A note on the Sorgenfrey line

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Abstract

In this paper,[1] by using Cantor's principle of nested intervals, we give a new and simple proof that the Sorgenfrey line is a topological space of the second Baire category. One application of this result in asymptotic analysis is also given.

The Sorgenfrey line (S_1) is the real line with the topology whose base is the family of intervals

$$[a, b), \ -\infty < a < b < \infty; \ a, b \in \mathbb{R}.$$

This space was introduced by R.H. Sorgenfrey [4] as an example of a normal topological space whose square is not normal. It is a non-metrizable, totally disconnected, paracompact, normal and Lindelöf topological space. The Sorgenfrey plane $S_2 = S_1 \times S_1$ is not a paracompact, normal and Lindelöf space.

A topological space X is of the first Baire category if it is a countable union of nowhere dense subsets of X; otherwise, X is of the second Baire category. It follows from the definition that a topological space X is of the second Baire category, or shorter Baire's space, if and only if the intersection of any countable family of open dense sets in X is dense in X. Classical examples of Baire's spaces are complete metric spaces and localy compact topological spaces. In this note we give a new proof that the Sorgenfrey line is a Baire space.

Proposition 1 *The Sorgenfrey line is a Baire space.*

Proof. Let $(A_n : n \in \mathbb{N})$ be a sequence of sets such that each of them is open and dense in the Sorgenfrey line and $A = \cap_{n \in \mathbb{N}} A_n$. Let a, b $(\infty < a < b < \infty)$ be arbitrary real numbers. We shall prove that the interval $[a, b)$ contains a point of A. The set A_1^c is nowhere dense in S_1 and so there exist real numbers $a_1, b_1 \in S_1$ such that

$$a < \frac{2a+b}{3} < a_1 < b_1 < \frac{a+2b}{3} < b \text{ and } [a_1, b_1) \subseteq A_1.$$

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Continuing this procedure we obtain sequences $\{a_k\}, \{b_k\}$ such that $[a_k, b_k) \subseteq A_k$ and

$$a_k < \frac{2a_k + b_k}{3} < a_{k+1} < b_{k+1} < \frac{a_k + 2b_k}{3} < b_k.$$

From the Cantor principle of nested intervals it follows that there exists only one real number λ such that

$$\lambda \in \bigcap_{k \in \mathbb{N}} [a_k, b_k].$$

So we have that

$$A \cap [a, b) \neq \emptyset$$

for any interval $[a, b)$ which implies that A is dense in S_1. ■

From this proposition it follows that the Sorgenfrey plane S_2 is a Baire space.

A function $f : X \longrightarrow \mathbb{R}$ from a topological space X into the real line is lower semicontinuous at a point $x_0 \in X$ if and only if

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

f is lower semicontinuous on a set $A \subseteq X$ if it has this property at each point of A. $f : X \longrightarrow \mathbb{R}$ is a lower semicontinuous function on X if and only if the set $\{x \in X : f(x) < r\}$ ($\{x \in X : f(x) > r\}$) is open for each $r \in \mathbb{R}$. The least upper bound of a family of continuous functions on a Baire space is lower semicontinuous, and the set of points in which it is bounded is open and dense in this space ([1], [3]).

The next statement is an extension of Theorem 2 from the paper [2].

Corollary 1 *Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers such that*

$$\limsup_{m,n \to \infty} \frac{a_{[\alpha m]}}{b_{[\beta n]}}$$

is finite for each $\alpha, \beta > 0$, and let I_1, I_2 be compact intervals in $(0, \infty)$. Then

$$\limsup_{m,n \to \infty} \sup_{\alpha \in I_1, \beta \in I_2} \frac{a_{[\alpha m]}}{b_{[\beta n]}} < \infty.$$

Proof. The function $R_{m,n} : [0, \infty) \to [0, \infty)$ defined by:

$$R_{m,n}(\alpha, \beta) = \frac{a_{[\alpha m]}}{b_{[\beta n]}}$$

is continuous on $[0,\infty)^2$ in the S_2 topology for any $m,n=1,2,\dots$. The family of functions $\{R_{m,n}\}$ is bounded for each $\alpha,\beta\in[0,\infty)$ which implies that the function $r:[0,\infty)^2\to[0,\infty)$ defined by

$$r(\alpha,\beta)=\limsup_{m,n\to\infty}\frac{a_{[\alpha m]}}{b_{[\beta n]}}$$

is lower semicontinuous on $[0,\infty)^2$ in the S_2 topology.

So for any compact interval I with $I^2\subseteq[0,\infty)^2$ there exists an open and dense subset $I'\subseteq I^2$ such that:

$$\sup_{(\alpha,\beta)\in I'} r(\alpha,\beta)<\infty.$$

If $(\alpha,\beta)\in I'$, there exists a sequence $\{(\alpha_m,\beta_n)\}\subseteq I'$ such that $(\alpha_m,\beta_n)\to(\alpha,\beta)$ and $[\alpha_m m]=[\alpha m]$, $[\beta_n n]=[\beta n]$. This implies that $(\alpha,\beta)\in I'$. So I' is a closed set which implies that $I^2=I'$. ■

References

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