Matrix representation of BUAR(1)

Biljana Č. Popović and Miroslav M. Ristić

Abstract

We consider the bivariate first order stationary autoregressive process $\{\mathbf{W}_t\}$,

$$\mathbf{W}_t = \mathbf{M}_t \mathbf{W}_{t-1} + \varepsilon(t)$$

with uniform marginal distribution defined by Ristić and Popović [8]. We pay our attention onto the proving procedure specified by Nicholls and Quinn [4][1].

1 Introduction

Uniform autoregressive time series was defined by Chernik [1] and the idea was developed later on by Chernik and Davis [2] and Lawrance [3]. In recent times Ristić and Popović [5],[6] and [8] contributed new results in this area. Here we consider the time series BUAR(1) in contexts of the general definition of the random coefficient autoregressive time series with the special attention on the proving procedure developed by Nicholls and Quinn [4] which becomes rather complicated comparing with this specified in Ristić and Popović [8].

Let us start with the definitions:

Definition 1.1 *Doubly-infinite vector valued time series $\{\mathbf{W}_t\}$ is $BUAR(1)$ iff*

$$\mathbf{W}_t = \mathbf{M}_t \mathbf{W}_{t-1} + \varepsilon(t) \tag{1}$$

where $\mathbf{W}'_t = [X_t, Y_t]$, $\mathbf{M}_t = \begin{bmatrix} U_{t1} & V_{t1} \\ U_{t2} & V_{t2} \end{bmatrix}$, $\varepsilon'(t) = [\varepsilon_1(t), \varepsilon_2(t)]$, $\{(U_{ti}, V_{ti})'\}$, $i = 1, 2$, are independent sequences of i.i.d. random vectors distributed as follows

$U_{ti} \backslash V_{ti}$	*0*	β_i
0	*0*	$\frac{-\beta_i}{\alpha_i - \beta_i}$
α_i	$\frac{\alpha_i}{\alpha_i - \beta_i}$	*0*

[1]Presented at the IMC "Filomat 2001", Niš, August 26–30, 2001

2000 Mathematics Subject Classification: 62M10
Keywords: Uniform autoregressive process, first-order, bivariate uniform autoregressive process, tensor product, stationarity.

with $-1 < \beta_i \leq 0 \leq \alpha_i < 1$, $\alpha_i - \beta_i > 0$, $1/(\alpha_i - \beta_i) \in \{1, 2, 3, \cdots\}$, $i = 1, 2$, $\{\varepsilon(t)\}$ *is the sequence of i.i.d. random vectors with independent components distributed as*

$$P\{\varepsilon_{ti} = j_i(\alpha_i - \beta_i) - \beta_i\} = \alpha_i - \beta_i;\ \ j_i = 0, 1, \ldots, \frac{1}{\alpha_i - \beta_i} - 1,\ \ i = 1, 2.$$

$\{X_t\}$ and $\{Y_t\}$ are the sequences of i.i.d. random variables distributed as X_t: $\mathcal{U}(0,1)$, $Y_t : \mathcal{U}(0,1)$, $\forall t$.

Here, we shall consider the case of independent vectors $\mathbf{M}'_{ti} = [U_{ti}, V_{ti}]$ and $\varepsilon(t)$ for any t.

The bivariate time series $\{\mathbf{W}_t\}$ satisfies the general definition of RCA(n) for $n = 1$ defined by Nicholls and Quinn [4] but with nonzero mean of the innovation sequence:

Definition 1.2 *RCA*(n) *p-variate random coefficient autoregression*

$$X(t) = \sum_{i=1}^{n} \{\beta_i + B_i(t)\} X(t-1) + \varepsilon(t) \tag{2}$$

where:

(i) $\{\varepsilon(t);\ t = 0, \pm 1, \pm 2, \cdots\}$ *is an independent sequence of p-variate random variables with mean zero and covariance matrix* $\mathbf{G}$.

(ii) *The* $p \times p$ *matrices* β_i, $i = 1, 2, \cdots, n$ *are constants.*

(iii) *Letting* $B(t) = \{B_n(t), \cdots, B_1(t)\}$, *then* $\{B(t)\}$ *is an independent sequence of* $p \times np$ *matrices with mean zero and* $E[B(t) \otimes B(t)] = C$. $\{B(t)\}$ *is also independent of* $\{\varepsilon(t)\}$.

(iv) *There is no non-zero* $p \times 1$ *constant vector* z *such that* $z'X(t)$ *is purely linearly deterministic.*

In this paper we use algebra of well known Kronecker's (tensor) product, defined as

Definition 1.3 *Let* $A = [a_{ij}]_{p\times q}$ *and* $B = [b_{ij}]_{m\times n}$. *Then Kronecker's product is defined as* $A \otimes B = [a_{ij}B]_{pm\times qn}$.

We consider the existence of a solution of the equation (1) and the stationarity of such possible solution under the assumption of the defined uniform distribution.

2 Stationarity

First of all we shall discuss the stationarity (wide sense) of the process $\{\mathbf{W}_t\}$. We shall use the procedure of Nicholls and Quinn to prove the stationarity: the (necessary and) sufficient conditions for stationarity of $\{\mathbf{W}_t\}$ will be expressed as

$$(\mathbf{I}_2 - \beta)\mu = \tilde{\epsilon} \tag{3}$$

and

$$(\mathbf{I}_4 - \mathbf{M})vecV = (\tilde{\epsilon} \otimes \beta + \beta \otimes \tilde{\epsilon})\mu + vec\tilde{\mathbf{G}} + vec(\tilde{\epsilon}\tilde{\epsilon}') \tag{4}$$

where $\beta = E\mathbf{M}_t$, $\mu = E\mathbf{W}_t$, $\mathbf{M} = E(\mathbf{M}_t \otimes \mathbf{M}_t)$, $\tilde{\epsilon} = E\varepsilon(t)$, $V = E(\mathbf{W}_t\mathbf{W}_t')$, $\tilde{\mathbf{G}} = E\left[(\varepsilon(t) - \tilde{\epsilon})(\varepsilon(t) - \tilde{\epsilon})'\right]$.

It is easy to verify that $\beta = \frac{1}{\alpha_1 - \beta_1}A_1 + \frac{1}{\alpha_2 - \beta_2}A_2$, $A_1 = \begin{bmatrix} \alpha_1^2 & -\beta_1^2 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ \alpha_2^2 & -\beta_2^2 \end{bmatrix}$, $\mu' = [1/2, 1/2]$, $\tilde{\epsilon} = \frac{1}{2}\begin{bmatrix} 1 - \alpha_1 - \beta_1 \\ 1 - \alpha_2 - \beta_2 \end{bmatrix}$ and (3) holds. Also,

$$\mathbf{M} = \begin{bmatrix} \alpha_1 u_1 & 0 & 0 & \beta_1 v_1 \\ u_1 u_2 & u_1 v_2 & u_2 v_1 & v_1 v_2 \\ u_1 u_2 & u_2 v_1 & u_1 v_2 & v_1 v_2 \\ \alpha_2 u_2 & 0 & 0 & \beta v_2 \end{bmatrix},$$

where $u_i = E(U_{ti})$ and $v_j = E(V_{tj})$, $i, j = 1, 2$,

$$(\tilde{\epsilon} \otimes \beta)' = \frac{1}{2}\left[\frac{1 - \alpha_1 - \beta_1}{\alpha_1 - \beta_1}A_1' + \frac{1 - \alpha_1 - \beta_1}{\alpha_2 - \beta_2}A_2', \frac{1 - \alpha_2 - \beta_2}{\alpha_1 - \beta_1}A_1' + \frac{1 - \alpha_2 - \beta_2}{\alpha_2 - \beta_2}A_2'\right],$$

similarly $\beta \otimes \tilde{\epsilon}$.

Further on

$$vec\tilde{\mathbf{G}} + vec(\tilde{\epsilon}\tilde{\epsilon}') = vecF,$$

$$vecF = vecE(\varepsilon(t)\varepsilon'(t)) = \begin{bmatrix} \frac{1}{6}(2 + \alpha_1^2 + 4\alpha_1\beta_1 + \beta_1^2 - 3\alpha_1 - 3\beta_1) \\ \frac{1}{4}(1 - \alpha_1 - \beta_1)(1 - \alpha_2 - \beta_2) \\ \frac{1}{4}(1 - \alpha_1 - \beta_1)(1 - \alpha_2 - \beta_2) \\ \frac{1}{6}(2 + \alpha_2^2 + 4\alpha_2\beta_2 + \beta_2^2 - 3\alpha_2 - 3\beta_2) \end{bmatrix}$$

and the result (4) follows. So, $\{\mathbf{W}_t\}$ is a wide sense stationery.

3 Uniqueness and σ_t-measurability

In order to prove the existence of the solution of (1), first of all, we shall prove that $\mathbf{M}$ has no unit eigenvalues. This solution will be stationary and σ_t–measurable, where σ_t is the σ–field generated by $\{(\mathbf{M}_s, \varepsilon(s)),\ s \le t\}$.

The characteristic equation of $\mathbf{M}$ is

$$(\lambda^2 - (\alpha_1 u_1 + \beta_2 v_2)\lambda + \alpha_1\beta_2 u_1 v_2 - \alpha_2\beta_1 u_2 v_1)((u_1 v_2 - \lambda)^2 - u_2^2 v_1^2) = 0\,.$$

So, its eigenvalues are

$$\begin{aligned} \lambda_{1,2} &= u_1 v_2 \pm u_2 v_1, \\ \lambda_{3,4} &= \frac{\alpha_1 u_1 + \beta_2 v_2 \pm \sqrt{(\alpha_1 u_1 - \beta_2 v_2)^2 + 4\alpha_2\beta_1 u_2 v_1}}{2}. \end{aligned}$$

After some cumbersome but simple calculations on these eigenvalues, it follows that $\mathbf{M}$ has no unit eigenvalue at all.

According to the Theorem 2.2 (Nicholls and Quinn [4]), as $\mathbf{M}$ has no eigenvalue on the unit circle, necessary and sufficient condition for existing unique, stationary, σ_t-measurable solution of (1) will be positive definiteness of matrix H, in our case, defined as

$$vec\mathbf{H} = vec\mathbf{G} + \mathbf{c}\sum_{j=0}^{\infty} \mathbf{M}^j vec\mathbf{G}.$$

To reach this term and to prove it, we need the transformation $\mathbf{Q}_t = \mathbf{W}_t - \mu$. After this, we have $E\mathbf{Q}_t = \mathbf{0}$, $\mathbf{Q}_t = \mathbf{M}_t\mathbf{Q}_{t-1} + \eta(t)$, $\eta(t) = \varepsilon(t) + (\mathbf{M}_t - \mathbf{I})\mu$, $E\eta(t) = \mathbf{0}$, $\mathbf{G} = E\left[\eta(t)\eta'(t)\right]$, $\mathbf{C} = E\left(\mathbf{B}_t \otimes \mathbf{B}_t\right)$, where $\mathbf{B}_t = \mathbf{M}_t - \beta$.

We will reach the proof in two steps:

1. Convergence of the series

$$\sum_{j=0}^{\infty} \mathbf{M}^j vec\mathbf{G}.$$

It is easy to verify that

$$\mathbf{Q}_t = \sum_{j=1}^{r} S_{t,j-1}^{Q}\eta(t-j) + \eta(t) + \mathbf{R}_{t,r}^{Q}$$

where $S_{t,j}^{Q} = \prod_{k=0}^{j} \mathbf{M}_{t-k}$ and $\mathbf{R}_{t,r}^{Q} = S_{t,r}^{Q}\mathbf{Q}_{t-r-1}$. If we set $\mathbf{W}_{t,r}^{q} = \mathbf{Q}_t - \mathbf{R}_{t,r}^{Q}$ and $S_{t,-1}^{Q} \overset{def}{=} \mathbf{I}_2$, then

$$\mathbf{Q}_t = \sum_{j=0}^{r} S_{t,j-1}^{Q}\eta(t-j) + \mathbf{R}_{t,r}^{Q}$$

and

$$\mathbf{W}_{t,r}^{Q} = \sum_{j=0}^{r} S_{t,j-1}^{Q}\eta(t-j).$$

After this we obtain

$$vecE\left[\mathbf{W}_{t,r}^{Q}\left(\mathbf{W}_{t,r}^{Q}\right)'\right] = vecE\left[\sum_{j=0}^{r} S_{t,j-1}^{Q}\eta(t-j)\eta'(t-j)\left(S_{t,j-1}^{Q}\right)'\right]$$

thanks to the fact that $\eta(t-j)$ and $\eta(t-i)$ are uncorrelated for $i \neq j$.

If we use the following properties of Kronecker's product

a) $vec(ABC) = (C' \otimes A)\, vecB$, and

b) $\left(\prod_{i=0}^{j} A_i\right) \otimes \left(\prod_{k=0}^{j} B_k\right) = \prod_{i=0}^{j} (A_i \otimes B_i)$

whenever these products are defined, it will be

$$vecE\left[\mathbf{W}_{t,r}^{Q}\left(\mathbf{W}_{t,r}^{Q}\right)'\right] = \sum_{j=0}^{r} \mathbf{M}^j vec\mathbf{G}.$$

Thanks to the no unit eigenvalues of $\mathbf{M}$ in our special case,
$\sum_{j=0}^{r} \mathbf{M}^j vec\mathbf{G}$ converges as $r \to \infty$.

So, we have proved mean square convergence and, as the consequence, the convergence in probability of $\mathbf{W}_{t,r}^{Q}$.

That implies that the solution of (1) will be of the form

$$\mathbf{W}_t = \sum_{j=1}^{\infty}\left(\prod_{k=0}^{j} \mathbf{M}_{t-k}\right)\left[\varepsilon(t-j) + (\mathbf{M}_{t-j} - \mathbf{I})\,\mu\right] + \varepsilon(t) + (\mathbf{M}_t - \mathbf{I}_4)\mu. \quad (5)$$

2. Positive definiteness of $\mathbf{H}$ is equivalent to the fact that for any nonzero vector $z_{2\times 1}$, $vec(z'\mathbf{H}z)$ is a nonzero vector. So, we shall discuss the last one. As the $vec\mathbf{G}$ is

$$\begin{aligned} vec\mathbf{G} &= vec\mathbf{F} + (\beta \otimes \tilde{\varepsilon} - \mathbf{I}_2 \otimes \tilde{\varepsilon} + \tilde{\varepsilon} \otimes \beta - \tilde{\varepsilon} \otimes \mathbf{I}_2)\mu + \\ &+ (\mathbf{M} - \mathbf{I}_2 \otimes \beta - \beta \otimes \mathbf{I}_2 + \mathbf{I}_4) vec(\mu\mu'). \end{aligned}$$

it will be

$$\begin{aligned} vec(z'\mathbf{H}z) &= vecz'\mathbf{F}z + z'\beta\mu\beta' z + z'\beta\mu\tilde{\varepsilon}' z + (z'\beta\mu)(z'\beta\mu)' - \\ &- z'\beta\mu\mu' z - (z'\beta\mu\mu' z)' + z' E\left[\mathbf{B}_t\mu\,(\mathbf{B}_t\mu)'\right] z + \\ &+ (z'\mu)(z'\mu)' + \sum_{j=0}^{\infty} E\left[z'\mathbf{B}_t\beta^j\mathbf{G}\left(\beta^j\right)'\mathbf{B}_t' z\right] - \\ &- (z' \otimes (\beta z)') vec\mu. \end{aligned}$$

Using the Sylvester's theorem for the positive definiteness of symmetric quadratic matrices and direct solving for the rest, it follows that $\mathbf{H}$ is positive definite.

So, (1) has the unique stationary σ_t-measurable solution (5).

References

[1] M. R. Chernick, *A limit theorem for the maximum of autoregressive processes with uniform marginal distributions*, Ann. Probab. (9) 1 (1981), 145–149.

[2] M. R.Chernick and R. A. Davis, *Extremes in autoregressive processes with uniform marginal distributions*, Statistics and Probability Letters 1 (1982), 85–88.

[3] A. J. Lawrance, *Uniformly distributed first-order autoregressive time series models and multiplicative congruential random number generators*, J. Appl. Prob. 29 (1992), 896–903.

[4] D. Nicholls and B. Quinn, *Random Coefficient Autoregressive Models: An Introduction*, Springer, New York-Berlin, Lecture Notes in Statistics, (1982)

[5] M. M. Ristić and B. Č. Popović, *A new uniform* $AR(1)$ *time series model* ($NUAR(1)$), Publications de L' Institut Mathématique - Beograd 68 (82) (2000), 145–152.

[6] M. M. Ristić and B. Č. Popović, *Parameter estimation for uniform autoregressive processes*, Novi Sad Journal of Mathematics, 30 (1) (2000), 89–95.

[7] M. M. Ristić and B. Č. Popović, *The uniform autoregressive process of the second order (UAR(2))*, Statistics and Probability Letters, accepted for publication

[8] M. M. Ristić and B. Č. Popović, *A bivariate uniform autoregressive process of the first-order (BUAR(1))*, submitted for publication

Faculty of Sciences and Mathematics
Department of Mathematics
University of Niš
Višegradska 33, 18000 Niš
PO.Box 224, Yugoslavia
biljanap@ban.junis.ni.ac.yu, miristic@ptt.yu