty |r^A_{nk}| N^{1/p_k} < \infty$ *for all* $N \in I\!N \setminus \{1\}$

(3.2) $\begin{cases} \text{there is a sequence } (\alpha_k)_{k=1}^\infty \text{ such that} \\ \lim_{n\to\infty} \left(\sum_{k=1}^\infty |r^A_{nk} - \alpha_k| N^{1/p_k}\right)^{q_n} = 0 \\ \text{for all } N \in I\!N \setminus \{1\} \end{cases}$

(3.3) $\lim_{n\to\infty} |a_{n1} - \alpha|^{q_n} = 0$ *for some* $\alpha \in \mathbb{C}$

(4.): *(4.1), (4.2), (1.3) where*

(4.1) $\sup_n \left(\sum_{k=1}^\infty |r^A_{nk}| N^{-1/p_k}\right)^{q_n} < \infty$ *for some* $N \in I\!N \setminus \{1\}$

(4.2) $\begin{cases} \text{for each } n \in I\!N \text{ there is } N_n \in I\!N \setminus \{1\} \text{ such that} \\ \sup_k |r^A_{nk}| \sum_{j=1}^{k-1} N_n^{-1/p_j} < \infty \end{cases}$

(5.): *(5.1), (5.2), (4.2), (2.2) where*
(5.1) $\lim_{n\to\infty} |r^A_{nk}|^{q_n} = 0$ *for all* k
(5.2) $\lim_{M\to\infty} \sup_n (\sum_{k=1}^{\infty} |r^A_{nk}| M^{-1/p_k})^{q_n} = 0$

(6.): *(6.1), (6.2), (6.3), (4.2), (3.3) where*
(6.1) $\sup_n \sum_{k=1}^{\infty} |r^A_{nk}| N^{-1/p_k} < \infty$ *for some* $N \in I\!N \setminus \{1\}$
(6.2) $\begin{cases} \text{there is a sequence } (\alpha_k)_{k=1}^{\infty} \text{ such that} \\ \lim_{M\to\infty} \sup_n (\sum_{k=1}^{\infty} |r^A_{nk} - \alpha_k| M^{-1/p_k})^{q_n} = 0 \end{cases}$
(6.3) $\begin{cases} \text{there is a sequence } (\beta_k)_{k=1}^{\infty} \text{ such that} \\ \lim_{n\to\infty} |r^A_{nk} - \beta_k|^{q_n} = 0 \text{ for all } k \end{cases}$

(7.): *(7.1), (4.1), (4.2), (1.3) where*
(7.1) $\sup_n |\sum_{k=1}^{\infty} k a_{nk}|^{q_n} < \infty$

(8.): *(8.1), (5.1), (5.2), (4.2), (2.2) where*
(8.1) $\lim_{n\to\infty} |\sum_{k=1}^{\infty} k a_{nk}|^{q_n} = 0$

(9.): *(9.1), (6.1), (6.2), (6.3), (4.2), (3.3) where*
(9.1) $\lim_{n\to\infty} |\sum_{k=1}^{\infty} k a_{nk} - \alpha|^{q_n} = 0$ *for some* $\alpha \in \mathbb{C}$

(10.): *(10.1), (10.2), (10.3), (10.4), (10.5) where*
(10.1) $\sup_n (\sum_{k=1}^{\infty} |r^B_{nk}| N^{1/p_k})^{q_n} < \infty$ *for all* $N \in I\!N \setminus \{1\}$
(10.2) $\begin{cases} \sup_k |r^B_{nk}| \sum_{j=1}^{k-1} N^{1/p_j} < \infty \\ \text{for all } N \in I\!N \setminus \{1\} \text{ and for all } n = 1, 2, \dots \end{cases}$
(10.3) $\sup_n |a_{n1} - a_{n+1,1}|^{q_n} < \infty$
(10.4) $\sum_{k=1}^{\infty} |r^A_{1k}| N^{1/p_k} < \infty$ *for all* $N \in I\!N \setminus \{1\}$
(10.5) $\sup_k |r^A_{1k}| \sum_{j=1}^{k-1} N^{1/p_j} < \infty$ *for all* $N \in I\!N \setminus \{1\}$

(11.): *(11.1), (10.2), (11.2), (10.4), (10.5) where*
(11.1) $\lim_{n\to\infty} (\sum_{k=1}^{\infty} |r^B_{nk}| N^{1/p_k})^{q_n} = 0$ *for all* $N \in I\!N \setminus \{1\}$
(11.2) $\lim_{n\to\infty} |a_{n1} - a_{n+1,1}|^{q_n} = 0$

(12.): *(12.1), (12.2), (10.2), (12.3), (10.4), (10.5) where*
(12.1) $\sup_n \sum_{k=0}^{\infty} |r^B_{nk}| N^{1/p_k} < \infty$ *for all* $N \in I\!N \setminus \{1\}$
(12.2) $\begin{cases} \text{there is a sequence } (\alpha_k)_{k=1}^{\infty} \text{ such that} \\ \lim_{n\to\infty} (\sum_{k=1}^{\infty} |r^B_{nk} - \alpha_k| N^{1/p_k})^{q_n} = 0 \\ \text{for all } N \in I\!N \setminus \{1\} \end{cases}$
(12.3) $\lim_{n\to\infty} |a_{n1} - a_{n+1,1} - \alpha|^{q_n} = 0$ *for some* $\alpha \in \mathbb{C}$

(13.): *(13.1), (13.2), (10.3), (13.3), (13.4) where*
(13.1) $\sup_n (\sum_{k=1}^{\infty} |r^B_{nk}| N^{-1/p_k})^{q_n} < \infty$ *for some* $N \in I\!N \setminus \{1\}$
(13.2) $\begin{cases} \text{for each } n \in I\!N \text{ there is } N_n \in I\!N \setminus \{1\} \text{ such that} \\ \sup_k |r^B_{nk}| \sum_{j=1}^{k-1} N_n^{-1/p_j} < \infty \end{cases}$
(13.3) $\sum_{k=1}^{\infty} |r^A_{1k}| N^{-1/p_k} < \infty$ *for some* $N \in I\!N \setminus \{1\}$
(13.4) $\sup_k |r^A_{1k}| \sum_{j=1}^{k-1} N^{-1/p_j} < \infty$ *for some* $N \in I\!N \setminus \{1\}$

(14.): *(14.1), (14.2), (13.2), (11.2), (13.3), (13.4) where*
(14.1) $\lim_{n\to\infty} |r^B_{nk}|^{q_n} = 0$ *for all* k
(14.2) $\lim_{M\to\infty} \sup_n (\sum_{k=1}^{\infty} |r^B_{nk}| M^{-1/p_k})^{q_n} = 0$

(15.): *(15.1), (15.2), (15.3), (13.2), (12.3), (13.3), (13.4) where*
(15.1) $\sup_n \sum_{k=1}^{\infty} |r^B_{nk}| N^{-1/p_k} < \infty$ *for some* $N \in I\!N \setminus \{1\}$
(15.2) $\begin{cases} \text{there is a sequence } (\alpha_k)_{k=1}^{\infty} \text{ such that} \\ \lim_{M\to\infty} \sup_n (\sum_{k=1}^{\infty} |r^B_{nk} - \alpha_k| M^{-1/p_k})^{q_n} = 0 \end{cases}$
(15.3) $\begin{cases} \text{there is a sequence } (\beta_k)_{k=1}^{\infty} \text{ such that} \\ \lim_{n\to\infty} |r^B_{nk} - \beta_k|^{q_n} = 0 \text{ for all } k \end{cases}$

(16.): *(16.1), (13.1), (13.2), (10.3), (13.3), (13.4), (16.2) where*
(16.1) $\sup_n |\sum_{k=1}^{\infty} k(a_{nk} - a_{n+1,k})|^{q_n} < 0$
(16.2) $\sum_{k=1}^{\infty} ka_{1k}$ *converges*

(17.): *((17.1), (14.1), (14.2), (13.2), (11.2),*
(13.3), (13.4), (16.2) where
(17.1) $\lim_{n\to\infty} |\sum_{k=1}^{\infty} k(a_{nk} - a_{n+1,k})|^{q_n} = 0$

(18.): *(18.1), (15.1),(15.2), (15.3), (13.2), (12.3),*
(13.3), (13.4), (16.2) where
(18.1) $\lim_{n\to\infty} |\sum_{k=1}^{\infty} k(a_{nk} - a_{n+1,k}) - \alpha|^{q_n} = 0$ *for some* $\alpha \in \mathbb{C}$

Proof. We apply Theorem 2, [2, Theorem 5.1] and Remark 1 to obtain the conditions in **(1.)** to **(9.)**. By Theorem 2 (b), we have to add condition (16) of Theorem 2 which is (7.1), (8.1) or (9.1) to the conditions in **(4.)**, **(5.)** of **(6.)**, respectively. Condition (19) in Remark 1 is (1.3) in **(1.)** and **(4.)**, (2.2) in **(2.)** and **(5.)** or (3.3) in **(3.)** and **(6.)**; condition (18) in Theorem 2 is (1.2) in **(1.)**, **(2.)** and **(3.)**; condition (14) in Theorem 2 is (4.2) in **(4.)**, **(5.)** and **(6.)**. The conditions for $R^A \in (\ell_\infty(p), Y)$ for $Y = \ell_\infty(q), c_0(q), c(q)$ are given in [2, Theorem 5.1, (15), (7), (11)] and those for $R^A \in (c_0(p), Y)$ for $Y = \ell_\infty(q), c_0(q), c(q)$ are given in [2, Theorem 5.1, (13), (5), (9)].
By Theorem 3, we have to add condition (20) in Theorem 3 in **(10.)** to **(18.)** which is (10.4) and (10.5) in **(10.)**, **(11.)** and **(12.)**, (13.3) and (13.4) in **(13.)**, **(14.)** and **(15.)** and (13.3), (13.4) and (16.2) in **(16.)**, **(17.)** and **(18.)** . Furthermore, we have to replace r_{nk}^A and a_{nk} in the conditions in **(1.)** to **(9.)** by r_{nk}^B and b_{nk} in the respective ones in **(10.)** to **(18.)**. ■

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