Nonlinear AR(1) and MA(1) time series models

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Abstract

The simple nonlinear models of autoregressive and moving average structure are analyzed in the paper[1]. In this paper we obtain some information on distributions of random parameters of two models.

1 An autoregressive process

In this section we shall deal with the following model for a stationary sequence

$$\{X_t\}, t \in \{0, \pm 1, \pm 2, \dots\} :$$

$$X_t = AX_{t-1} + B\xi_t \tag{1}$$

where A and B are random coefficients with distributions

$$A : \begin{pmatrix} \alpha & \beta & \gamma \\ p & r & q-r \end{pmatrix}$$

$$B : \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}.$$

p, q, r are probabilities with $p+q = 1$, $0 < \alpha < \beta < \gamma < 1$ and ξ_t are independent identically distributed random variables.

We assume A is independent of B, A, B are independent of X_t and A, B are independent of ξ_t, all t. We also assume X_t is independent of ξ_s, for all $s > t$.

To derive moments of X_t from model (1):

$$X_{t+1} = \begin{cases} \alpha X_t, & \text{w.p.}pq \\ \beta X_t, & \text{w.p.}rq \\ \gamma X_t, & \text{w.p.}(q-r)q \\ \alpha X_t + \xi_{t+1}, & \text{w.p.}p^2 \\ \beta X_t + \xi_{t+1}, & \text{w.p.}rp \\ \gamma X_t + \xi_{t+1}, & \text{w.p.}(q-r)p \end{cases} \tag{2}$$

we use the Laplace transforms

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$$\mathcal{L}_x(s) = E\left(\exp(-sX)\right) ,$$
$$\mathcal{L}_\xi(s) = E\left(\exp(-s\xi)\right) .$$

Transforming both sides of (2) gives:

$$\begin{aligned}\mathcal{L}_x(s) &= pq\mathcal{L}_x(\alpha s) + rq\mathcal{L}_x(\beta s) + (q-r)q\mathcal{L}_x(\gamma s) + p^2\mathcal{L}_x(\alpha s)\mathcal{L}_\xi(s) + \\ &+ rp\mathcal{L}_x(\beta s)\mathcal{L}_\xi(s) + (q-r)p\mathcal{L}_x(\gamma s)\mathcal{L}_\xi(s)\,,\end{aligned} \tag{3}$$

and from (3) follows

$$EX = -\mathcal{L}'_X(0) = \frac{pE\xi}{1-p\alpha-r\beta-(q-r)\gamma}\,,$$

$$EX^2 = \mathcal{L}''_X(0) = \frac{2pE\xi EX(p\alpha+r\beta+(q-r)\gamma)+pE\xi^2}{1-p\alpha^2-r\beta^2-(q-r)\gamma^2}\,.$$

The method of moments is quite complicated for estimating unknown parameters p, q, r and we shall estimate p, q and r using the method of maximum likelihood.

For model (2) we can consider the conditional probability:

$$\psi(s|s_t) = P(X_{t+1} < s | s_t \le X_t < s_t + h)$$

and the conditional density given by:

$$\begin{aligned}g(s|s_t) &= \frac{d}{ds}P\left(X_{t+1} < s | X_t = s_t\right) = \\ &= pq\delta(s-\alpha s_t) + rq\delta(s-\beta s_t) + (q-r)q\delta(s-\gamma s_t) + \\ &+ p^2 g_\xi(s-\alpha s_t)H(s-\alpha s_t) + rpg_\xi(s-\beta s_t)H(s-\beta s_t) + \\ &+ (q-r)pg_\xi(s-\gamma s_t)H(s-\gamma s_t)\end{aligned}$$

when $h \to 0$.

$\delta(.)$ is the Dirac delta function, $H(.)$ is the Heaviside function defined by:

$$H_{(0)}(s-\alpha s_t) = \begin{cases} 1, & s = \alpha s_t \\ 0, & s \ne \alpha s_t \end{cases} ,$$

$$H(s-\alpha s_t) = H_{(0,\infty)}(s-\alpha s_t) = \begin{cases} 1, & s > \alpha s_t \\ 0, & s \le \alpha s_t \end{cases} ,$$

and $g_\xi(.)$ is the density of random variables ξ_t for all t.

An alternative form for the conditional density is:

$$\begin{aligned}g(s|s_t) &= [pq\delta(s-\alpha s_t)]^{H_{(0)}(s-\alpha s_t)}[rq\delta(s-\beta s_t)]^{H_{(0)}(s-\beta s_t)} \cdot \\ &\cdot\ [(q-r)q\delta(s-\gamma s_t)]^{H_{(0)}(s-\gamma s_t)}[p^2 g_\xi(s-\alpha s_t)]^{H(s-\alpha s_t)} \cdot \\ &\cdot\ [rpg_\xi(s-\beta s_t)]^{H(s-\beta s_t)}[(q-r)pg_\xi(s-\gamma s_t)]^{H(s-\gamma s_t)}.\end{aligned}$$

Having observed $(X_2, \ldots, X_{n+1})$ and fixed $X_1 = s_1$ from the model (2) we can estimate the parameters p, q, r of the model (2) using conditional likelihood function:

$$L(p,r) = \prod_{t=1}^{n} g(s_{t+1}|s_t)\,.$$

For fixed α, β, γ the maximum likelihood estimators of p, q and r are:

$$\hat{p} = \frac{A_1 + 2A_4 + A_5 + A_6}{2n}$$

$$\hat{r} = \frac{(A_2 + A_5)(A_1 + 2A_2 + 2A_3 + A_5 + A_6)}{(A_2 + A_3 + A_5 + A_6)2n}$$

$$\hat{q} = 1 - \hat{p} = \frac{A_1 + 2A_2 + 2A_3 + A_5 + A_6}{2n},$$

where

$$A_1 = \sum_{t=1}^{n} H_{(0)}(s_{t+1} - \alpha s_t); \qquad A_2 = \sum_{t=1}^{n} H_{(0)}(s_{t+1} - \beta s_t);$$

$$A_3 = \sum_{t=1}^{n} H_{(0)}(s_{t+1} - \gamma s_t); \qquad A_4 = \sum_{t=1}^{n} H_{(0,\infty)}(s_{t+1} - \alpha s_t);$$

$$A_5 = \sum_{t=1}^{n} H_{(0,\infty)}(s_{t+1} - \beta s_t); \qquad A_6 = \sum_{t=1}^{n} H_{(0,\infty)}(s_{t+1} - \gamma s_t).$$

If the exponential distribution with mean λ^{-1} is used for the sequence $\{\xi_t\}$ then the maximum likelihood estimator of λ is:

$$\hat{\lambda} = \frac{\sum_{t=1}^{n}[H_{(0,\infty)}(s_{t+1} - \alpha s_t) + H_{(0,\infty)}(s_{t+1} - \beta s_t) + H_{(0,\infty)}(s_{t+1} - \gamma s_t)]}{S}$$

where

$$\begin{aligned} S \;&=\; \sum_{t=1}^{n} \Big[(s_{t+1} - \alpha s_t)H_{(0,\infty)}(s_{t+1} - \alpha s_t) + \\ &+\; (s_{t+1} - \beta s_t)H_{(0,\infty)}(s_{t+1} - \beta s_t) + \\ &+\; (s_{t+1} - \gamma s_t)H_{(0,\infty)}(s_{t+1} - \gamma s_t)\Big]. \end{aligned}$$

2 A moving-average process

Let us consider a first order moving-average process given by

$$X_{t+1} = \begin{cases} \alpha\xi_{t+1}, & \text{w.p. } p_1 q_1 \\ \beta\xi_{t+1}, & \text{w.p. } q_1^2 \\ \alpha\xi_{t+1} + \xi_t, & \text{w.p. p}_1^2 \\ \beta\xi_{t+1} + \xi_t, & \text{w.p. } p_1 q_1 \end{cases} \tag{4}$$

where ξ_t are i.i.d. random variables with exponential $\varepsilon(\lambda)$ distribution (with mean λ^{-1}), and p_1, q_1 are probabilities with $p_1 + q_1 = 1$ and $0 < \alpha < \beta < 1$. We assume X_t is independent of ξ_s, for all $s > t$.

Using Laplace transform of (4) we can derive:

$$EX = \frac{p_1\alpha + q_1\beta + p_1}{\lambda}\,.$$

Let's denote $\hat{p}_1$ and $\hat{q}_1$ the estimators of p_1 and q_1 produced by the method of moments (for fixed α, β). Then we have

$$\hat{p}_1 = \frac{\lambda\bar{X}_n - \beta}{\alpha - \beta + 1}\,,$$

and

$$\hat{q}_1 = 1 - \hat{p}_1 = \frac{-\lambda\bar{X}_n + \alpha + 1}{\alpha - \beta + 1}\,,$$

where $\bar{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ for given sample $(X_1, \ldots, X_n)$. It can be easily shown that $\hat{p}_1$ and $\hat{q}_1$ are consistent estimators for $n \to \infty$.

For model (4) we can also consider the conditional probability:

$$\psi(s|s_t) = P(X_{t+1} < s | s_t \le X_t < s_t + h).$$

We can not give the maximum likelihood estimators of p_1 and q_1 here. In one of 16 ($4 \cdot 4$ combinations in(4)) different cases (where, for example $X_{t+1} = \alpha\xi_{t+1} + \xi_t$ and, for example, $X_t = \alpha\xi_t + \xi_{t-1}$) we have very complicated form of the conditional density:

$$g(s|s_t) = \frac{\lambda(\alpha - 1)\exp(-\lambda\frac{s}{\alpha})(\exp(-\lambda\frac{s_t}{\alpha})\exp(\lambda\frac{s_t}{\alpha^2}) - \exp(-\lambda s_t))}{(\alpha^2 - \alpha + 1)(\exp(-\lambda\frac{s_t}{\alpha}) - \exp(-\lambda s_t))}$$

for $s > \frac{s_t + h}{\alpha}$ and $h \to 0$. Then it is difficult to derive the likelihood and estimate unknown parameters.

Some other aspects of such a process are under investigation.

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References

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