p-adic Feynman's path integrals

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Abstract

The Feynman path integral method plays even more important role in p-adic and adelic quantum mechanics than in ordinary quantum theory. p-Adic path integral is defined as a natural generalization of the standard one. Here[1], we give a brief review of our recent investigations on this subject. In particular, the path integrals for two-dimensional models with quadratic Lagrangians are evaluated.

1 Introduction

The invention of the path integral [8] is one of the major achievements in theoretical physics. Originally developed as a space-time approach to non-relativistic quantum mechanics, Feynman's path integrals became very soon of great importance in quantum electrodynamics. Presently, it is very useful tool and sometimes inevitable ingredient of many modern physical theories (as superstring theory and quantum cosmology).

During the last 15 years applications of p-adic numbers and adeles have attracted a significant interest, mainly in mathematical and high energy physics devoted to the Planck scale processes and very early universe [1, 11]. There is a quite common belief that the usual picture of space-time as a smooth pseudo-Riemannian manifold should be essentially changed at the Planck scale. Besides an idea of noncommutativity [2], nonarchimedean space-time is also an attractive mathematical background for the fundamental physical theory.

There is not p-adic Schrödinger equation. Nevertheless, p-adic generalization of the Feynman path integration is possible [12, 4], and for one-dimensional quadratic systems the corresponding propagator is completely determined [5]. During the last few years, besides a few standard and very interesting one-dimensional models [6], some multidimensional systems have been also treated by the same technique [7].

The main aim of this paper is to examine two-dimensional p-adic path integrals and to find the corresponding propagator for the models with quadratic actions. We restrict ourselves to the systems which classical trajectories are

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represented by analytic functions. During calculations we will see that three- or higher-dimensional generalization at least in principle is possible but, as in the real case [9], it is mainly a problem of increasing computational complexity.

2 p-adic numbers and related analysis

Let us recall that all numerical experimental results belong to the field of rational numbers Q. The completion of this field with respect to the standard norm $|\ |_\infty$ (absolute value) leads to the field of real numbers $R = Q_\infty$. According to the Ostrowski theorem, each non-trivial norm (valuation) on Q is equivalent either to a p-adic norm (p is a prime number) $|\ |_p$ or to the absolute value function. Completion of Q with respect to the p-adic norms yields the fields of p-adic numbers Q_p.

Any p-adic number $x \in Q_p$ can be presented as an expansion [11]

$$x = x^\nu(x_0 + x_1 p + x_2 p^2 + \cdots), \quad \nu \in Z, \tag{1}$$

where $x_i = 0, 1, ..., p-1$. p-Adic norm of any term $x_i p^{\nu+i}$ in (1) is $p^{-(\nu+i)}$. The p-adic norm is the nonarchimedean (ultrametric) one, i.e. $|x+y|_p \leq max\{|x|_p, |y|_p\}$ and, as a consequence, there are a lot of exotic features of p-adic spaces. For example, any point of a disc $B_\nu(a) = \{x \in Q_p : |x-a|_p \leq p^\nu\}$ can be treated as its center. It also leads to the total disconnectedness of p-adic spaces.

There is no natural ordering on Q_p, but one can define a linear order as follows: $x < y$ if $|x|_p < |y|_p$, or when $|x|_p = |y|_p$, there exists an index $m \geq 0$ such that following is satisfied: $x_0 = y_0, x_1 = y_1, \cdots, x_{m-1} = y_{m-1}, x_m < y_m$. Generally speaking, there are mainly two analyses over Q_p. One of them is connected with map $\phi : Q_p \to Q_p$, and the second one is related to the map $\psi : Q_p \to C$.

In the case of p-adic valued function, derivatives of $\phi(x)$ are defined as in the real case, but using p-adic norm instead of the absolute value. p-Adic valued definite integrals are defined for analytic functions

$$\phi(t) = \sum_{n=0}^{\infty} \phi_n t^n, \qquad \phi_n, t \in Q_p, \tag{2}$$

as follows:

$$\int_a^b \phi(t)dt = \sum_{n=0}^{\infty} \frac{\phi_n}{n+1}(b^{n+1} - a^{n+1}). \tag{3}$$

In the case of mapping $Q_p \to C$, standard derivatives are not possible, and some types of pseudodifferential operators have been introduced [11, 3]. However, there is a well defined integral with the Haar measure. Of a special importance is Gauss integral

$$\int_{Q_p} \chi_p(\alpha x^2 + \beta x)dx = \lambda_p(\alpha)|2\alpha|_p^{-1/2}\chi_p\left(-\frac{\beta^2}{4\alpha}\right), \ \ \alpha \neq 0, \tag{4}$$

where $\chi_p(u) = \exp(2\pi i\{u\}_p)$ is a p-adic additive character, and $\{u\}_p$ denotes the fractional part of $u \in Q_p$. $\lambda_v(\alpha)$ is an arithmetic complex-valued function

$$\lambda_p(0) = 1, \tag{5}$$

$$\lambda_p(x) = \begin{cases} 1, & \nu = 2k \quad , \quad p \neq 2, \\ \left(\frac{x_0}{p}\right), & \nu = 2k+1, \quad p \equiv 1(\text{mod } 4), \\ i\left(\frac{x_0}{p}\right) & \nu = 2k+1, \quad p \equiv 3(\text{mod } 4), \end{cases} \tag{6}$$

$$\lambda_2(x) = \begin{cases} \frac{1}{\sqrt{2}}[1+(-1)^{x_1}i], & \nu = 2k \ , \\ \frac{1}{\sqrt{2}}(-1)^{x_1+x_2}[1+(-1)^{x_1}i], & \nu = 2k+1, \end{cases} \tag{7}$$

with the following basic properties:

$$\lambda_p(a^2\alpha) = \lambda_p(\alpha), \ \lambda_p(\alpha)\lambda_p(\beta) = \lambda_p(\alpha+\beta)\lambda_p(\alpha^{-1}+\beta^{-1}), \ |\lambda_p(\alpha)|_\infty = 1. \tag{8}$$

3 Path integral in ordinary quantum mechanics

According to Feynman's idea [8], quantum transition from a space-time point (x', t') to another (x'', t'') is a superposition of motions along all possible paths connecting these two points. The corresponding probability amplitude is $\langle x'', t''|x', t'\rangle = \sum_q e^{\frac{2\pi i}{h}S[q]}$, where $S[q]$ is the action along the corresponding trajectory q. Dynamical evolution of any quantum-mechanical system, described by a wave function $\psi(x,t)$, is given by

$$\psi(x'', t'') = \int_{Q_\infty} \mathcal{K}(x'', t''; x', t')\psi(x', t')dx', \tag{9}$$

where $\mathcal{K}(x'', t''; x', t')$ is a kernel of the unitary evolution operator $U(t'', t')$.

In Feynman's formulation of quantum mechanics, the transition amplitude $\mathcal{K}(x'', t''; x', t')$ was postulated to be the path integral

$$\mathcal{K}(x'', t''; x', t') = \int_{(x',t')}^{(x'',t'')} \exp\left(\frac{2\pi i}{h}\int_{t'}^{t''} L(q, \dot{q}, t)dt\right)\mathcal{D}q, \tag{10}$$

where $x'' = q(t'')$ and $x' = q(t')$, and h is the Planck constant. $\mathcal{K}(x'', t''; x', t')$ is also called the quantum-mechanical propagator. One can easily deduce the following three general properties:

$$\mathcal{K}(x'', t''; x', t') = \int_{Q_\infty} \mathcal{K}(x'', t''; x, t)\mathcal{K}(x, t; x', t')dx, \tag{11}$$

$$\int_{Q_\infty} \mathcal{K}^*(x'',t'';x',t')\mathcal{K}(z,t'';x',t')dx' = \delta(x''-z), \tag{12}$$

$$\mathcal{K}(x'',t;x',t) = \delta(x''-x'). \tag{13}$$

For a classical action $\bar{S}(x'',t'';x',t')$, which is a polynomial quadratic in x'' and x', it has been shown [10] that in ordinary one-dimensional quantum mechanics

$$\mathcal{K}(x'',t'';x',t') = \left(\frac{i}{h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right)^{\frac{1}{2}} \exp\left(\frac{2\pi i}{h}\bar{S}(x'',t'';x',t')\right). \tag{14}$$

It can be rewritten in the form

$$\mathcal{K}_\infty(x'',t'';x',t') = \lambda_\infty\left(-\frac{1}{2h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right)\left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right|_\infty^{1/2}\chi_\infty\left(-\frac{1}{h}\bar{S}\right), \tag{15}$$

where $\sqrt{ia} = \sqrt{i\ sign\ a\ |a|_\infty} = |a|_\infty^{1/2}\lambda_\infty(-a)$. In (15), $\chi_\infty(a) = \exp(-2\pi i a)$ is an additive character of the field of real numbers R. D-dimensional generalization of the transition amplitude for a quadratic classical action contains a determinant:

$$\mathcal{K}_\infty(x'',t'';x',t') = \lambda_\infty\left(\det\left(-\frac{1}{2h}\frac{\partial^2\bar{S}}{\partial x''_a\partial x'_b}\right)\right)\left|\det\left(-\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x''_a\partial x'_b}\right)\right|_\infty^{1/2}$$

$$\times\chi_\infty\left(-\frac{1}{h}\bar{S}(x'',t'';x',t')\right), \tag{16}$$

where we defined

$$\lambda_\infty\left(\det\left(-\frac{1}{2h}\frac{\partial^2\bar{S}}{\partial x''_a\partial x'_b}\right)\right) = \sqrt{\frac{1}{i^D} sign\det\left(-\frac{1}{2h}\frac{\partial^2\bar{S}}{\partial x''_a\partial x'_b}\right)}, \tag{17}$$

and $x = (x_a), \quad a = 1,2,\cdots,D$. Defining $\lambda_\infty(0) = 1$ one can easily show that this λ_∞-function satisfies all properties stated for λ_p in (8).

4 Path integral in p-adic quantum mechanics

In p-adic quantum mechanics does not exist dynamical differential equation of the Schrödinger type and p-adic quantum dynamics is defined by the kernel $\mathcal{K}_p(x'',t'';x',t')$ of the evolution operator:

$$\psi_p(x'',t'') = \int_{Q_p}\mathcal{K}_p(x'',t'';x',t')\psi_p(x',t')dx'. \tag{18}$$

We evaluate a general expression for this kernel as in standard quantum mechanics. All general properties which hold for the kernel $\mathcal{K}(x'',t'';x',t')$ in standard quantum mechanics also hold in p-adic case, where integration in (11) and (12) is now over Q_p.

p-Adic generalization of (10) for a harmonic oscillator was done in [12] starting from

$$\mathcal{K}_p(x'',t'';x',t') = \int_{(x',t')}^{(x'',t'')} \chi_p\left(-\frac{1}{h}\int_{t'}^{t''} L(q,\dot{q},t)dt\right)\prod_t dq(t) \tag{19}$$

($h \in Q$ and $q,t \in Q_p$). In (19), $dq(t)$ is the Haar measure and p-adic path integral is the limit of a multiple Haar integral. This approach has been also used in [4].

We extend here our previous one-dimensional investigations of p-adic path integrals [5] to the two-dimensional case

$$\mathcal{K}_p(x'',y'',t'';x',y',t') = \int_{(x',y',t')}^{(x'',y'',t'')} \chi_p\left(-\int_{t'}^{t''} L(q_1,q_2,\dot{q}_1,\dot{q}_2,t)dt\right)\mathcal{D}q_1\mathcal{D}q_2, \tag{20}$$

($h=1$) for a system which Lagrangian is a quadratic polynomial with respect to q_i and $\dot{q}_i,\ i=1,2$. Classical action is related to the classical p-adic trajectory $\bar{q}_i(t)$, i.e.

$$S[\bar{q}_1,\bar{q}_2] = \bar{S}(x'',y'',t'';x',y',t') = \int_{t'}^{t''} L(\bar{q}_1,\bar{q}_2,\dot{\bar{q}}_1,\dot{\bar{q}}_2,t)dt, \tag{21}$$

with

$$x''=\bar{q}_1(t''),\ \ y''=\bar{q}_2(t''),\ \ x'=\bar{q}_1(t'),\ \ y'=\bar{q}_2(t'). \tag{22}$$

We regard any p-adic quantum path $q_i=q_i(t)$ as a deformation of the classical p-adic trajectory: $q_i(t)=\bar{q}_i(t)+h_i(t)$ with conditions $h_i(t'')=h_i(t')=0$.

Using the Taylor expansion of $S[q_1,q_2]$ around the classical path $\bar{q}_i(t)$ (with $\delta S[\bar{q}_1,\bar{q}_2]=0$), we obtain p-adic analogue of the Feynman theorem, namely

$$\begin{gathered}\mathcal{K}_p(x'',y'',t'';x',y',t') = \chi_p\left(-S[\bar{q}_1,\bar{q}_2]\right)\\ \times\int_{(0,0,t')}^{(0,0,t'')} \chi_p\left(-\frac{1}{2}\int_{t'}^{t''}\left(h_1\frac{\partial}{\partial q_1}+h_2\frac{\partial}{\partial q_2}+\dot{h}_1\frac{\partial}{\partial \dot{q}_1}+\dot{h}_2\frac{\partial}{\partial \dot{q}_2}\right)^2 Ldt\right)\mathcal{D}h_1\mathcal{D}h_2.\end{gathered} \tag{23}$$

Since expression of the above integral depends only on t'' and t', we denote it $N_p(t'',t')$, i.e.

$$\mathcal{K}_p(x'',y'',t'';x',y',t') = N_p(t'',t')\chi_p(-\bar{S}(x'',y'',t'';x',y',t')). \tag{24}$$

Applying property (12) to the kernel (24), performing expansion of the classical action around the classical path , integrating over x' and y' and using standard properties of the δ_p function, we obtain

$$|N_p(t'',t')|_\infty = \left|\det\begin{pmatrix} -\frac{\partial^2 \bar{S}}{\partial x' \partial x''} & -\frac{\partial^2 \bar{S}}{\partial x' \partial y''} \\ -\frac{\partial^2 \bar{S}}{\partial y' \partial x''} & -\frac{\partial^2 \bar{S}}{\partial y' \partial y''} \end{pmatrix}\right|_p^{1/2}. \tag{25}$$

Since the factor $N_p(t'',t')$ can be presented as $N_p(t'',t') = A_p(t'',t')|N_p(t'',t')|_\infty$ we have to investigate the form of the factor $A_p(t'',t')$ in two-dimensional case.

The general form of the quadratic Lagrangian for a system with two degrees of freedom can be expressed in the following form:

$$\begin{aligned} &L(q_1,q_2,\dot{q}_1,\dot{q}_2,t) \\ &= L_0 + \frac{\partial L_0}{\partial q_1}q_1 + \frac{\partial L_0}{\partial q_2}q_2 + \frac{\partial L_0}{\partial \dot{q}_1}\dot{q}_1 + \frac{\partial L_0}{\partial \dot{q}_2}\dot{q}_2 + \frac{1}{2}\frac{\partial^2 L_0}{\partial q_1^2}q_1^2 + \frac{\partial^2 L_0}{\partial q_1 \partial \dot{q}_1}q_1\dot{q}_1 \\ &+ \frac{1}{2}\frac{\partial^2 L_0}{\partial \dot{q}_1^2}\dot{q}_1^2 + \frac{1}{2}\frac{\partial^2 L_0}{\partial q_2^2}q_2^2 + \frac{\partial^2 L_0}{\partial q_2 \partial \dot{q}_2}q_2\dot{q}_2 + \frac{1}{2}\frac{\partial^2 L_0}{\partial \dot{q}_2^2}\dot{q}_2^2 \\ &+ \frac{\partial^2 L_0}{\partial q_1 \partial q_2}q_1 q_2 + \frac{\partial^2 L_0}{\partial q_1 \partial \dot{q}_2}q_1\dot{q}_2 + \frac{\partial^2 L_0}{\partial \dot{q}_1 \partial q_2}\dot{q}_1 q_2 + \frac{\partial^2 L_0}{\partial \dot{q}_1 \partial \dot{q}_2}\dot{q}_1\dot{q}_2. \end{aligned} \tag{26}$$

The general solution of the classical equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,\ \ i = 1,2, \tag{27}$$

can be expressed as

$$q_1(t) = \sum_{j=1}^{4} C_i f_i(t) + X(t),\ \ q_2(t) = \sum_{j=1}^{4} C_i g_i(t) + Y(t) \tag{28}$$

where $X(t)$ and $Y(t)$ are two particular solutions of the two coupled inhomogeneous differential equations (27). C_i are four constants of integration and f_i are four linearly independent particular solutions of the homogeneous linear resolvent equation of the fourth order for $q_1(t)$. When solutions f_i are chosen the corresponding g-functions are uniquely determined as a consequence of the coupling between the equations (27). Functions f_i, g_i, X, Y are some analytic functions of time. The classical path is given by

$$x(t) = \sum_{j=1}^{4} C_j(t'',t') f_j(t) + X(t),\ \ y(t) = \sum_{j=1}^{4} C_j(t'',t') g_j(t) + Y(t). \tag{29}$$

Determination of the constants of integration from the conditions $x'' = x(t'')$, $x' = x(t')$, $y'' = y(t'')$, $y' = y(t')$, gives

$$C_j(t'',t') = \frac{1}{\Delta(t'',t')}\left([x' - X(t')]\,\Delta_{1j}(t'',t') + [x'' - X(t'')]\,\Delta_{2j}(t'',t')\right)$$

$$+\frac{1}{\Delta(t'',t')}\left([y'-Y(t')]\,\Delta_{3j}(t'',t')+[y''-Y(t'')]\,\Delta_{4j}(t'',t')\right), \qquad (30)$$

with determinant

$$\Delta(t'',t')=\begin{vmatrix} f_1' & f_2' & f_3' & f_4' \\ f_1'' & f_2'' & f_3'' & f_4'' \\ g_1' & g_2' & g_3' & g_4' \\ g_1'' & g_2'' & g_3'' & g_4'' \end{vmatrix}, \qquad (31)$$

and $\Delta_{jk}(t'',t')$ represents the "algebraic cofactor" of the element in the jth row and kth column of $\Delta(t'',t')$.

Due to the equations of motion (27) and the expansion of the Lagrangian (26), we get

$$\bar{S}(x'',y'',t'';x',y',t')=\int_{t'}^{t''}L(q_1,q_2,\dot{q}_1,\dot{q}_2,t)dt$$

$$=\frac{1}{2}\left[\frac{\partial^2 L_0}{\partial\dot{q}_1^2}\dot{x}x+\frac{\partial^2 L_0}{\partial\dot{q}_1\partial\dot{q}_2}\dot{y}x+\frac{\partial^2 L_0}{\partial q_1\partial\dot{q}_1}x^2+\frac{\partial^2 L_0}{\partial q_2\partial\dot{q}_1}xy\right]_{t'}^{t''}$$

$$+\frac{1}{2}\left[\frac{\partial^2 L_0}{\partial\dot{q}_2^2}\dot{y}y+\frac{\partial^2 L_0}{\partial\dot{q}_1\partial\dot{q}_2}\dot{x}y+\frac{\partial^2 L_0}{\partial q_2\partial\dot{q}_2}y^2+\frac{\partial^2 L_0}{\partial q_1\partial\dot{q}_2}xy\right]_{t'}^{t''}+S_{lin}(x'',t'';x',t'). \qquad (32)$$

After substitution of (29) for $x, y, \dot{x}$ and $\dot{y}$ into (32), we obtain expressions for the partial derivatives of the classical action with respect to its arguments x', y', x'', y''.

We now apply property of the form (11) to the kernel (24) and perform the corresponding Gauss integration. At the end we derive the phase

$$A_p(t'',t')=\lambda_p\left(\det\begin{pmatrix} -\frac{1}{2}\frac{\partial^2\bar{S}}{\partial x'\partial x''} & -\frac{1}{2}\frac{\partial^2\bar{S}}{\partial x'\partial y''} \\ -\frac{1}{2}\frac{\partial^2\bar{S}}{\partial y'\partial x''} & -\frac{1}{2}\frac{\partial^2\bar{S}}{\partial y'\partial y''}\end{pmatrix}\right). \qquad (33)$$

Finally, we obtain that the propagator in two dimensional p-adic case is

$$\mathcal{K}_p(x'',y'',t'';x',y',t')=\lambda_p\left(\det\begin{pmatrix} -\frac{1}{2}\frac{\partial^2\bar{S}}{\partial x'\partial x''} & -\frac{1}{2}\frac{\partial^2\bar{S}}{\partial x'\partial y''} \\ -\frac{1}{2}\frac{\partial^2\bar{S}}{\partial y'\partial x''} & -\frac{1}{2}\frac{\partial^2\bar{S}}{\partial y'\partial y''}\end{pmatrix}\right)$$

$$\times\left|\det\begin{pmatrix} -\frac{\partial^2\bar{S}}{\partial x'\partial x''} & -\frac{\partial^2\bar{S}}{\partial x'\partial y''} \\ -\frac{\partial^2\bar{S}}{\partial y'\partial x''} & -\frac{\partial^2\bar{S}}{\partial y'\partial y''}\end{pmatrix}\right|_p^{1/2}\chi_p\left(-\bar{S}(x'',y'',t'';x',y',t')\right). \qquad (34)$$

Note that obtained p-adic result (34) has the same form as (16) in the real case. It is natural to expect that also higher-dimensional p-adic propagator will maintain this form.

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