# $\mathcal{O}$–regular variability and power series

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**Abstract**

In this paper[1] we prove some theorems of Abelian and Tauberian type for power series.

## 1 Introduction

In the papers [5], [1] etc, J. Karamata and V. Avakumović have founded the theory of $\mathcal{O}$–regularly varying mappings (functions and sequences), which very soon became a very developed theory. In particular, this theory has found the applications in many other areas of mathematics.

**Definition 1** A sequence $(c_n)$ $(n \in \mathbb{N}_0)$, $c_0 = 0$ and $c_n > 0$ $(n \in \mathbb{N})$ is called *$\mathcal{O}$–regularly varying* if

$$k_c(\lambda) = \overline{\lim_{n \to +\infty}} \frac{c_{[\lambda = n]}}{c_n} < +\infty$$

for every $\lambda > 0$. The class of all $\mathcal{O}$–regularly varying sequences is denoted $ORV$.

The class $ORV$ has many applications in the asymptotic analysis, and in particular in the Fourier analysis (see e.g. [7], [4], [8], [9]).

**Definition 2** A function $f : [A, +\infty) \mapsto (0, +\infty)$ $(A > 0)$ is called *$\mathcal{O}$–regularly varying* if it is measurable and

$$k_f(\lambda) \overline{\lim_{x \to +\infty}} \frac{f(\lambda x)}{f(x)} < +\infty$$

for all $\lambda > 0$. The class of all $\mathcal{O}$–regularly varying functions is also denoted $ORV$.

**Lemma 1.1** *If $(c_n)$, $c_0 = 0$, $c_n > 0$ $(n \in \mathbb{N})$ is a nondecreasing sequence, then the next statements are equivalent:*
*(a)* $(c_n) \in ORV$*;*
*(b)* $f(x) = c_{[x]} \in ORV$ *for* $x \geq 1$.

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**Proof.** $(a) \Rightarrow (b)$: If $(c_n) \in ORV$, then $f(x) = c_{[x]}$ $(x \geq 1)$ is a piecewise constant function, which is thus measurable on the interval $[0, +\infty)$, $f(x) > 0$ for all $x \geq 1$, and $f(x) = 0$ for all $x \in [0, 1)$. Next we have that

$$\overline{\lim_{n\to+\infty}} \sup_{\lambda\in[a,b]} \frac{c_{[\lambda n]}}{c_n} = \overline{\lim_{n\to+\infty}} \frac{c_{[bn]}}{c_n} = k_c(b) < +\infty$$

for every fixed interval $[a, b] \subset (0, +\infty)$. Since $(\lambda\, x)/[\lambda\, [x]] \in [1, 2]$ for every $\lambda$ and all sufficiently large $x$, there is a $\lambda > 0$ such that

$$\begin{aligned} \overline{\lim_{x\to+\infty}} \frac{f(\lambda\, x)}{f(x)} &= \overline{\lim}_{x\to+\infty} \tfrac{c_{[\lambda\, x]}}{c_{[x]}} \leq \\ &\leq \overline{\lim}_{x\to+\infty} \tfrac{c_{[\lambda\, [x]]}}{c_{[x]}} \cdot \overline{\lim}_{x\to+\infty} \tfrac{c_{[\lambda\, x]}}{c_{[\lambda\, [x]]}} \leq \\ &\leq k_c(\lambda) \cdot k_c(2) < +\infty. \end{aligned}$$

This proves that $f \in ORV$.

$(b) \Rightarrow (a)$ is trivial since $k_c(\lambda) < k_f(\lambda)$ for every $\lambda > 0$. ∎

# 2 Main result

In this section we shall prove two theorems of Abelian and Tauberian type for power series, in which a central role have the sequences from the class $ORV$.

**Theorem 2.1** (a) (Abelian statement) *For any nondecreasing sequence* $(c_n) \in ORV$*, and the corresponding function*

$$f(t) = \sum_{n=1}^{\infty} (c_n - c_{n-1})\, t^n, \qquad 0 < t < 1,$$

*the function* $g(x) = f(1 - 1/x)$ $(x \geq 2)$ *belongs to the class* $ORV$*.*

(b) (Tauberian statement) *If* $(c_n)$ *is a nondecreasing sequence of positive numbers,* $c_0 = 0$*,* $f(t)$ *and* $g(x)$ *are defined as above, and* $g(x) \in ORV$*, then* $(c_n) \in ORV$*.*

**Proof.** (a) The function $\alpha(x) = c_{[x]}$ $(x \geq 1)$ is piecewise constant, nondecreasing, so by Lemma 1.1 $\alpha(x) \in ORV$. Here $\alpha(x) = 0$ for every $x \in [0, 1)$. Consider the function

$$\hat{\alpha}(s) = \int_0^{+\infty} e^{-sk}\, d\alpha(k) = \sum_{n=1}^{+\infty} (c_n - c_{n-1}) e^{-sn}.$$

Since $\alpha \in ORV$, the function $\hat{\alpha}(s)$ is defined for every $s > 0$, because by a result from [2] we have that $\hat{\alpha}(1/s) \asymp \alpha(s)$ $(s \to +\infty)$, so $\hat{\alpha}(s)$ is defined for all $s \in (0, \delta)$ $(\delta > 0)$. It is also defined for all $s \in [\delta, +\infty)$ since it is positive and decreasing.

By the same result the function $\hat{\alpha}(1/s)$ $(s>0)$ belongs to the class $ORV$. Taking $t=e^{-s}$ $(s>0)$ we have that

$$\hat{\alpha}(-log\ t)=\sum_{n=1}^{\infty}=(c_n-c_{n-1})\,t^n \qquad (0<t<1).$$

Defining $f(t)=\hat{\alpha}(-log\ t)$ $(0<t<1)$ and $g(t)=f(1-1/t)$ $(1<t<+\infty)$, we have that $g(t)$ is a positive and increasing function on the interval $(1,+\infty)$. Besides, since $\hat{\alpha}(1/s)$ $(s>0)$ is an $ORV$ function, we find that for every $\lambda>1$

$$\begin{aligned}\overline{\lim_{t\to+\infty}}\frac{g(\lambda\,t)}{g(t)} &= \overline{\lim}_{t\to+\infty}\tfrac{f(1-1/\lambda\,t)}{f(1-1/t)}=\\ &=\overline{\lim}_{t\to+\infty}\tfrac{\hat{\alpha}(-log\ (1-1/\lambda\,t))}{\hat{\alpha}(-log\ (1-1/t))}\le\\ &\le\overline{\lim}_{t\to+\infty}\tfrac{\hat{\alpha}\left(\frac{1}{2\,\lambda}(-log\ (1-1/t))\right)}{\hat{\alpha}(-log\ (1-1/t))}=\\ &=\overline{\lim}_{s\to+\infty}\tfrac{\hat{\alpha}\left(\frac{1}{2\,\lambda}\frac{1}{s}\right)}{\hat{\alpha}(1/s)}<+\infty.\end{aligned}$$

In this calculations we used that the function $\hat{\alpha}(s)$ is decreasing for $s>0$, and

$$\frac{log\ (1-1/\lambda\,t)}{log\ (1-1/t)}\in[1/2\,\lambda,1]$$

for all sufficiently large $t$ and the considered $\lambda$. Consequently, the function $g(x)=f(1-1/x)$ $(x\ge 2)$ is $ORV$.

(b) Since the function $g(x)=f(1-1/x)$ $(x\ge 2)$ is $ORV$, we have that

$$\overline{\lim_{x\to+\infty}}\frac{f(1-1/\lambda\,x)}{f(1-1/x)}=\overline{\lim_{x\to+\infty}}\frac{g(\lambda\,x)}{g(x)}<+\infty$$

for all $\lambda>1$. Letting $s=-(log\ (1-1/x))^{-1}$ we find

$$\begin{aligned}\overline{\lim_{x\to+\infty}}\frac{f(1-1/\lambda\,x)}{f(1-1/x)} &= \overline{\lim}_{s\to+\infty}\tfrac{f(1-\frac{e^{-1/s}-1}{\lambda})}{f(e^{-1/s})}=\\ &=\overline{\lim}_{s\to+\infty}\frac{f\left(\frac{1-\frac{e^{1/s}-1}{\lambda}}{e^{-1/\lambda\,s}}\,e^{-1/\lambda\,s}=\right)}{f(e^{-1/s})}>\\ &>\overline{\lim}_{s\to+\infty}\tfrac{f(e^{-1/\lambda\,s})}{f(e^{-1/s})},\quad \lambda>1.\end{aligned}$$

In the previous calculations we used that for any $\lambda>1$

$$\frac{1-\frac{e^{-1/s}-1}{\lambda}}{e^{-1/\lambda\,s}}\to 1 \qquad (s\to+\infty),$$

the function $h(\lambda,s)=(1-\frac{e^{-1/s}-1}{\lambda})e^{1/\lambda\,s}$ $(\lambda>1, s\ge 1/log\ 2)$ is nonincreasing in $s$, and the function

$$f(t)=\sum_{n=1}^{+\infty}(c_n-c_{n-1})\,t^n$$

is nondecreasing on $(0,1)$.

Since $f$ is continuous and positive on $[1/2,1)$, we have that the function $p(s)=f(e^{-1/s})$ $(s\geq 1/log\,2)$ is $ORV$. Defining $\alpha(t)=c_{[t]}$ $(t>0)$, we find that

$$f(e^{-1/s})=\sum_{n=1}^{+\infty}(c_n-c_{n-1})e^{-n/s}=\hat{\alpha}(1/s)=\int_0^{+\infty}e^{-t/s}\,d\alpha(t)\qquad(s\geq 1/log\ 2)$$

is $ORV$. Therefore, a result from [2] gives that the function $\alpha(t)$ is $ORV$, and Lemma 1.1 gives that $(c_n)\in ORV$. ∎

As an immediate consequence, we obtain the next theorem.

**Theorem 2.2** (a) (Abelian statement) *If $(c_n)$ is a nondecreasing ORV sequence, and*

$$f(t)=\sum_{n=1}^{+\infty}c_n\,t^n\qquad(0<t<1),\tag{1}$$

*then $g(x)=f(1-1/x)$ $(x\geq 2)$ is $ORV$.*

(b) (Tauberian statement) *If $(c_n)$ is a nondecreasing sequence of positive numbers, $c_0=0$, $f(t)$ is defined by (1) and $g(x)=f(1-1/x)$ $(x\geq 2)$ is $ORV$, then $(c_n)$ is $ORV$.*

**Proof.** (a) Define the sequence $(d_n)$ such that $d_0=0$ and $d_n=\sum_{k=1}^{n}c_k$. Then $0=d_0<d_1<d_2<\cdots$, and $c_n=d_n-d_{n-1}$ $(n\in\mathbb{N})$. Besides, the function $f(x)=c_{[x]}$ $(x\geq 1)$ is $ORV$. Since it is positive for $x\geq 1$ and nondecreasing in $x$, redefining $f(x)$ with $f(x)=0$ for $x\in[0,1)$ by [2] we have

$$F(x)=\int_0^x f(x)\,dx\qquad(x\geq 1+\delta,\delta>0)$$

is $ORV$. Since

$$F(x)=\sum_{k=1}^{[x]-1}c_k+c_{[x]}(x-[x])\qquad(x\geq 1+\delta)$$

it is easily seen that the sequence $F(n)=\sum_{k=1}^{n-1}c_k$ $(n\geq 2)$ is $ORV$ if we redefine $F(0)=0$ and $F(1)=c>0$. Since

$$\overline{\lim_{n\to+\infty}}\frac{F([\lambda\,n])}{F(n)}<+\infty$$

for every $\lambda>1$, we have that

$$\overline{\lim_{n\to\infty}}\frac{F([\lambda(n+1)])}{F(n+1)}<+\infty$$

for all $\lambda > 1$. Hence the sequence $\phi_n = F(n+1)$ $(n \in \mathbb{N})$, $\phi_0 = 0$ is $ORV$. Moreover we have that $\phi_n = d_n$ $(n \geq 0)$. Finally, Theorem 2.1 gives the statement.

(b) Consider the sequence $(d_n)$ such that $d_0 = 0$ and $d_n = \sum_{k=1}^{n} c_k$ $(n \geq 1)$. Then $0 = d_0 < d_1 < d_2 < \cdots$, and $c_n = d_n - d_{n-1}$ $(n \geq 1)$, and by Theorem 2.1 $(d_n)$ is $ORV$. Consequently, the sequence $(\phi_n)$ defined by $\phi_0 = 0$ and $\phi_n = d_{n+1}$ $(n \in N)$ is also $ORV$. Lemma 1.1 gives that the functions

$$f(x) = \sum_{k=1}^{[x]} c_k \quad , \quad g(x) = \sum_{k=1}^{[x+1]} c_k, \qquad x \geq 1$$

are $ORV$.

Next consider the function $h(x) = \sum_{k=1}^{[x]} c_k + c_{[x]+1}(x - [x])$, $x \geq 1$. It is obviously nondecreasing, positive and continuous for every $x \geq 1$. Since $f(x) \leq h(x) \leq g(x)$ for every $x \geq 1$, we obtain

$$\begin{aligned} \overline{\lim_{x\to+\infty}} \frac{h(\lambda x)}{h(x)} &\leq \overline{\lim}_{x\to+\infty} \tfrac{g(\lambda x)}{f(x)} = \\ &= \overline{\lim}_{x\to+\infty} \tfrac{f(\lambda x+1)}{f(x)} = \\ &= \overline{\lim}_{x\to+\infty} \tfrac{f((\lambda+1/x)x)}{f(x)}. \end{aligned}$$

The last limit superior is finite by the Uniform convergence theorem for the $ORV$ functions on some interval $[\lambda - \delta, \lambda + \delta]$, with a fixed $\lambda > 0$ and some $\delta \in (0, \lambda)$ (see e.g. [2]).

This proves that $h$ is $ORV$. Since $h(x) = \int_0^{x+1} c_{[t]}\, dt$ and $\varphi(t) = c_{[t]}$, $t \geq 1$ is positive and monotone nondecreasing, we find that its left Matuszewska index $k_M(\varphi) \geq 0$, so its left Karamata index $k_K(\varphi) > -\infty$. Since $h$ is $ORV$ we have that the right Karamata index $k^K(h) < +\infty$. By [6] we now have that $k_K(h) \geq k_M(h) \geq 1 + k_M(\varphi) > 0$, what by [2] gives that

$$\varphi(x+1) \asymp \frac{h(x)}{x+1}, \quad x \to +\infty,$$

that is $\varphi(x+1) \asymp h(x)/x$, $x \to +\infty$. This gives that $p(x) = \varphi(x+1)$, $x \geq 1$, is $ORV$; thus $\varphi(x)$ is also $ORV$. Finally, since $(c_n)$ is the restriction of the function $\varphi$ to $\mathbb{N}$, by Lemma 1.1 $(c_n)$ is $ORV$. ■

**Remark.** It is easy to see that all nondecreasing regularly varying sequences ([2]), all $*$–regularly varying sequences ([8]), and all nondecreasing sequences which are restrictions of functions from the Matuszewska class or the class $CRV$ to the set $\mathbb{N}$ ([2] and [3]) are $ORV$, so they satisfy the conditions of the Theorems 2.1(a) and 2.2(a).

**An open question.** Describe all the functions $\varphi(x)$, $x \geq 2$ which can substitute the function $r(x) = 1 - 1/x$, $x \geq 2$, in Theorems 2.1 and 2.2, so that these theorems remain true.

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