On the investigation of separability of a bipartite system observable

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Abstract

We point out and discuss[1] some subtleties of the recently proposed method for investigating the so-called separability of an observable of a bipartite quantum system. The relevance of the topic for the decoherence theory has also been made.

1 Introduction

The last two decades of the 20th century characterizes significant increase of interest in some fundamental issues of quantum mechanics, such as quantum measurement, open quantum systems, the problem of irreversibility, the problem of the "transition from quantum to classical", quantum mechanical behavior of the macroscopic systems, and so on; cf. e.g., Cvitanović et al 1991, Grigolini 1993, Wheeler and Zurek 1982, Zurek 1982, 1991, 1993, Giulini et al 1996, Leggett 1980, Leggett and Garg 1985. For most of these topics, the issue of *decoherence* is of the central interest. Actually, it is sometimes claimed (Giulini et al 1996, Zurek 1991) that the decoherence theory gives a background for setting the solutions to most of the above mentioned problems.

Recently, existence of the necessary conditions for the occurrence of decoherence has been pointed out (Dugić 1996, 1997), and the so-called *separability* of the interaction Hamiltonian (of the composite system "(open) system plus the environment") has been stressed. To this end, a method for investigating the separability has been formulated (Dugić 1997). However, for most of the operational tasks in the decoherence theory, the investigation of the separability due to the method bears some subtleties. So, here, we point out some subtleties of the method as a background of the general investigations of the effect of decoherence.

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2 On decoherence

The effect of decoherence is *not identical* with: quantum measurement(s), quantum dissipation, quantum difusion, or with the "dephasing" effect. By *decoherence*, it is usually assumed the so-called *decoherence-induced decoherence*, i.e. the occurrence of the effective superselection rules for an open quantum system. To this end, usually, by decoherence one means destruction of the initial coherence in the course of interaction of an open system S with its environment E, as presented by:

$$\sum_n C_n|\phi_n|_S \to \hat{\rho}_S = \sum_n |C_n|^2|\phi_n\rangle_{S\,S}\langle\phi_n|, \tag{1}$$

where $\hat{\rho}_S$ represents the subsystem's density matrix. While this definition emphasizing the DEcoherence (OFFcoherence) for the open system S is correct, it is by far incomplete. Therefore, we shall give the precise definition of decoherence.

Let us first define the main concepts.

A bipartite quantum system $S + E$ is defined by the Hamiltonian:

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{int}, \tag{2}$$

where $\hat{H}_{int}$ represents the interaction Hamiltonian. Assuming universal validity of the Schrodinger law for the composite system $S + E$, one defines the subsystem's (S's) density matrix as:

$$\hat{\rho}_S = tr_E \hat{U}(t)\hat{\rho}_{SE}(t=0)\hat{U}^\dagger(t), \tag{3}$$

where $\hat{U}(t)$ represents the unitary in time evolution operator for the composite system. Then the decoherence effect is defined as follows.

Definition 1 The effect of decoherence is defined by the following conditions for the off-diagonal elements of the subsystem's S's density matrix:

$$\lim_{t\to\infty} \rho_{mn}(t) = 0, m \neq n \tag{4}$$

while the time averages satisfy

$$\lim_{t\to\infty} \langle\rho_{mn}\rangle_t = 0, \tag{5}$$

and the standard deviation of the off-diagonal elements satisfies:

$$\Delta_t \rho_{mn} \propto N^{-1/2}, \tag{6}$$

where Δ_t denotes the standard deviation over the time interval t, and N is a real number proportional to the number of particles in the environment E, *as well* *as* the "robustness" of some states of the open system, relative to the influence of the environment as presented by:

$$\hat{U}(t)|\phi_m\rangle_S|\chi\rangle_E = |\phi_n\rangle_S|\chi(t)\rangle_E. \tag{7}$$

Physically, the above definition stems that, in an open system there may occur the effective superselection rules (Zurek 1982). Actually, the above indices (m, n) determine a decomposition of the Hilbert state space for S:

$$H^{(S)} = \sum_p H^{(p)}, \tag{8}$$

where the states belonging to the different superselection sectors $H^{(p)}$ satisfy the conditions of Definition 1. So, effectively, these superselection sectors appear as the carriers of the effective, approximate *classical behavior* of an open system: certain coherent superpositions are forbidden. Similarly, the expression (4d) distinguishes the *robustness* of the states $|\phi_n\rangle_S$; the orthonormalized basis $\{|\phi_n\rangle_S\}$ adapted to the decomposition (5) is referred to as the *"pointer basis"*. The robustness of classical states is a main characteristic of the classical (macroscopic) systems. So, sometimes, the decoherence effect is considered as a basis for setting a solution to the famous problem of the "transition from quantum to classical" (cf., e.g., Zurek 1991, Grigolini 1993, Giulini et al 1986).

However, recently, it was pointed out existence of the (effective) necessary conditions for the occurrence of decoherence in nonrelativistic quantum mechanics–cf. Appendix I. The deeper physical meaning of these conditions can be found in Dugić 1997 [3], [4]. Here, we want briefly to present some subtleties of the method that might be of interest in some operational tasks of the decoherence theory.

3 The method

Definition 2 A composite-system's observable $\hat{A}_{12}$ is of the separable kind if any of the following mutually equivalent conditions is fulfilled:

(i) there exists an orthonormalized basis in $H^{(S)}$ which diagonalizes the observable, *and* there is an orthonormalized basis in $H^{(E)}$ that diagonalizes the observable; $H^{(S)}$ and $H^{(E)}$ represent the Hilbert state spaces of S, and of E, respectively;

(ii) the observable is diagonalizable in a *noncorrelated basis* (tensor product states) of the Hilbert state space of the composite system $S + E$;

(iii) the observable can be given the following spectral form:

$$\hat{A}_{12} = \sum_{p.q} \gamma_{pq} \hat{P}_{Sp} \otimes \hat{\Pi}_{Eq}, \tag{9}$$

where $\hat{P}_{Sp}$ and $\hat{\Pi}_{Eq}$ represent the orthogonal projectors for S, and for E, respectively;

(iv) for a particular form of the observable of the type:

$$\hat{A}_{12} = \sum_k \hat{C}_{Sk} \otimes \hat{D}_{Ek}, \tag{10}$$

where both sets of the observables on the r.h.s. of (6) bear linear independence, one may state the compatibilities:

$$[\hat{C}_{Sk}, \hat{C}_{Sk'}] = 0, \forall k, k', [\hat{D}_{Ek}, \hat{D}_{Ek'}] = 0, \forall k, k'. \tag{11}$$

Equivalence of the points (i)-(iii) is rather obvious. On the other side, equivalence of, e.g., point (iv) with the point (i) has been proved in Dugić [3]. A method has also been developed for rewriting arbitrary form of a composite-system observable into a particular form of the type (6). Whilst there might be many different forms of an observable of the type (6), the compatibilities (7a,b) guarantee the separability of the observable. If there is noncommutativity for any of the pairs of the observables, we say the observable is of the nonseparable kind. The interaction Hamiltonian is a typical observable of a bipartite quantum system.

In presenting the method of Dugić it is convenient to use a few simple examples.

Example 1 Let us consider an observable of a pair of the one-dimensional particles defined as follows:

$$\hat{A}_{12} = C_1 \hat{x}_1 \otimes \hat{X}_2 + C_2 \hat{x}_1 \otimes \hat{P}_2 + C_3 \hat{p}_1 \otimes \hat{X}_2 + C_4 \hat{p}_1 \otimes \hat{P}_2. \tag{12}$$

Now, if one may write:

$$C_1 C_4 = C_2 C_3, \tag{13}$$

the observable can be re-written in a trivial-linear form of the type (6):

$$\hat{A}_{12} = \hat{C}_{11} \otimes \hat{D}_{21}, \tag{14}$$

where $C_{11} = (C_2/C_3)\hat{x}_1 + \hat{p}_1$, and $\hat{D}_{21} = C_3 \hat{X}_2 + C_4 \hat{P}_2$. Needless to say, the observable is of the separable kind.

Example 2 As in the Example 1, except that $C_4 = 0$:

$$\hat{A}'_{12} = C_1 \hat{x}_1 \otimes \hat{X}_2 + C_2 \hat{x}_1 \otimes \hat{P}_2 + C_3 \hat{p}_1 \otimes \hat{X}_2. \tag{15}$$

Now, having in mind the Example 1, one can proceed as follows: to add and subtract the missing term, with properly chosen constant, and then to apply the result of the preceding example. Actually, one can write:

$$\hat{A}'_{12} = \hat{A}'_{12} + (C_2 C_3/C_4)\hat{p}_1 \otimes \hat{P}_2 = \hat{C}'_{11} \otimes \hat{D}'_{21} + \hat{C}'_{12} \otimes \hat{D}'_{22}, \tag{16}$$

where $C'_{11} = \hat{x}_1 + (C_3/C_1\hat{p}_1)$, $\hat{C}'_{21} = (C_2C_3/C_4)\hat{p}_1$, and $\hat{D}_{21} = C_1\hat{X}_2 + C_2\hat{P}_2$ and $\hat{D}'_{22} = \hat{P}_2$.

The question of separability of the observable on the r.h.s. of (12) will be discussed below. In this section we want to emphasize that the Example 2 brings the main idea of the method.

Actually, for arbitrary form of the observable:

$$\hat{A}_{12} = \sum_{i,j} \sum_{m_i, n_j} C_{m_i n_j} \hat{A}_{1i}^{m_i} \otimes \hat{B}_{2j}^{n_j}, \tag{17}$$

one can choose a subsum of the original sum (13), and to test if the equalities of the type (9) are fulfilled. Then, if one may write:

$$C_{m_i n_j} C_{m'_{i'} n'_{j'}} = C_{m i n'_{j'}} C_{m'_{i'} n_j}, \tag{18}$$

the observable can be written in a trivial-linear form of the type (11). However, if this is not the case, one should choose a subsum of the original sum (13), and then, in analogy with the Example 2, to add and subtract the "missing" terms. Then the added terms plus the chosen subsum sum up to a trivial-linear form. For the rest of the original sum, one should apply the same procedure, and then to eliminate the linear dependence in either both sets of the observables. And the whole procedure should be applied until there would not be linear dependence, as well as that one cannot choose a subsum. As to the later, the criterion for choosing a subsum is the so-called "convergency criterion":

$$M < n - 1, \tag{19}$$

where n is the number of elements in the chosen subsum, while M represents the number of the added (and also subtracted) terms. So, the first stage of the method–choosing the subsum, adding (and subtracting) the terms–breaks before $M = n - 1$.

This procedure guarantees that the number of the terms in the (final) sum is smaller than in the original sum. For some special cases, the method is applicable even for the infinite sums (13)–cf. Dugić 1998. Then, in testing the (non)separability, there remains to test the (non)compatibilities (7a,b). However, for most models, the application of the method might bear some subtleties, which is the matter of the next section.

4 Some subtleties of the method

First, if an observable is given in a form in which all the observables (of both S and E) mutually commute, one needs not apply the method–due to Definition 2(iv), the observable is of the separable kind. Certainly, if the observable can be rewritten in a form bearing incompatibilities, there is linear dependence in either set of the observables; then, eliminating the linear dependencies, one would obtain complete commutativity in both sets of the observables.

Second, if the "condition of converegency" is not fulfilled, the application of the method leads to increase in the number of terms in the sum represented the observable. So, the choice of the subsum (cf. above) is two-fold: it determines the efficacy of the method as well as the final form of the observable. Certainly, the "final" form of the observable is not unique. However, as long as the assumptions of Definition 2(iv) are fulfilled, the conclusion about the (non)separability of the observable is unique.

Third, from the operational point of view, the choice of the subsum should be guided by the requirement of as bigger the number n as possible, with as smaller the number M as possible. Then the resulting number of terms in the sum is smaller, and eventually (if at all) might lead to the trivial-linear form of the observable. Needless to say, the choice of the subsum(s) is partly a matter of experience, and a general advice in this regard can hardly be formulated (but see Dugić 1998 for some examples).

Fourth, as regards the occurrence of decoherence, the two should be identified (Zurek 1982): exact and only approximate separability (of the Hamiltonian). The *exact* separability is defined by Definition 2. However, the *approximate* separability is defined by the following expression:

$$\hat{H}_{int} = \hat{H}^{(sep)} + \hat{h}^{(nonsep)}, \tag{20}$$

where the first term on the r.h.s. of (16) is of the separable kind, while the second term is of the nonseparable kind, while:

$$\|\hat{H}^{(sep)}\| \gg \|\hat{h}^{(nonsep)}\|. \tag{21}$$

So, *physically*, purely mathematical conclusions on the separability need not be useful. To this end, we give the following example.

Let us back to the above Example 1, and let us assume that the equality (9) is not fulfilled. Then, the straightforward application of Definition 2(iv) leads to the conclusion that the observable is of the nonseparable kind. And this is correct as regards the non-correlated basis that are the tensor products of the unnormalizable states–i.e. of the eigenstates of the "continuous observables". However, the opposite conclusion follows in the case of the normalizable states; i.e. the observable is of the separable kind in respect to the noncorrelated bases consisting of the normalizable states. This can be proved as follows.

As it was shown by von Neumann (von Neumann 1955, Sec. 5.4), the pairs of the conjugate observables can be approximated by mutually commuting observables with the fully discrete spectra. The error of these approximations (i.e. of the "coarse graining") is proportional to the Planck constant. That is, instead of the pair $(\hat{x}, .\hat{p})$, one has the pair $(\hat{\xi}, \hat{\pi})$, such that:

$$\|(\hat{x} - \hat{\xi})\Psi_{\mu\nu}\| < C\epsilon, \|(\hat{p} - \hat{\pi})\Psi_{\mu\nu}\| < C\eta, \tag{22}$$

where

$$\epsilon\eta = \hbar/2. \tag{23}$$

Now, inclusion of these approximations in (8) gives:

$$\hat{A}_{12} = C_1\hat{\xi}_1 \otimes \hat{\Xi}_2 + C_2\hat{\xi}_1 \otimes \hat{\Pi}_2 + C_3\hat{\pi}_1 \otimes \hat{\Xi}_2 + C_4\hat{\pi}_1 \otimes \hat{\Pi}_2 + \hat{h}, \tag{24}$$

where appears the small term $\hat{h}$. Now, the observable $\hat{A}_{12}$ is of the approximately separable kind, and the occurrence of decoherence is not forbidden–the pointer basis consists of the common eigenstates of $\hat{\xi}_1$ and π_1 which follow from the orthonormalization procedure for the set of the "coherent states".

So, the concept of separability, although mathematically quite clear, physically distinguishes between the unnormalizable and normalizable states, while not distinguishing between the exact and only approximate separability. So, one should be careful in dealing with the separability in the context of the decoherence theory.

5 Conclusion

We point out the relevance of the concept of separability of a composite-system observable in the context of the decoherence theory. To this end, we point out the relevance of the task of investigating the spearability of a composite-system observable (e.g. of the interaction Hamiltonian). So, our main task, here, is to present the subtleties of a method for investigating the separability as a technical, mathematical task of wide interest in the operational tasks of the decoherence theory. This is achieved by a brief overview of the method, and with the pointing out the main subtleties that are of relevance in the technical dealing with the composite-systems' observables.

6 Appendix

The logical structure of the proof of the separability of $\hat{H}_{int}$ as a necessary condition for the occurrence of decoherence is as follows. We prove the following implications:

(a) "(4d)" $\Rightarrow$ "diagonalizability of $\hat{H}_{int}$ in $H^{(S)}$",

(b) "nondiagonalizability of $\hat{H}_{int}$ in $H^{(E)}$" $\Rightarrow$ "nonvalidity of the expressions (4a-c)",

Since the point (b) is logically equivalent with:

(c) "validity of (4a-c)" $\Rightarrow$ "diagonalizability of $\hat{H}_{int}$ in $H^{(E)}$",

the following implication is proved:

"Definition 1" $\Rightarrow$ "Definition 2".

It is worth emphasizing: the implication (b) is not exact. Actually, it proves exact in the context of the Zurek's theory, but not necessarily in a hypothetical wider (more general) theory. To this end, the exceptions from the point (b) have not been disproved–rather, existence of the exceptions from (b) have not been proved to exist. So, we deal with the separability as with an *effective* necessary condition for the occurrence of decoherence. In a more general theory (not known yet), the exceptions from (b) might occur. However, it seems that such exceptions–if exist at all–can hardly be considered physically relevant, as pointed out in Dugić 1998.

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