# On a modification of the AOR method

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**Abstract**

In this paper[1] a modification of the Accelerated Overrelaxation (AOR) method is investigated. Some improvements of the results in [2] are presented.

## 1 Introduction

We consider a system of linear equations

$$Ax = b,$$

where $A = [a_{ij}] \in \mathcal{R}^{n,n}$ is a nonsingular matrix with nonzero diagonal entries, and $x, b \in \mathcal{R}^n$. From now on, without losing generality, we can suppose that $a_{ii} = 1$, where $i = 1, \ldots, n$. Let $A = I - L - U$ be the decomposition of $A$ into its diagonal, strictly lower i strictly upper triangular parts, respectively, and let $\omega, \sigma \in \mathcal{R}, \omega \neq 0$. The associated AOR method has the form as in [1]

$$x_{\nu+1} = M_{\sigma,\omega} x_\nu + d, \qquad x_0 \in \mathcal{R}^n,$$

where $M_{\sigma,\omega} = (I - \sigma L)^{-1}((1-\omega)I + (\omega - \sigma)L + \omega U)$, and $d = \omega(I - \sigma L)^{-1} b$. In the paper [2] the AOR method was combined with the method of averaging functional corrections (AFC), [4], [5]. The AFC method has the form

$$x_{\nu+1} = M(x_\nu + y_\nu) + f,$$

where $y_\nu = s_\nu [1, 1, \ldots, 1]^T \in \mathcal{R}^n, s_\nu = \frac{1}{n}\sum_{i=1}^n (x_i^{(\nu+1)} - x_i^{(\nu)})$. In the paper [4] is shown that the AFC method has an equivalent form:

**Algorithm.**

Step 0. Calculate $a = \sum_{i=1}^n \sum_{j=1}^n a_{ij}$.

Step 1. If $n \leq a$ stop.

Step 2. Choose $x_0$.

Step 3. Calculate $s_0 = \frac{1}{n-a}\sum_{i=1}^n f_i$.

Step 4. Calculate $x_{\nu+1}$.

Step 5. Calculate $s_{\nu+1} = \frac{1}{n-a}\sum_{i=1}^n a_{ij}(x_j^{(\nu+1)} - x_j^{(\nu)} - s_\nu)$.

Step 6. $\nu := \nu + 1$ and return to Step 4.

If $M = M_{\sigma,\omega}, f = d$, then we obtain the AOR+AFC method which was introduced in [2].

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## 2 The convergence of the AOR+AFC method

In the further we need the following result from [3].

**Theorem 1** *Let* $M \geq 0$, $|||M|||_1 \leq 1$, *and* $|||M|||_\infty < 1$. *Then the AFC method converges for any* $x_0 \in \mathcal{R}^n$.

Now, we give the following result concerning to the convergence of AOR + AFC method for the system $Ax = b$.

**Theorem 2** *Let* $A = [a_{ij}] \in \mathcal{R}^{n,n}$ *be a nonsingular matrix,* $a_{ii} = 1$, $a_{ij} \leq 0$ *for* $i \neq j$,

$$max(|||L+U|||_1, |||L+U|||_\infty) < \omega \leq 1,$$

$$0 \leq \sigma < min\left(\omega, \frac{\omega - |||L+U|||_\infty}{|||L|||_\infty}, \frac{\omega - |||L+U|||_1}{|||L|||_1}\right).$$

*Then converges the AOR+AFC method for any* $x_0 \in \mathcal{R}^n$.

**Proof.** Since $|||\sigma L|||_\infty \leq |||L|||_\infty \leq |||L+U|||_\infty < 1$ we have

$$(I - \sigma L)^{-1} = \sum_{i=0}^{n-1} (\sigma L)^i \geq 0.$$

From the conditions of the Theorem 2 it follows

$$(1-\omega)I + (\omega - \sigma)L + \omega U \geq 0.$$

So, $M_{\sigma,\omega} \geq 0$. It is easy to see that

$$\begin{aligned} |||M_{\sigma,\omega}|||_\infty &\leq \frac{|||(1-\omega)I + (\omega-\sigma)L + \omega U|||_\infty}{1 - \sigma|||L|||_\infty} \\ &= \frac{max_i\left(1 - \omega + (\omega - \sigma)\sum_{j=1}^{i-1}|a_{ij}| + \omega\sum_{j=i+1}^{n}|a_{ij}|\right)}{1-\sigma|||L|||_\infty} \\ &= \frac{1 - \omega + |||L+U|||_\infty}{1 - \sigma|||L|||_\infty} \end{aligned}$$

Analogously, we have $|||M_{\sigma,\omega}|||_1 < 1$. ∎

**Theorem 3** *Let* $A = [a_{ij}] \in \mathcal{R}^{n,n}$ *be a nonsingular matrix,* $a_{ij} \leq 0$ *for* $i \neq j$, $a_{ii} = 1$, $max(|||L|||_\infty + |||U|||_\infty, |||L|||_1 + |||U|||_1) < 1$, $1 \leq \sigma \leq \omega \leq 1$, $\omega \neq 0$. *Then converges the AOR+AFC method for any* $x_0 \in \mathcal{R}^n$.

**Proof.** In the same way as in Theorem 2 we have $M_{\sigma,\omega} \geq 0$. Now

$$|||M_{\sigma,\omega}|||_\infty \leq \frac{1 - \sigma|||L|||_\infty + \omega(|||L|||_\infty + |||U|||_\infty - 1)}{1 - \sigma|||L|||_\infty} < 1.$$

Analogously $|||M_{\sigma,\omega}|||_1 < 1$. ∎

Now we compare our Theorem 3 with the Theorem 3 form [2]. namely, in Theorem 3 [2] the following conditions for $|||L|||_\infty = l$, and $|||U|||_\infty = u$ are given:

$$D_1 = \{(u,l) : l + u < \frac{1}{3}, l \geq 0, u \geq 0\}$$

$$D_2 = \{(u,l) : l + 3u < 1, l \geq 0, u \geq 0\}.$$

For our Theorem 3 we have

$$D_3 = \{(u,l) : l + u < 1, l \geq 0, u \geq 0\}.$$

We conclude that $D_1 \subset D_2 \subset D_3$, i.e. the area of convergence in $lu$ plane for our Theorem 3 is bigger than ones in Theorem 3 from [2].

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