Integrated semigroups of unbounded linear operators

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Abstract

Appropriate assumptions on a family of unbounded linear operators in a Banach space E ensure that this family forms integrated semigroups on a suitable net of subspaces of E. The structure of such family analized. Infinitesimal generator related to this family is analyzed. Hille-Yosida type theorems are give for an operator determining an integrated semigroup of unbounded linear operators.

0 Introduction

Arendt invented in [1] and [2] a class of so called integrated semigroups. Since then a huge number of paper has been devoted to the analysis of this class (cf. [3], [5], [6], [9], [15], [16], [19]) as well as of some classes of distribution semigroups ([3], [14], [20]). On the other hand, Huges [8] studied families of unbounded linear operators on a Banach space E generating C_0-semigroups on a suitable nested family of subspaces of E.

In this paper[1] we are interested in a family of unbounded linear operator $(S(t))_{t\geq 0}$ forming integrated semigroups on a nested family of subspaces of E. The significance of the theory of integrated semigroups lies in the fact that their infinitesimal generators need not to be densely defined. This is the main motivation of our investigations. Moreover, the composition low for an integrated semigroup and the fact $S(0)x = 0,\ x \in E$ make the theory more complex than in the case of C_0-semigroups. The construction of a nested family $E_\omega,\ \omega > 0$ demands assumptions on $(S(t))_{t\geq 0}$ which enable us to develope in Section 1 and 2 the similar theory as in [8], related to the structure of basic space and infinitesimal generators $A^\omega,\ \omega > 0$. Example 1 illustrates our approach. We emphasize that $(S(t))_{t\geq 0}$ forms integrated semigroups on every $E_\omega,\ \omega > 0$ while in the case of a semigroup of unbounded linear operators forms of C_0-semigroups the domain in every $E_\omega,\ \omega > 0$, has to be shrinked.

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Hille-Yosida theorems reflect the differences between assumptions related to the infinitesimal generator A generating integrated semigroups on a nested family of Banach spaces and assumptions on A generating C_0-semigroups on a nested family of Banach spaces ([8]). We characterize in Section 2 operators A defined families of unbounded linear operators forming integrated semigroups of bounded linear operators on suitable domains. The fact that operators $A^\omega,\ \omega > 0$, need not to be densely defined enable us to formulate conditions in Theorem 7 which are considerably different from assumptions in the corresponding Theorem 2.27 of [8]. It implies that we can develop the theory of integrated semigroup of unbounded linear operators startings with Banach spaces E_ω and thus, without assumptions (1) and (2) which were the main assumptions of Sections 1-2. But in order to compare the integrated semigroup determined by A and the most extended integrated semigroup which can be constructed by the procedure of Sections 1-2, one has to assume assumptions which can be compared with the ones in cited theorem in [8]. Moreover, one has to prove a theorem on "maximal unicity" similary as in [8]. We note that in Theorem 8 we exclude the condition: For $x \in D_\omega,\ \|\lambda R(\lambda)x - x\|_{R^\omega} \to 0,\ \lambda \to \infty$ appearing in [8] Theorem 2.27, condition (ii) and which is superflous.

1 Integrated semigroup of unbounded operators

Let $(S(t)_{t\geq 0}$ be a family of unbounded linear operators in a Banach space $(E, \|\cdot\|)$. Denote by $D(S(t))$ a domain of $S(t)$ and define a set D to consist of elements $x \in \bigcap_{s,t\geq 0} D(S(s)S(t))$ such that the following condition hold:

(i) $S(0)x = 0, t \to S(t)x, t \in [0,\infty)$ is strongly continuous,

(ii) $S(s)S(t)x = \int\limits_0^s (S(r+t) - S(r))xdr = S(t)S(s)x$, for $s,t \geq 0$.

If $D \neq \{0\}$, then $(S(t))_{t\geq 0}$ is said do be *an integrated semigroup of unbounded linear operators* in E.

A semigroup of unbounded linear operators $(S(t))_{t\geq 0}$ is called *non-degenerate* if $\mathcal{N} = \{x \in D; S(t)x = 0, t \geq 0\} = \{0\}$ and *degenerate*, otherwise. We will assume that $(S(t))_{t\geq 0}$ is non-degenerate.

Let $\omega \in \mathbb{R}^+ = (0,\infty)$. Then

$$E_\omega := \{x \in D; \|x\|_\omega < \infty\},\ \text{where } \|x\|_\omega; = \sup_{t\geq 0} e^{-\omega t}\|S(t)x\|.$$

$\overline{E}_\omega$ denoted the closure of the set E_ω under the norm $\|\cdot\|$ and $S(t)|\overline{E}_\omega$ is the part of $S(t)$ with domain $D(S(t)|\overline{E}_\omega) = \{x \in \overline{E}_\omega; x \in D(S(t))$ and $S(t)x \in \overline{E}_\omega\}$.

Remark 1 One can simply prove

$$\|S(t)x\|_\omega \leq \frac{2e^{\omega t}}{\omega}\|x\|_\omega \text{ and } \|S(s)S(t)x\| \leq \frac{2e^{\omega(s+t)}}{\omega}\|x\|_\omega, x \in E_\omega.$$

We assume additionally:

(1) For every $\omega > 0$ there exists $C_\omega > 0$ such that $\|x\|_\omega \geq C_\omega \|x\|, x \in E_\omega$;

$$S(t)|\overline{E}_\omega \text{ closed in } \overline{E}_\omega, \text{ for } t \geq 0 \text{ and } \omega > 0. \tag{2}$$

This implies $\overline{E}_\omega^{\|\cdot\|_\omega} = E_\omega$ and E_ω is a Banach space (cf. [10]).
For fixed $\omega > 0$ and $\lambda \in \mathbb{C}, \mathcal{R}e\lambda > \omega$, define

$$R^\omega(\lambda) = \lambda \int_0^\infty e^{-\lambda t} S(t)x dt, x \in E_\omega.$$

In general, $R^\omega(\lambda)$ is unbounded in $(E, \|\cdot\|)$. The family $(R^\omega(\lambda))_{Re\lambda>\omega}$ on E_ω satisfies the first resolvent equation. Since $(S(t))_{t\geq 0}$ is non-degenerate it follows that $R^\omega(\lambda)$ is injective.

Theorem 1 *[10] Let $\omega > 0$ and $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$. Then*
a) $R^\omega(\lambda)x \in D(S(t))$ and $S(t)R^\omega(\lambda)x = R^\omega(\lambda)S(t)x, t > 0, x \in E_\omega$.
b) i) $R^\omega(\lambda)(E_\omega) \subset E_\omega$ Moreover,

$$\frac{\omega(Re\lambda - \omega)}{2|\lambda|} \|R^\omega(\lambda)x\|_\omega \leq \|x\|_\omega, x \in E_\omega, \tag{3}$$

ii) For every $x \in E_\omega, \|x\|_{R^\omega} < \infty$, where $\|\cdot\|_{R^\omega}$ is the norm

$$\|x\|_{R^\omega} :\equiv \sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega} \frac{(\lambda-\omega)^{n+1}}{n!} \Big\| \Big(\frac{R^\omega(\lambda)}{\lambda}\Big)^{(n)} x \Big\|, \lambda > \omega,$$

and for every $\omega > 0$, $\frac{\omega C_\omega}{2}\|x\|_{R^\omega} \leq \|x\|_\omega \leq \|x\|_{R^\omega}$,
c)

$$\|R^\omega(\lambda)x\|_{R^\omega} \leq \frac{4\lambda}{\omega^2 C_\omega(\lambda-\omega)} \|x\|_{R^\omega}, x \in E.$$

2 Infinitesimal generator

The family $(R^\omega(\lambda))_{Re\lambda>\omega}$ is a resolvent of a closed linear operator A^ω in $(E_\omega, \|\cdot\|_\omega)$ and $A^\omega = \lambda I - (R^\omega(\lambda))^{-1}$ (I is the identity operator) for $\lambda \in \mathbb{C}, Re\lambda > \omega, D(A^\omega) = Range(R^\omega(\lambda))$. Note, operators $A^\omega, \omega > 0$, are not closed in $(E, \|\cdot\|)$. We have $D(A^{\omega_1}) \subset D(A^{\omega_2})$ or $D(A^{\omega_2}) \subset D(A^{\omega_1})$, where $\omega_1, \omega_2 \in (0, \infty)$.

Denote $D(A) = \bigcup_{\omega>0} D(A^\omega)$. For $x \in D(A)$, let $\omega > 0$ such that $x \in D(A^\omega)$. There exists $y \in E_\omega$ such that $x = R^\omega(\lambda)y$ for $Re\lambda > \omega$. We define $Ax := \lambda x - y$ and we call A *the infinitesimal generator* of the integrated semigroup $(S(t))_{t>0}$. Thus, $Ax = A^\omega x$ for $x \in D(A^\omega)$ and it is easy to prove that this definition does not depend on ω with the property $x \in D(A^\omega)$, i. e. if $x \in D(A^\omega)$, then $Ax = A^\omega x$ ([10]). Clearly A is a linear operator on $D(A)$. Theorem 1 implies

Corollary 1 *a) For $x \in E_\omega$, the resolvent equention $(\lambda I - A)y = x, Re\lambda > \omega$, has a unique solution belonging to E_ω and $y = R^\omega(\lambda)x$.*

b) Let $\omega > 0$. Then, for $t \geq 0, S(t)(D(A^\omega)) \subset D(A^\omega)$ and $S(t)A^\omega x = A^\omega S(t)x, x \in D(A^\omega)$.

c) If $x \in D(A)$, then there exists $\omega' > 0$ such that $R^\omega(\lambda)Ax = AR^\omega(\lambda)x$, $\omega \geq \omega'$.

Theorem 2 *a) For $x \in D(A), t \to S(t)x, t \geq 0$, is a differentiable function on E_ω with respect to $\|\cdot\|$ and $S'(t)x - x = S(t)Ax$ or equivalently*

$$S(t)x - tx = \int_0^t S(s)Axds. \tag{4}$$

b) If $x \in E' = \bigcup_{\omega>0} E_\omega$, then $\int_0^t S(s)xds \in D(A)$ and $A\int_0^t S(s)xds = S(t)x - tx$.

Proof. a) If $x \in D(A)$, then $x \in D(A^\omega)$ and $Ax = A^\omega x$ for some $\omega > 0$. Fix $\omega > 0$ and prove (4) for A^ω. Let $x \in D(A^\omega)$. Then $x = R^\omega(\lambda)y$ for some $y \in E_\omega, Re\lambda > \omega$ and $A^\omega x = \lambda x - y$. We have

$$\frac{S(t+h)x - S(t)x}{h} = \frac{\lambda}{h}\Big(S(t+h)\int_0^\infty e^{-\lambda s}S(s)yds - S(t)\int_0^\infty e^{-\lambda s}S(s)yds\Big) \tag{5}$$

$$= \frac{\lambda}{h}\Big(\int_0^\infty e^{-\lambda s}\int_{t+s}^{t+h+s} S(v)ydvds - \frac{1}{\lambda}\int_t^{t+h} S(r)ydr\Big)$$

Fubini's theorem implies

$$\int_0^\infty e^{-\lambda s}\int_{t+s}^{t+h+s} S(v)ydvds = \tag{6}$$

$$= -\frac{1}{\lambda}\int_t^{t+h} e^{-\lambda(v-t)}S(v)ydv + \frac{1}{\lambda}\int_t^{t+h} S(v)ydv$$

$$+\frac{e^{\lambda(t+h)}}{\lambda}\int_{t+h}^\infty e^{-\lambda v}S(v)ydv - \frac{e^{\lambda t}}{\lambda}\int_{t+h}^\infty e^{-\lambda v}S(v)ydv.$$

Then (5) and (6) imply

$$\frac{S(t+h)x - S(t)x}{h} = \frac{e^{\lambda h}-1}{h}e^{\lambda t}\int_0^\infty e^{-\lambda v}S(v)ydv$$

$$-\frac{e^{\lambda(t+h)}}{h}\int_0^{t+h} e^{-\lambda v}S(v)ydv+\frac{e^{\lambda t}}{h}\int_0^t e^{-\lambda v}S(v)ydv.$$

Now, by letting $h\to 0$, we obtain

$$S'(t)x=e^{\lambda t}x-f'(t), \tag{7}$$

where $f(t)=e^{\lambda t}\int_0^t e^{-\lambda v}S(v)ydv$. Differentating f, it follows

$$f'(t)=\lambda e^{\lambda t}\int_0^t e^{-\lambda v}S(v)ydv+e^{\lambda t}e^{-\lambda t}S(t)y$$

and (7) implies

$$S'(t)x=e^{\lambda t}x-\lambda e^{\lambda t}\int_0^t e^{-\lambda v}S(v)ydv-S(t)y. \tag{8}$$

Therefore

$$e^{\lambda t}x-\lambda e^{\lambda t}\int_0^t e^{-\lambda v}S(v)ydv=\lambda e^{\lambda t}\Big(\int_0^\infty e^{-\lambda v}S(v)ydv- \tag{9}$$

$$-\int_0^t e^{-\lambda v}S(v)ydv\Big)=x+\lambda S(t)\int_0^\infty e^{-\lambda p}S'(p)ydp.$$

Integrating by parts we have

$$\int_0^\infty e^{-\lambda p}S'(p)ydp=e^{-\lambda p}S(p)y|_0^\infty+\lambda\int_0^\infty e^{-\lambda p}S(p)ydp=x \tag{10}$$

Using (8), (9) and (10), we obtain

$$S'(t)x=x+\lambda S(t)x-S(t)y \text{ and } S'(t)x=x+S(t)A^\omega x.$$

For $x\in D(A^\omega), Ax=A^\omega x$ and $S'(t)x=x+S(t)Ax$. Integrating, we have

$$S(t)x-tx=\int_0^t S(s)Axds.$$

b) Let $x \in E_\omega$. Then $S(s)x \in E_\omega$ and

$$\int_0^t S(s)xds = \int_0^t S(s)(\lambda I - A^\omega)R^\omega(\lambda)xds$$

$$= \lambda R^\omega(\lambda) \int_0^t S(s)xds - S(t)R^\omega(\lambda)x + tR^\omega(\lambda)x.$$

This implies $\int_0^t S(s)xds \in D(A^\omega)$ and since $D(A) = \bigcup_{\omega>0} D(A^\omega)$, we have $\int_0^t S(s)xds \in D(A)$. By using

$$(\lambda I - A^\omega) \int_0^t S(s)xds = (\lambda I - A^\omega)\Big(\lambda R^\omega(\lambda) \int_0^t S(s)xds-$$

$$-R^\omega(\lambda)S(t)x + tR^\omega(\lambda)x\Big) = \lambda \int_0^t S(s)xds - S(t)x + tx,$$

it follows $A^\omega \int_0^t S(s)xds = S(t)x - tx$ and $A \int_0^t S(s)xds = S(t)x - tx$ because $A = A^\omega$ on $D(A^\omega)$.

Note that Theorem 2 a) implies if $x \in D(A^n)$ and $t \geq 0$, then

$$S^{(n)}(t)x = A^{n-1}x + S(t)A^n x, n \in \mathbb{N}.$$

Let $\omega > 0$ and set

$$D(A_1^\omega) = \left\{ x \in E_\omega \;\middle|\; \begin{array}{l} \text{(i) } S(t)x \text{ is differentiable for } t \geq 0 \text{ with respect to } \|\cdot\|, \\ \text{(ii) } \exists y \in E_\omega \text{ such that } S'(t)x - x = S(t)y \end{array} \right\}.$$

For $x \in D(A_1^\omega)$ define $A_1^\omega x := y$, where y satisfies (ii).

Let $D(A_1) = \bigcup_{\omega>0} D(A_1^\omega)$ and for $x \in D(A_1^\omega)$ we define $A_1 x := A_1^\omega x$. Then A_1 is well defined because S is nondegenerate on E_ω. Moreover, $A_1^\omega = A_1|E_\omega$ where $A_1|E_\omega$ is defined by $D(A_1|E_\omega) = \{x \in E_\omega : x \in D(A_1) \text{ and } A_1 x \in E_\omega\}$.

Theorem 3 *Let A be the infinitesimal generator of the integrated semigroup of unbounded linear operators $(S(t))_{t\geq 0}$. Then $A = A_1$.*

Proof. We show that for $\omega > 0, A^\omega = A_1^\omega$. For this it sufficies to prove that $R^\omega(\lambda) = R(\lambda, A^\omega)$ is also the resolvent of A_1^ω.

Theorem 2 implies that $D(A^\omega) \subset D(A_1^\omega)$ for if $x \in D(A^\omega)$ and $y = A^\omega x$, we have $S'(t)x - x = S(t)A^\omega x, y \in E_\omega$ and $S(t)x$ is differentable. This implies $A^\omega x = A_1^\omega x$ and $A^\omega \subset A_1^\omega$. Now, if $x \in E_\omega$, then $R^\omega(\lambda)x \in D(A^\omega) \subset D(A_1^\omega)$, and for $Re\lambda > \omega$.

$$(\lambda I - A_1^\omega)R^\omega(\lambda)x = (\lambda I - A^\omega)R^\omega(\lambda)x = x.$$

We claim that $R^\omega(\lambda)x$ is a unique of the resolvent equation $(\lambda I - A_1^\omega)y = x$, where $x \in E_\omega, Re\lambda > \omega$. Thus $R^\omega(\lambda)$ will be the resolvent of A_1^ω. Indeed, suppose $\lambda y - A_1^\omega y = 0$ for some $y \in D(A_1^\omega)$ and $y \neq 0$. Then, for $t \geq 0$.

$$\lambda S(t)y - S(t)A_1^\omega y = 0 \tag{11}$$

where $S(t)A_1^\omega y$ exists since $A_1^\omega y \in E_\omega$. But $y \in D(A_1^\omega)$, so that $S(t)y$ is differentiable for $t \geq 0$ and

$$S'(t)y - y = S(t)A_1^\omega y. \tag{12}$$

From (11) and (12) we have $S'(t)y - y = \lambda S(t)y$ for fixed $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$. This implies $S(t)y + \frac{1}{\lambda}y = ke^{\lambda t}y$. Since $y \neq 0$ and $S(0) = 0$ we have $k = \frac{1}{\lambda}$ and $S(t)y = \frac{1}{\lambda}(e^{\lambda t} - 1)y$ is the unique solution (for fixed $\lambda \in \mathbb{C}$ and $Re\lambda > \omega$). Then

$$\|y\|_\omega = \sup_{t\geq 0} e^{-\omega t}\|S(t)y\| = \sup_{t\geq 0} e^{-\omega t}\|\frac{1}{\lambda}(e^{\lambda t} - 1)y\| \geq$$

$$\geq \frac{1}{|\lambda|}\sup_{t\geq 0} e^{-\omega t}(e^{Re\lambda t} - 1)\|y\| = \infty$$

and this is in contradiction with $y \in E_\omega$. Thus $y = 0$ is a unique solution of (11) and $R^\omega(\lambda) = R(\lambda, A_1^\omega)$. This implies $A^\omega = A_1^\omega$ and $A = A_1$.

Example 1 Let $E = L^2(\mathbb{R}^n)$ with the usual L^2 norm $\|\cdot\|_2$ and $p(D) = \sum_{\alpha=0}^{m} a_\alpha D^\alpha, a_\alpha \in \mathbb{C}, |\alpha| \leq m$, where

$$D^\alpha = \frac{1}{i^{|\alpha|}}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}.$$

Not $\widehat{p(D)f}(x) = p(x)\widehat{f}(x), x \in \mathcal{R}$, where $\widehat{f}$ is the Fourier transform of f.

Define a family of operators $(T_t)_{t\geq 0}$ on $D(T_t) = \{f; e^{tp(x)}\widehat{f} \in L^2(\mathbb{R}^n), t \geq 0\}$ by $\widehat{T_t f(x)} = e^{tp(x)}\widehat{f}(x)$. Define $D = \{f; e^{tp(x)}\widehat{f} \in L^2(\mathbb{R}^n)$ for every $t \geq 0\}$. We have.

$$(i) \quad T_0 f = f. \quad (ii) T_t T_s f = T_{t+s} f, t, s \geq 0, f \in D.$$

Since $(e^{tp(x)}-1)\widehat{f}(x) \to 0$ as $t \to 0$ (a.e.) in $\mathbb{R}^n$ and $|(e^{tp(x)}-1)\widehat{f}(x)| \leq |e^{p(x)}\widehat{f}(x)+2|$ (a.e) $t \in (0,1)$ we have that $[0,\infty) \ni t \to T_t f$ is strongly continuous on D.

Let

$$\Sigma_\omega = \{f \in D; \sup_{t\geq 0} e^{-\omega t}\|T_t f\|_{L^2} < \infty\},$$

$$\overline{\Sigma}_\omega = \overline{\Sigma}_\omega^{\|\cdot\|} = \{f \in L^2; f \overset{L^2}{=} \lim_{n\to\infty} f_n; f_n \in \Sigma_\omega\},$$

$$D(T_t|\overline{\Sigma}_\omega) = \{f \in \overline{\Sigma}_\omega; f \in D(T_t) \text{ and } T_t f \in \overline{\Sigma}_\omega\}, T_t|\overline{\Sigma}_\omega : D(T_t|\overline{\Sigma}_\omega) \to \overline{\Sigma}_\omega,$$

we have that $(T_t)_{t\geq 0}$ satisfies all the assumptions of a semigroup of unbounded linear operators considered in [8].

Now we consider an integrated semigroup generated by $p(D)$ with additional assumptions.

Let $p(D)$ be an elliptic operator. It is well known that the set of zero $V = \{x; p(x) = 0\}$ is a compact set $K \subset\subset \mathbb{R}^n$. We well assume that

$$Re|p(x)| \to \infty \text{ as } |x| \to \infty.$$

Let $\phi_t(x) = \int\limits_0^t e^{p(x)s}ds, x \in \mathbb{R}^n$. A family of operators defined by

$$S_t f(x) = \phi_t \widehat{f}(x) = \left\{ \begin{array}{ll} \dfrac{e^{tp(x)}-1}{p(x)}\widehat{f}(x), & p(x) \neq 0 \\ & \\ t\widehat{f}(x), & p(x) = 0, \end{array} \right\}.$$

$f \in D(S_t) = \{f \in L^2; S_t f \in L^2\}$, constitutes an integrated semigroup of unbounded operators on $L^2(\mathbb{R}^n)$.

Theorem 4 *[12] Let $\omega > 0$ be fixed. Then $(S^\omega(t))_{t\geq 0} = (S(t)|E_\omega)_{t\geq 0}$ is an exponentially bounded integrated semigroup on $(E_\omega, \|\cdot\|_\omega)$ with the infinitesimal generator A^ω.*

Let $\omega > 0$. Put $\mathcal{D}_\omega := \overline{D(A^\omega)}^{\|\cdot\|_\omega}$ i.e. the closure in the $\|\cdot\|_\omega$ norm; $A^\omega|\mathcal{D}_\omega$ denotes the part of A^ω in $\mathcal{D}_\omega$ with the domain

$$D(A^\omega|\mathcal{D}_\omega) = \{x \in \mathcal{D}_\omega; x \in D(A^\omega) \text{ and } A^\omega x \in \mathcal{D}_\omega\}.$$

Theorem 5 *[12] a) For all $x \in \mathcal{D}_\omega$, $\lim_{\lambda\to\infty} \|\lambda R^\omega(\lambda)x - x\|_\omega = 0$.*

b) $\overline{D((A^\omega)^n)}^{\|\cdot\|_\omega} = \mathcal{D}_\omega, \overline{D((A^\omega)^n)} = \overline{E}_\omega, n \in \mathbb{N}$ and $\overline{D(A)} = \overline{E'}$, where $E' = \bigcup_{\omega>0} E_\omega$.

c) Let $\omega > 0$ be fixed. Then, for $t \geq 0, S(t)(\mathcal{D}_\omega) \subset \mathcal{D}_\omega$ and $(S^\omega(t))_{t\geq 0} = (S(t)|\mathcal{D}_\omega)_{t\geq 0}$ is an exponentially bounded integrated semigroup on $(\mathcal{D}_\omega, \|\cdot\|_\omega)$ with the densely defined infinitesimal generator equals $A^\omega|\mathcal{D}_\omega$.

3 Hille-Yosida type theorems

Let A be the infinitesimal generator of an integrated semigroup of unbounded linear operators $(S(t))_{t\geq 0}$ with assumptions given in Section 1. We need the following result.

Theorem 6 *Let $\omega > 0, t \geq 0,$ and $\gamma > \omega$. Then*

a) For $x \in D((A)^2)$

$$S^\omega(t)x = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R^\omega(\lambda)x\frac{d\lambda}{\lambda}$$

and the integral converges in the $\|\cdot\|_\omega$ norm.

b) Let $x \in D((A^\omega)^3)$ and

$$I_{\omega,t}(x) = \frac{1}{2\pi i}\int\limits_0^t\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda s}R^\omega(\lambda)A^\omega x\frac{d\lambda}{\lambda}ds, t \geq 0.$$

Then the operator $I_{\omega,t}$ is well defined and closable (in the norm $\|\cdot\|$).

c) Let $\tilde{\mathcal{D}}_\omega = \{x \in E_\omega; \|S'(t)x - x\|_\omega \to 0$ as $t \to 0_+\}$. Then $\tilde{\mathcal{D}}_\omega = \mathcal{D}_\omega$.

d) Let $(S(t))_{t\geq 0}$ be a nondegenerate exponentially bounded integrated semigroup with the densely defined infinitesimal generator A. Then $\lim\limits_{\lambda\to\infty}\|\lambda R(\lambda)x - x\| = 0, x \in E$ where

$$R(\lambda)x = \lambda\int\limits_0^\infty e^{-\lambda t}S(t)xdx.$$

Proof. a) The proof follows by using Theorem 6.3.1 in [7].

b) Let $x \in D((A^\omega)^3)$. Then $A^\omega x \in D((A^\omega)^2)$ and by a)

$$S(t)A^\omega x = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R^\omega(\lambda)A^\omega x\frac{d\lambda}{\lambda}, t \geq 0, \gamma > \omega.$$

The integral $\int\limits_0^t S(s)A^\omega xds$ exists because $A^\omega x \in E_\omega$ and $t \to S(t)A^\omega x, t \geq 0,$ is a continuous function. Thus $I_{\omega,t}$ is well defined on $D((A^\omega)^3)$. Let $\{x_n\} \subset D((A^\omega)^3)$ such that $\|x_n\| \to 0$ and $\|I_{\omega,t}(x_n) - y\| \to 0$ as $n \to \infty$ for some $y \in E$. By a) we have

$$S(t)x_n - tx_n = \frac{1}{2\pi i}\int\limits_0^t\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda s}R^\omega(\lambda)A^\omega x_n\frac{d\lambda}{\lambda}ds = I_{\omega,t}(x_n).$$

Observe that $I_{\omega,t}$ is independent of the choice of γ. Then $\|S(t)x_n - tx_n - y\| \to 0$ as $n \to \infty$ and since $S(t)|\overline{E}_\omega$ is a closed linear operator in $\overline{E}_\omega$ (in the norm $\|\cdot\|$) it follows $y = 0$. Thus $I_{\omega,t}$ is closable.

c) For $x \in \tilde{\mathcal{D}}_\omega$ it is sufficient to prove $\|\lambda R^\omega(\lambda)x - x\|_\omega \to 0$ as $\lambda \to \infty$. From

$$e^{-\omega t}\|S(t)(\lambda R^\omega(\lambda)x - x)\| = e^{-\omega t}\|S(t)A^\omega R^\omega(\lambda)x\| = e^{-\omega t}\|S'(t)R^\omega(\lambda)x - R^\omega(\lambda)x\|$$

$$\leq \frac{1}{C_\omega}\|R^\omega(\lambda)\|_\omega\|S'(t)x - x\|_\omega < \frac{1}{C_\omega}\|R^\omega(\lambda)\|_\omega\varepsilon \ \text{ for } 0 \leq t < \delta.$$

Then $\overline{\lim}_{\lambda\to\infty}\|\lambda R^\omega(\lambda)x - x\|_\omega \leq \varepsilon_1$ and $\|\lambda R^\omega(\lambda)x - x\|_\omega \to 0$ as $\lambda \to \infty$. This implies $\tilde{\mathcal{D}}_\omega \subset \mathcal{D}_\omega$.

Conversely, for $x \in D(A^\omega)$ we have $S'(t)x - x = S^\omega(t)A^\omega x$ and such that $t \to S^\omega(t)x, t \geq 0$ is strongly continuous under the norm $\|\cdot\|_\omega$, it follows

$$\|S^{'\omega}(t)x - x\|_\omega \to 0, t \to 0_+. \tag{13}$$

By $\overline{\mathcal{D}(A^\omega)}^{\|\cdot\|_\omega} = \mathcal{D}_\omega$, (13) holds for every $x \in \mathcal{D}_\omega$ Moreover $S(t)|\mathcal{D}_\omega = S^\omega(t)$ and this implies

$$\|S(t)x - x\|_\omega \to 0, t \to 0_+$$

for every $x \in \mathcal{D}_\omega$.

Then $\tilde{\mathcal{D}}_\omega = \mathcal{D}_\omega$.

d) For $x \in D(A)$ and $\lambda > \omega$ we have

$$\|\lambda R(\lambda)x - x\| = \|R(\lambda)Ax\| = \left\|\lambda\int_0^\infty e^{-\lambda t}S(t)Axdt\right\|$$

$$\leq \lambda\int_0^\infty e^{-\lambda t}\|S(t)Ax\|dt = \lambda\int_0^\infty e^{-\lambda t}\|S'(t)x - x\|dt$$

$$= \lambda\int_0^\delta e^{-\lambda t}\|S'(t)x - x\|dt + \int_\delta^\infty e^{-\lambda t}\|S(t)Ax\|dt < \lambda\varepsilon\int_0^\delta e^{-\lambda t}dt$$

$$+\lambda M\int_\delta^\infty e^{(\omega-\lambda)t}\|Ax\|dt = (1 - e^{-\lambda\delta})\varepsilon + \frac{\lambda M}{\lambda - \omega}e^{(\omega-\lambda)\delta}\|Ax\| \to 0$$

as $\lambda \to \infty$ because for $\varepsilon > 0$ exists $\delta > 0$ such that $\|S'(t)x - x\| < \varepsilon$ for $0 \leq t < \delta$. Since $\overline{D(A)} = E$, it holds for $x \in E$.

The question is, when a linear operator in E is an infinitesimal generator of an integrated semigroup of unbounded linear operators?

Theorem 7 *Let E be a Banach space and $(E_\omega)_{\omega>0}$ be a nested family of non - trival subspaces of E such that $\omega_1 \le \omega_2$ implies $E_{\omega_1} \subset E_{\omega_2}$. Let A be a linear operator with the domain and range in $\bigcup_{\omega>0} E_\omega$ and A^ω denote the part of A in E_ω. Assume:*

(i) For every $x \in E_\omega$, the resolvent equation $(\lambda I - A)y = x, \lambda > \omega$ has a unique solution $y = R^\omega(\lambda)x \in E_\omega$.

(ii) $\|x\|_{R^\omega} < \infty, x \in E_\omega$, where $\|x\|_{R^\omega} = \sup\limits_{n\ge 0}\sup\limits_{\lambda>\omega} \left\| \frac{(\lambda-\omega)^{n+1}}{n!} \left(\frac{R^\omega(\lambda)}{\lambda}\right)^{(n)} x \right\|$, and $(E_\omega, \|\cdot\|_{R^\omega})$ is a Banach space.

(iii) There exists $K_\omega > 0$ such that $\|x\|_{R^\omega} \ge K_\omega \|x\|, x \in E_\omega$.

(iv) There exists $M > 0$ such that

$$\left\| \frac{(\lambda-\omega)^{n+1}}{n!} (R^\omega(\lambda))^{(n)} \right\|_{R^\omega} \le M, n \in \mathbb{N}_0, \lambda > \omega.$$

Then there exists a family $(S(t))_{t\ge 0}$ of linear operators in E, in general unbounded, with the domain containing $\bigcup_{\omega>0} E_\omega$ such that all the assertion of Sections 1-2 hold true (except the second part of Theorem 5.b)).

Proof. Let $\omega > 0$. Then $R^\omega(\lambda), \lambda > \omega$ is the resolvent of A^ω. By assumption (iv) $R^\omega(\lambda)$ is bounded on the Banach space $(E_\omega, \|\cdot\|_{R^\omega})$. This implies that A^ω is the closed operator in the $\|\cdot\|_{R^\omega}$ norm topology.

Let $\omega_1 \le \omega_2, x \in E_{\omega_1}$ and $\lambda > \omega_2$. Then $R^{\omega_1}(\lambda)x = R^{\omega_2}(\lambda)x$ because both are solutions of the resolvent equation $(\lambda I - A)y = x$ belonging to E_{ω_2}. Thus $R^{\omega_1}(\lambda) \subset R^{\omega_2}(\lambda)$ whenever $\lambda > \omega_2$ and this implies $A^{\omega_1} \subset A^{\omega_2}$.

If $\omega_1 \le \omega_2$ and $x \in E_{\omega_1}$, then $\|\cdot\|_{R^{\omega_2}} \le \|\cdot\|_{R^{\omega_1}}$ since $R^{\omega_2}(\lambda)x = R^{\omega_1}(\lambda)x$ for $\lambda > \omega_2$.

Condition (iv), Theorem 2.4. and Corollary 2.3 in [9] imply that A^ω is the generator of a locally Lipschitz continuous integrated semigroup $(S^\omega(t))_{t\ge 0}$ on E_ω and $\|S^\omega(t)\|_{R^\omega}$ on E_ω and $\|S^\omega(t)\|_{R^\omega} \le Me^{\omega t}, \omega > 0$. Condition (iii) implies that $\|x\|_\omega := \sup\limits_{t\ge 0} e^{-\omega t}\|S^\omega(t)x\| < \infty, x \in E_\omega$ and that the norms $\|\cdot\|_{R^\omega}$ and $\|\cdot\|_\omega, x \in E_\omega$ are equivalent. In fact we have to use that $(\lambda I - A)^{-1} = \lambda \int\limits_0^\infty e^{-\lambda t} S(t)x dt$ ([2] Corollary 1.2) and the arguments as in the proof of Theorem 1 b) (ii) (cf.[4]).

We define for $x \in \bigcup_{\omega>0} E_\omega, t \ge 0, S(t)x := S^\omega(t)x$, if $x \in E_\omega$. It follows that $S(t)$ is well defined for $t \ge 0$, bacause $\omega_1 \le \omega_2$ implies $E_{\omega_1} \subset E_{\omega_2}$. Since for $\lambda > \omega_2$, both $R^{\omega_1}(\lambda)$ and $R^{\omega_2}(\lambda)$ are defined and equal, it follows

$$\frac{R^{\omega_1}(\lambda)}{\lambda} x = \int\limits_0^\infty e^{-\lambda t} S^{\omega_1}(t) x dt = \frac{R^{\omega_2}(\lambda)}{\lambda} x = \int\limits_0^\infty e^{-\lambda t} S^{\omega_2}(t) x dt.$$

By the uniquenes theorem for the Laplace transform, it follows $S^{\omega_1}(t)x = S^{\omega_2}(t)x$. Thus $S^{\omega_1}(t) \subset S^{\omega_2}(t)$ and $S(t)$ is well defined.

The family of operators $(S(t))_{t\geq 0}$ on $\bigcup_{\omega>0} E_\omega$ satisfies; $S(0) = 0, t \to S(t)x, x \in \bigcup_{\omega>0} E_\omega$, is strongly continuous $(t \geq 0)$ and $S(s)S(t)x = \int\limits_0^s (S(r+t) - S(r))x dr = S(t)S(s)x$ for $s, t \geq 0$.

Now by inspecting the proofs of assertions in Sections 1-2 one can show that these assertions hold true.

Remark 2 The closedness of $S(t)|\overline{E}_\omega$ in $\overline{E}_\omega$ is not considered in the given theorem. It is used in order to prove the completeness of E_ω in the norm $\|\cdot\|_\omega$. Here this is a consequence of equivalence of norms $\|\cdot\|_{R^\omega}$ and $\|\cdot\|_\omega$.

Theorem 8 *[12] Assume the same conditions (i), (ii),(iii) as in Theorem 7 and the following ones:*

(iv) There exists $M > 0$ such that $\left\| \frac{(\lambda-\omega)^{n+1}}{n!} \Big(\frac{R(\lambda)}{\lambda}\Big)^{(n)} \right\|_{R^\omega} \leq M$, $n \in \mathbb{N}_0$, $\lambda > \omega$.

(v) For $\omega > 0, t \geq 0$, the operator

$$I_{\omega,t}(x) = \frac{1}{2\pi i} \int\limits_0^t \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda s} R^\omega(\lambda) A^\omega x \frac{d\lambda}{\lambda} ds, \gamma > \omega,$$

defined on $D((A^\omega)^3)$ is closable with respect to $\|\cdot\|$. Then there exists a family $(S(t))_{t\geq 0}$ of linear operators in E, in general unbounded, on $\bigcup_{\omega>0} \mathcal{D}_\omega$ where $\mathcal{D}_\omega := \overline{D(A^\omega)}^{\|\cdot\|_{R^\omega}}$, such that, $\omega > 0$,

1. $\bigcup_{\omega>0} \mathcal{D}_\omega \subset D$ (D is defined in first section).

2. $S(t)|\overline{E}_\omega$ is closed operator in $\overline{E}_\omega$ for every $t \geq 0 (\omega > 0)$,

3. If $(\tilde{E}_\omega, \|\cdot\|_\omega), \tilde{R}^\omega(\lambda), \lambda > \omega, \tilde{A}^\omega$ and $\tilde{D}_\omega = \overline{D(\tilde{A}^\omega)}^{\|\cdot\|_\omega} (\omega > 0)$ are defined as in Sections 1,2 for $(S(t))_{t\geq 0}$ and if conditions (2) and (3) hold for $(\tilde{E}_\omega, \|\cdot\|_\omega)$, then

$$\mathcal{D}_\omega \subset \tilde{\mathcal{D}}_\omega \text{ and } A^\omega|\tilde{\mathcal{D}}_\omega = \tilde{A}^\omega|\tilde{\mathcal{D}}_\omega \text{ for every } \omega > 0.$$

Theorem 9 *[12] Let A be the infinitesimal generator of an integrated semigroup of unbounded linear operators $(S(t))_{t\geq 0}$. Then A is "maximal unique" in the following sense:*

Assume $\left(E'_\omega\right)_{\omega>0}$ is a nested family of subspaces on E and A' is a linear operator with domain and range in $\bigcup_{\omega>0} E'_\omega = E'$, Let A'^ω denote the part of A' to E'_ω.Assume

(i) for every $x \in E'_\omega$, the resolvent equation $(\lambda I - A')y = x, \lambda > \omega$, has a unique solution in E'_ω equals $R'^\omega(\lambda)x$;

(ii) E'_ω is a Banach space with the norm

$$\|x\|_{R'^\omega} = \sup_{n\geq 0} \sup_{\lambda>\omega} \left\| \frac{(\lambda-\omega)^{n+1}}{n!} \Big(\frac{R'^\omega(\lambda)}{\lambda}\Big)^{(n)} x \right\|, \lambda > \omega$$

and there exists $K_\omega > 0$, *such that* $\|x\|_{R'^\omega} \geq K_\omega \|x\|, x \in E'_\omega$;

(iii) if $\mathcal{D}'_\omega = \overline{D(A'^\omega)}^{\|\cdot\|_{R'^\omega}}$, *then* $\mathcal{D}'_\omega \subset D, S(t)(\mathcal{D}'_\omega) \subset \mathcal{D}'_\omega$ *for* $t \geq 0$ *and* $(S(t)|\mathcal{D}'_\omega)_{t\geq 0}$ *is an integrated semigroup of exponentially bounded linear operators in a Banach space* $(\mathcal{D}'_\omega, \|\cdot\|_{R'^\omega})$ *with an infinitesimal generator* $A'^\omega|\mathcal{D}'_\omega$ *which is densely defined.*

Then $\mathcal{D}'_\omega \subset \mathcal{D}_\omega$ *and* $A^\omega|\mathcal{D}'_\omega = A'^\omega|\mathcal{D}'_\omega$.

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