Stability and convergence of difference schemes for parabolic interface problems

Boško S. Jovanović and Lubin G. Vulkov*

Abstract

In this paper[1] we report results on stability and convergence of two-level difference schemes for parabolic interface equations. Energy norms that rely on spectral problems containing the eigenvalue in boundary conditions or in conditions on conjugation are introduced. Necessary and sufficient stability conditions in these norms for weighted difference schemes are established. Convergence rate estimates of difference schemes compatible with the smoothness of the differential problems solutions are presented. The introducing of intrinsic discrete norms enable us to precise the values of the mesh steps that control stability and the rate of convergence of the difference schemes.

1 Introduction

Mathematically, interface problems lead to differential equations whose input data and solutions have discontinuities or non-smoothness across one or several interfaces which have lower dimension than that of the space where the problem is defined. The numerical methods designed for smooth solutions do not work efficiently for interface problems.

Interface problems occur in many applications and therefore, various forms of jump (interface) relations satisfied by the solution and its derivatives are known.

In [4] a review of results on numerical solutions of elliptic and parabolic interface problems in the recent three decades is presented.

Also, there, convergence of FEM for elliptic and parabolic problems with jump conditions of the form

$$[u] = 0, \ \ [k\partial u/\partial n] = g$$

is studied. These relations correspond to singular source type right-hand side in the equation.

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In the present paper we discuss stability and convergence of difference schemes (DS) for parabolic problems with other type of interfaces, see the next section.

The rest of this paper is organized as follows. In the next section we provide an abstract formulation of the problems. In Section 3, we discuss our previously results and we present new ones concerning stability of DS for $2D$ problems. Results on convergence of difference schemes are reported in Section 4.

2 The Differential problems

The problems solved below can be written as an abstract Cauchy problem

$$B\frac{du}{dt} + Au = f(t), \quad 0 < t < T, \quad u(0) = u_0, \tag{2.1}$$

where A and B are unbounded selfadjoint positive definite linear operators, with domains dense in Hilbert space H, u_0 is given element of energy space H_B, $f(t) \in L_2((0,T), H_{A^{-1}})$ and $u(t)$ is the unknown function from $(0,T)$ into H_A. Existence, uniqueness of solution and a priori estimates (2.1) can be found in [3, 8-10]

2.1 1D problems

Let us consider the initial boundary value problem for the heat equation with concentrated capacity at the interior point $x = \xi$ [2-4, 7-13]:

$$\big[c(x) + K\,\delta(x-\xi)\big]\,\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t), \quad (x,t) \in Q = (0,\,1) \times (0,\,T), \tag{2.2}$$

$$u(0,\,t) = 0, \quad u(1,\,t) = 0, \quad 0 < t < T; \quad u(x,\,0) = u_0(x), \quad x \in (0,\,1), \tag{2.3}$$

where $K > 0$, and $\delta(x)$ is the Dirac distribution. If $0 < c_3 \le c(x)$ we shall refer the problem (2.2), (2.3) as Dirichlet concentrated capacity problem (DCCP) It follows from (2.2), that the solution of this problem satisfies at $(x,t) \in Q_1 = (0,\,\xi) \times (0,\,T)$ and $(x,t) \in Q_2 = (\xi,\,1) \times (0,\,T)$ the equation

$$c(x)\,\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t)\,,$$

and at $x = \xi$ – the conditions of conjugation

$$[u]_{x=\xi} \equiv u(\xi+0,t) - u(\xi-0,t) = 0\,, \qquad \left[a\,\frac{\partial u}{\partial x}\right]_{x=\xi} = K\,\frac{\partial u(\xi,t)}{\partial t}\,. \tag{2.4}$$

It is easy to see that the DCCP can be reduced in the form (2.1) letting $H = L_2(0,1)$, $Au = -\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big)$ and $Bu = \big[c(x) + K\,\delta(x-\xi)\big]\,u(x,\,t)$.

Then $H_A = \overset{\circ}{W}{}_2^1(0,1)$, $\|w\|_A^2 = \int_0^1 a(x)\,[w'(x)]^2\,dx$, $\|w\|_B^2 = \int_0^1 c(x)\,w^2(x)\,dx + K\,w^2(\xi)$.

When $\xi = 0$ in (2.2) we come to initial–boundary value problem for the heat equation with dynamical boundary condition (DBCP) at $x = 0$ (cf. [3]):

Let suppose now in (2.2) $c(x) \equiv 0$. Then the solution for $(x,t) \in Q_1$ and $(x,t) \in Q_2$ satisfies the equation

$$-\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t)\,,$$

and for $x = \xi$ – the conjugation conditions (2.4)

Therefore, at fixed t, the equation is elliptic on $(0,\xi)$ and $(\xi,1)$, it is parabolic in the point $x = \xi$ and we will refer it as weakly parabolic problem (WPP).

This problem also has the form (2.1), where $Au = -\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big)$ and $Bu = K\,\delta(x-\xi)\,u(x,t)$.

The operator A is positively definite in the space $H_A = \overset{\circ}{W}{}_2^1(0,1)$. The operator B is nonnegative in H_A and $\|w\|_B = \sqrt{K}\,|w(\xi)|$.

2.2 2D problems

We consider the following linear model problem :

$$\big[c(x) + \delta_\Gamma(x)K(x)\big]\frac{\partial u}{\partial t} - Lu = f(x,T), \quad x = (x_1,x_2), \quad (x,t) \in Q_T, \qquad (2.5)$$

$$Lu = \frac{\partial}{\partial x_1}\Big(k_1(x)\frac{\partial u}{\partial x_1}\Big) + \frac{\partial}{\partial x_1}\Big(k_2(x)\frac{\partial u}{\partial x_2}\Big)$$

with initial and boundary condition, respectively

$$u(x,0) = u_0(x), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega, \;\; 0 < t < T. \qquad (2.6)$$

Here $\Omega \subset R^2$ is an open domain with C^1-piecewise smooth boundary $\Omega_1 \subset \Omega$ is an open domain with C^1-piecewise smooth boundary $\partial\Omega_1 = \Gamma$, $\Omega_2 = \Omega \setminus \Omega_1$, $\partial\Omega_2 = \partial\Omega$ Next, $Q_T = \Omega \times (0,T)$, $\partial u/\partial n_k = \sum_{i=1}^2 k_i(x)\frac{\partial u}{\partial x_i}\cos(n,x_i)$ is the normal derivative and δ_Γ is the Dirac-delta function concentrated on Γ.

Physically [3], two cases of the equation (2.5) are important. In the first one we suppose that $c(x) \ge c_0 > 0$, $x \in \Omega$, $K(x) \ge K_0 > 0$, $x \in \Gamma$ and we come to $2D$ DCCP. Under certain smoothness and compatibility conditions on the data (initial condition, coefficients, right-hand side, the boundaries $\partial\Omega_1, \partial\Omega_2$), it follows from (2.5) that the $2D$-DCCP is equivalent to the following one:

$$\frac{\partial u}{\partial t} - Lu = f(x,t) \;\text{ in }\; Q_T \setminus \Gamma_T; \quad u = 0, \quad (x,t) \in \partial Q_T; \quad u(x,0) = u_0(x), \quad x \in \Omega \qquad (2.7)$$

and on the interface $\Gamma_T = \Gamma \times [0,T]$, the jump conditions hold

$$[u]_{\Gamma_T} = 0, \quad \left[\frac{\partial u}{\partial n_k}\right]_{\Gamma_T} = K(x)\frac{\partial u}{\partial t}\bigg|_{\Gamma_T}. \tag{2.8}$$

The second important case is when $c(x) \equiv 0, \quad K(x) \geq K_0 > 0$. Then we come to $2D$-WPP, which is equivalent to the following one:

$$-Lu = f(x,t) \;\text{ in }\; Q_T \setminus \Gamma_T; \;\; u(x,t) = 0, \;\; (x,t) \in \partial Q_T; \;\; u(x,0) = u_0(x), \;\; x \in \Gamma \tag{2.9}$$

and on the interface Γ_T, the jump conditions (2.8) hold. The problems $2D$-DCCP, $2D$-WPP also can be written in the abstract form (2.1).

3 Stability of difference schemes

3.1 Preliminary results

Consider the variational problems

$$\frac{1}{\lambda} = \sup_{w \in H_A} \frac{\|w\|_B^2}{\|w\|_A^2}, \quad \frac{1}{\mu} = \sup_{v \in H_A} \frac{\|v\|^2}{\|v\|_A^2}. \tag{3.1}$$

The solutions of (3.1) satisfy the spectral problems

$$Aw = \lambda Bw \quad \text{and} \quad Av = \mu v \tag{3.2}$$

respectively. Suppose that the the operator $T = A^{-1}B$ is compact. Then the spectra of (3.2) are discrete, all eigenvalues are positive: $0 < \lambda_1 < \lambda_2 < ...; \;\; 0 < \mu_1 < \mu_2 < \ldots$. Here we present a simple estimate for stability of the solution of problem (2.1), which will be a landmark at investigation of the corresponding DS for stability with respect to initial data and right–hand side.

Lemma 1 *Let H, H_A, H_B and u_0, f are as above. Then (1.1) has a unique solution $u \in L^2((0,T); H_A) \cap H^1((0,T); H_{A^{-1}})$ (In fact $u \in C((0,T); H)$, cf. [13]. If $u_0 \in H_B$, then u satisfies the estimate*

$$E(t) \leq e^{-2\lambda_1 ct}\left(E(0) + \frac{1}{2\varepsilon}\int_0^t \|f(\rho)\|^2 e^{2\lambda_1 c\rho} d\rho\right), \quad 0 < t < T, \tag{3.3}$$

where $\varepsilon > 0$, $0 < c < 1$ and

$$E(t) = \frac{1}{2}(Bu, u), \quad E(0) = \frac{1}{2}(Bu_0, u_0).$$

We shall make use of abstract results in [5,11]. The interval $[0,T]$ is replaced by the grid $\bar{\omega}_\tau = \{t_n = n\tau, \;\; n = 0, 1, ...M, \;\; M\tau = T\}$. Let us set $\omega_\tau = \bar{\omega}_\tau \cap (0,T)$, $\omega_\tau^- = \omega_\tau \cup \{0\}$ and $\omega_\tau^+ = \omega_\tau \cup \{T\}$. A two–level scheme in operator form is given by

$$Bv_t + Av = \varphi, \tag{3.4}$$

where $\varphi = \varphi_{h,\tau}(t_n)$ is a given function of $t_n \in \omega_\tau^-$ with values in a finite–dimensional Hilbert space H, $v = v_{h\tau}(t_n) = v_n$ is unknown function from $\bar{\omega}_\tau$ into H and A, B are linear operators, defined on H. Let $D : H \to H$ is a selfadjoint linear positive definite operator. The linear space H equipped with the inner product $(v, w)_D = (Dv, w)$ and the norm $\|w\|_D = \sqrt{(w, w)_D}$ is called the energy space H_D.

Theorem 1 *Let A be a selfadjoint positive operator independent of n and let B^{-1} exist. The scheme* (3.4) *is stable in the space H_A if and only if*

$$B \geq 0.5\tau A \ \text{ or at least } \ B \geq \varepsilon E + 0.5\tau A \ \text{ for some } \ \varepsilon > 0, \tag{3.5}$$

where $E : H \to H$ is identity operator.

Theorem 2 *Let $B = E + \tau\sigma A$, $\sigma^* = \sigma$, $A^* = A$, and let B^{-1} exist and operator A is represented in the form $A = L^* L$, where $L : H \to \bar{H}_D$, $L^* : \bar{H}_D \to H$, $\bar{H}_D$ is Euclid space with inner product $(.,.)_{\bar{H}}$. If the FDS (3.4) is stable in some space H_D, then the operator inequality*

$$E + \tau L \mu L^* \geq 0 \ \text{ in } \ \bar{H}, \ \text{ where } \ \mu = \sigma - 0.5E \tag{3.6}$$

is valid. Conversely, if condition (3.6) is satisfied, then the scheme (3.4) is stable in $H_{B_* B}$. But if in addition, A^{-1} exists, then the scheme (3.4) is stable in H_{A^2} as well.

3.2 Difference schemes with constant and variable weights

For simplicity we shall take $\Omega = (0,1) \times (0,1)$. Let introduce on Ω the non-uniform grid $\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2$, $\bar{\omega}_1 = \{x_{1,i} = x_{1,i-1} + h_{1,i},\ \ i = 1, ..., N_1,\ \ x_{1,0} = 0,\ x_{1,N_1} = 1\}$, $\bar{\omega}_2 = \{x_{2,k} = x_{2,k-1} + h_{2,k},\ k = 1, ..., N_2,\ x_{2,0} = 0,\ x_{2,N_2} = 1\}$, $\bar{\omega} = \omega \cup \partial\omega$, where ω is the set of the internal nodes of the mesh, $\partial\omega$ – the set of boundary nodes. Together with the basic mesh, we introduce the flow mesh $\widetilde{\omega} = \widetilde{\omega}_1 \times \widetilde{\omega}_2$, $\widetilde{\omega}_1 = \{\widetilde{x}_{1,i} = 0.5(x_{1,i-1} + x_{1,i}),\ \ i = 1, ..., N_1\}$, $\widetilde{\omega}_2 = \{\widetilde{x}_{2,k} = 0.5(x_{2,k-1} + x_{2,k}),\ \ k = 1, ..., N_2\}$. To each node $x = (x_{1,i}, x_{2,k})$ we attach the rectangle $e(x) = \{\xi = (\xi_1, \xi_2)\ \mid\ \widetilde{x}_{1,i} < \xi_1 < \widetilde{x}_{1,i+1},\ \ \widetilde{x}_{2,k} < \xi_2 < \widetilde{x}_{2,k+1}\}$. A node $x \in \omega$ for which $\Gamma \cap e(x) \neq \emptyset$ we shall call irregular (or interface) node. We approximate the $2D$ DCCP by the scheme [11]

$$\frac{y_{ik}^{n+1} - y_{ik}^n}{\tau} = \frac{\sigma_{ik}}{p_{ik}} \Lambda^h y_{ik}^{n+1} + \frac{1 - \sigma_{ik}}{p_{ik}} \Lambda_h y_{ik}^n + \varphi, \quad x \in \omega, \tag{3.7}$$

$(\Lambda^h y)_{ik} = [(a_1 y_{\bar{x}_1})_{\hat{x}_1}]_{ik} + [(a_2 y_{\bar{x}_2})_{\hat{x}_2}]_{ik}$,
$y_{ik}^n = 0, \quad x \in \partial\omega, \quad n \geq 0; \quad y_{ik}^0 = u_0(x), \quad x \in \bar{\omega}$.

Here y_{ik}^n is the difference solution in the space mesh node (i,k) and

$$p_{ik} = p(x) = c(x), \qquad x = (x_{1,i}, x_{2,k}) \in \hat{\omega} \ \ \text{(regular node)},$$

$$p_{ik} = p(x) = c(x) + \frac{K(x)}{\bar{h}_{1,i} \bar{h}_{2,k}} l(\Gamma \cap e(x)), \quad x = (x_{1,i}, x_{2,k}) \in \hat{\omega} \ \ \text{(irregular node)},$$

where $l(\Gamma \cap e(x))$ is the length of the part of the curve Γ closed into the cell $e(x)$.

Stability of DS with constant weights for the $1D$ DCCP, $1D$ DBCP and $1,2D$ WPP have been investigated on the base of Theorem 2 in [2,10,13]. In [10], following inequality (3.3), energy norms, that rely on spectral problems of type (3.2), containing the eigenvalues in the boundary conditions on conjugation are introduced. This enable us to precise the values of the mesh steps that control the stability of DS. Here, we shall show how this methodology works on $2D$ CCP. First, let consider the problem of type (3.2)

$$Lv + \lambda\big[c(x) + K(x)\delta_\Gamma(x)\big]\, v = 0, \;\; x \in \Omega; \quad v|_{\partial\Omega} = 0$$

and its discretization $\wedge z + \lambda^h p(x) z = 0, \;\; x \in \omega, \;\; z|_{\partial\omega} = 0$. This problem has finite number of eigenvalues and we denote by $\lambda^h_{\max}$ the greatest eigenvalue. The following theorem for DS with constant weights holds.

Theorem 3 *Suppose that*

$$\sigma \geq \frac{1}{2} - \frac{1}{\tau\lambda^h_{\max}}.$$

Then (3.7) is stable with respect to initial data. Suppose that

$$\sigma \geq \sigma_0 = \frac{1}{2} - \frac{1-\varepsilon}{\tau\lambda^h_{\max}}, \;\; \varepsilon > 0.$$

Then the scheme (3.7) is stable with respect to right hand side.

The scheme (3.7) is written in the operator form (3.4), where $(Ay^n)_{ik} = \frac{(\Lambda^n y^n)_{ik}}{p_{ik}}$, and $B = E + \tau\sigma A$. Then, the operator A is selfadjoint in the linear space H equipped with the inner product and the corresponding norm

$$(u,v)_\Gamma = \sum_{i=1}^{N_1-1}\sum_{k=1}^{N_1-1} p_{ik} u_{ik} v_{ik} \bar{h}_{1,i} \bar{h}_{2,k}, \quad \|u\|_\Gamma = \sqrt{(u,v)}_\Gamma$$

Now, on the base of Theorem 3, we discuss (3.7) in the case of variable weights. Operator $Q = E + \tau L\mu L^*$ is self-adjoint. Therefore condition (3.6) is equivalent to the requirement of nonnegative eigenvalue of operator Q. For our problem we involve H_1 and H_2 – Euclidian spaces with dimension: $\dim H_1 = N_1(N_2-1)$, $\dim H_2 = (N_2-1)N_1$ and usual inner products [11]. Next, we introduce the following operators: $L_1 : H \to H_1$, $L_2 : H \to H_2$,

$$(L_1 y)_{ik} = (\sqrt{a_1} y_{\bar{x}_1})_{ik}, \;\; y_{0k} = y_{N_1 k} = 0, \;\; i = 1,...,N_1; \;\; k = 1,...,N_2-1, \quad (3.8)$$

$$(L_2 y)_{ik} = (\sqrt{a_2} y_{\bar{x}_2})_{ik}, \;\; y_{i0} = y_{iN_2} = 0, \;\; i = 1,...,N_1-1; \;\; k = 1,...,N_2; \quad (3.9)$$

and $L_1^* : H \to H_1$, $L_2^* : H \to H_2$,

$$(L_1^* v)_{ik} = -\frac{1}{p_{ik}}(\sqrt{a_1} v_{\hat{x}_1})_{ik}, \;\; i = 1,...,N_1-1; \;\; k = 1,...,N_2-1, \qquad (3.10)$$

$$(L_2 v)_{ik} = -\frac{1}{p_{ik}}(\sqrt{a_2} v_{\hat{x}_2})_{ik}, \quad i = 1, ..., N_1 - 1; \quad k = 1, ..., N_2 - 1. \qquad (3.11)$$

From (3.8), (3.9) and (3.10, 3.11) we obtain $L_1^* L_1 : H \to H$, $L_2^* L_2 : H \to H$ and therefore $A = L_1^* L_1 + L_2^* L_2$, $A = L^* L$, where $L = (L_1, L_2)^T$, $L^* = (L_1^*, L_2^*)^T$, $L^* : \bar{H} \to H$, $\bar{H} = H_1 \oplus H_2$. Further analysis of (3.7) will be given in a forthcoming paper of the authors.

4 Convergence of difference schemes on generalized solutions

Convergence on classical solutions for $1D$-DCC, DBC, WPP is studied in [1], [2], [12]. More difficult is this question on generalized (weak) solutions. Then, rate of convergence estimates of the form

$$\|u - v\|_{W_{2,h,\tau}^{k,k/2}} \leq C(h + \sqrt{\tau})^{s-k} \|u\|_{W_2^{s,s/2}}, \quad s > k$$

where τ is time step and h - space mesh step are of great interest. Such estimates for $1D$ DCCP, $1D$ DBCP, $1D$ WPP are established in [7-9].

The conception of studying in [7-9] is as follows. The problems considered are treated as a first order abstract evolution (2.1), with selfadjoint operators A (positive), B (positive or nonnegative), defined in Hilbert space H and then to use energy methods from the theory of Hilbert spaces. Discrete analog of appropriate subspaces of the Sobolev spaces are used and yet that allow the discrete operators to be selfadjoint. In this first stage we obtain a priori estimates for the discrete solutions. The second important idea of this method consists in constructing the special integral representations of the error of the difference scheme. This allows us by applying imbedding Sobolev's theorems to obtain more accurate estimates. We do not make use of the accurate Bramble-Hilbert lemma. Here, we present a result of this type for $2D$ WPP.

We approximate the $2D$ WPP with constant coefficients and line interface $\Gamma = \{x = (x_1, x_2) \mid x_2 = x_2^0, \ 0 \leq x_1 \leq 1\}$ on the mesh $\bar{Q}_{h\tau} = \bar{\omega}_1 \times \bar{\omega}_2 \times \bar{\omega}_\tau$ by the implicit difference scheme with averaged right hand side

$$\delta_\gamma v_{\bar{t}} - \Delta_h v = T_1^2 T_2^2 T_t^- \varphi, \qquad (x, t) \in \omega \times \omega_\tau^+$$

$$v(x, t) = 0, \quad (x, t) \in \partial\omega \times \omega_\tau^+; \qquad v(x, 0) = u_0(x), \quad x \in \gamma = \omega \cap \Gamma,$$

where $\Delta_h v = v_{\bar{x}_1 x_1} + v_{\bar{x}_2 x_2}$ $\quad \delta_\gamma(x) = \delta_{h_2}(x_2 - x_2^0) = \begin{cases} 0, & x \in \omega \setminus \gamma \\ 1/h_2, & x \in \gamma \end{cases}$ and T_1^2, T_2^2, T_1^- are the Steklov averaging operators [6].

Theorem 4 *Let $f \in L_2(Q_T)$, $u_0 \in \overset{\circ}{W}{}_2^1(\Gamma)$ and the solution of $2D$ WPP with constant coefficients belongs to*

$$L_2(0, T; W_2^2(\Omega_1)) \cap L_2(0, T; W_2^2(\Omega_2)) \cap W_2^1(0, T; L_2(\Gamma)).$$

Then, the error $z = u - v$ satisfies the estimate

$$\left\{\tau \sum_{t\in\omega_\tau^+} \|z(\cdot,t)\|^2_{W_2^1(\omega)} + \tau^2 \sum_{t\in\bar\omega_\tau} \sum_{t'\in\bar\omega_\tau,\ t'\neq t} \frac{\|z(\cdot,t)-z(\cdot,t')\|^2_{L_2(\gamma)}}{|t-t'|^2} + \right.$$
$$\left. + \max_{t\in\omega_\tau^+} \|z(\cdot,t)\|^2_{L_2(\gamma)}\right\}^{1/2} \leq C\,(h_1^2+h_2^2+\tau)\,\sqrt{\ln 1/\tau}\,\Big(\|u_0\|_{W_2^2(\Gamma)} +$$
$$+ \|f(\cdot,0)\|_{L_2(\Omega)} + \|f\|_{L_2(0,T;W_2^1(\Omega))} + \|f\|_{W_2^1(0,T;W_2^{-1}(\Omega))}\Big).$$

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Faculty of Mathematics, University of Belgrade,
11000 Belgrade, Yugoslavia
bosko@matf.bg.ac.yu

University of Rousse, Department of Mathematics,
7017 Rousse, Bulgaria
vulkov@ami.ru.acad.bg