Measures of noncompactness on uniform spaces – the axiomatic approach

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Abstract

The theory of measures of noncompactness has many applications in Topology, Functional analysis and Operator theory. There are many nonequivalent definitions of this notion on metric and topological spaces. First of them was introduced by Kuratowski in 1930. In this paper[1] we consider one axiomatic approach to this notion which includes the most important classical definitions. We also give some new properties of the inner Hausdorff measure of noncompactness on metric linear spaces.

The theory of measures of noncompactness has many applications in Topology, Functional analysis and Operator theory (see [1], [6], [9] and [10]).

By $\mathcal{P}(X)$ we denote the set of all subsets of a set X and by $\mathrm{diam}(A)$ the diameter of the set A.

We introduce the following definition:

Definition 1 Let X be a uniform space. A regular measure of noncompactness on X is an arbitrary function $\phi : \mathcal{P}(X) \to [0, \infty]$ which satisfies the following conditions:

1) $\phi(A) = \infty$ if and only if the set A is unbounded;

2) $\phi(A) = \phi(\overline{A})$;

3) from $\phi(A) = 0$ it follows that A is a totally bounded set;

4) if X is a complete space, and if $\{B_n\}_{n\in N}$ is a sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $n \in N$ and $\lim_{n\to\infty} \phi(B_n) = 0$, then $K = \bigcap_{n\in N} B_n$ is a nonempty compact set;

5) from $A \subseteq B$ it follows $\phi(A) \leq \phi(B)$.

From the next examples we can see that the conditions in the above definition are logically independent.

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Example 1 Let $X = R$, $d(x, y) = |x - y|$,

$$f(x) = \begin{cases} 0 & x < 0, \\ [x] & x \geq 0. \end{cases}$$

The function $\psi : \mathcal{P}(R) \to [0, \infty]$ defined by:

$$\psi(A) = \sup_{x \in A} f(x)$$

satisfies the conditions 1), 3), 4) and 5) but not the condition 2). If $A = [0, 1)$, then $\psi(A) = 0$ and $\psi(\overline{A}) = 1$.

Example 2 Let $X = R$ and $d(x, y) = |x - y|$. The function $\psi : \mathcal{P}(R) \to [0, \infty]$ defined by:

$$\psi(A) = \begin{cases} \inf_{x \in A} |x| & \text{if } A \text{ is bounded,} \\ \infty & \text{if } A \text{ is unbounded} \end{cases}$$

satisfies 1), 2), 3) and 4) but not the condition 5).

Example 3 Let $X = R$ and

$$d(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$$

The function $\psi : \mathcal{P}(R) \to [0, \infty]$ defined by:

$$\psi(A) = \begin{cases} \sup_{x \in A} e^{-x} & A \text{ is infinite,} \\ 0 & A \text{ is finite} \end{cases}$$

satisfies 2), 3) and 5) but does not satisfy the conditions 1) and 4). If $B_n = [n, \infty)$, then $\cap_{n \in N} B_n = \emptyset$.

Proposition 1 *Let X be a metric space and let $\phi : \mathcal{P}(X) \to [0, \infty]$ satisfy the conditions* 1)-3) *and* 5) *of Definition 1 and the condition: for each $A \subseteq X$ and $x \in X$ $\phi(A \cup \{x\}) = \phi(A)$. Then ϕ is a regular measure of noncompactness on X.*

Proof. Let X be a complete space, and let $\{B_n\}_{n \in N}$ be a sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $n \in N$ and $\lim_{n \to \infty} \phi(B_n) = 0$. We can choose elements $x_n \in B_n$, $n \in N$. Let us consider the set $D = \{x_n : n \in N\}$. Then we have

$$\phi(D) = \phi(\cup_{n=1}^{\infty} \{x_n\} = \phi(\cup_{n=k}^{\infty} \{x_n\} \leq \phi(B_k).$$

So $\phi(D) = 0$. All the limits of the convergent subsequences of $\{x_n\}$ which satisfy $x_n \in B_n$ are contained in $K = \bigcap_{n \in N} B_n$. So K is a nonempty compact set. ■

Definition 2 Let X be a uniform space. Measure of noncompactness on X is an arbitrary function $\phi : \mathcal{P}(X) \to [0, \infty]$, which satisfies the following conditions:

1) $\phi(A) = \phi(\overline{A})$;

2) there exist regular measures of noncompactness ϕ_- i ϕ_+ on X such that for any $A \subseteq X$, $\phi_-(A) \leq \phi(A) \leq \phi_+(A)$.

Proposition 2 *Let X be a uniform space and ϕ a measure of noncompactness on X. Then ϕ satisfies the conditions* 1)–4) *of Definition 1.*

Proof. 1): From unboundness of A it follows $\phi_-(A) = \infty$, which implies $\phi(A) = \infty$. If $\phi(A) = \infty$ then $\phi_+(A) = \infty$ which implies unboundness of A.

2): It follows from the definition of ϕ.

3): Totally boundness of A implies $\phi_+(A) = 0$ and so $\phi(A) = 0$. From $\phi(A) = 0$ it follows $\phi_-(A) = 0$ which implies totally boundness of A.

4): From $\lim_{n\to\infty} \phi(B_n) = 0$, it follows $\lim_{n\to\infty} \phi_-(B_n) = 0$, which implies that $K = \bigcap_{n\in N} B_n$ is a nonempty compact set. ∎

The most important examples of measures of noncompactness are:

Definition 3 (classical measures of noncompactness) Let (X, d) be a pseudometric space. Then:

1) Kuratowski's measure

$$\alpha(A) = \inf\{r > 0 : A \subseteq \bigcup_{i=1}^{n} S_i, S_i \subseteq X,\ \mathrm{diam}(S_i) < r, 1 \leq i \leq n, n \in N\};$$

2) Hausdorff's measure

$$\chi(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{-net in } X\};$$

3) inner measure of Hausdorff

$$\chi_i(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{-net in } A\};$$

4) Istratescu's measure

$$I(A) = \inf\{\varepsilon > 0 : A \text{ contains no infinite } \varepsilon\text{-discrete set in } A\}.$$

Relations between this functions are given by the following inequalities, which are obtained by Danes [7]:

$$\chi(A) \leq \chi_i(A) \leq I(A) \leq \alpha(A) \leq 2\chi(A).$$

The functions α, χ and I are regular measures of noncompactness (see Banás and Goebel [1], Rakočević [10]), and the function χ_i is measure of noncompactness which follows from the Danes inequality. χ_i is not regular because it does not satisfy the condition 5) of Definition 1. I. Arandjelović [2] defined measure of noncompactness on uniform spaces indenticaly as we defined regular measure of noncompactness. We think that any axiomatic definition of this notion must include the function χ_i which is important in applications (see for example O. Hadžić [8]).

In the next proposition we give two common properties of classical measures of noncompactness on metric linear spaces.

Proposition 3 *If Q, Q_1 and Q_2 are bounded subsets of a metric linear space X, $x \in X$ and $\gamma \in \{\alpha, \chi, \chi_i, I\}$, then:*

1) $\gamma(Q_1 + Q_2) \le \gamma(Q_1) + \gamma(Q_2)$;

2) $\gamma(x + Q) = \gamma(Q)$.

Proof. For $\gamma = \alpha$ see [5], for $\gamma = \chi$ see [4] and for $\gamma = I$ see [3]. Let $\delta > 0$ be an arbitrary positive real number. From

$$Q_1 + Q_2 \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} [B(x_i, \chi_i(Q_1) + \delta) + B(y_j, \chi_i(Q_2) + \delta)]$$

it follows that $z \in Q_1 + Q_2$ implies that there exist $z_1 \in Q_1$ and $z_2 \in Q_2$ such that $z = z_1 + z_2$, $d(z_1, x_i) < \chi_i(Q_1) + \delta$ and $d(z_2, y_j) < \chi_i(Q_2) + \delta$, for some i, j $(1 \le i \le n;\ 1 \le j \le m)$. Since

$$\begin{aligned} d(z, x_i + y_j) &= d(z_1 + z_2, x_i + y_j) = d(z_1 - x_i, y_j - z_2) \\ &\le d(z_1 - x_i, 0) + d(z_2 - y_j, 0) = d(z_1, x_i) + d(z_2, y_j) \\ &\le \chi_i(Q_1) + \chi_i(Q_2) + 2\delta, \end{aligned}$$

when $\delta \to 0^+$, we have $\chi_i(Q_1 + Q_2) \le \chi_i(Q_1) + \chi_i(Q_2)$.

From 1) we have $\chi_i(x + Q) \le \chi_i(\{x\}) + \chi_i(Q) = \chi_i(Q)$, which implies

$$\chi_i(Q) = \chi_i(-x + x + Q) \le \chi_i(x + Q).$$

So $\chi_i(x + Q) = \chi_i(Q)$. ∎

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