On the permutation products of torus

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Abstract

In the present paper[1] it is proved that the m-th permutation product of torus $T^{(m)}$ is bundle over the torus T with fibre CP^{m-1}.

1 Introduction

First we give some basic arguments concerning the permutation products on manifolds. We assume everywhere that $m > 1$.

Let M be an arbitrary set. In the Cartesian product M^m we define a relation $\approx$ as follows

$$(x_1, \cdots, x_m) \approx (y_1, \cdots, y_m) \Leftrightarrow$$

there exists a permutation $\theta : \{1, 2, \cdots, m\} \to \{1, 2, \cdots, m\}$ such that

$$y_i = x_{\theta(i)} \qquad (1 \le i \le m).$$

This is a relation of equivalence and the class represented by $(x_1, \cdots, x_m)$ will be denoted by $(x_1, \cdots, x_m)/\approx$ and the set $M^m/\approx$ will be denoted by $M^{(m)}$. The set $M^{(m)}$ is called *permutation product* of M. Note that some authors call it *symmetric product* of M.

If M is a topological space, then $M^{(m)}$ is also a topological space. The space $M^{(m)}$ is introduced quite early [1], but mainly it was studied in [4]. If M is an arbitrary connected manifold and $m > 1$, then it is proved in [1] that

$$\pi_1(M^{(m)}) \cong H_1(M, Z).$$

Another important result [4] is that $(R^n)^{(m)}$ is a manifold only for $n = 2$. Indeed it is proved that if $n \ne 2$ and $m > 1$, then the tangent space is not homeomorphic to the Euclidean space R^{nm} and hence $(R^n)^{(m)}$ is not a manifold. If $n = 2$, then $(R^2)^{(m)} = C^{(m)}$ is homeomorphic to C^m. Indeed, using that C is algebraically closed field, it is obvious that the mapping $\varphi : C^{(m)} \to C^m$ defined by

$$\varphi((z_1, \cdots, z_m)/\approx) = (\sigma_1, \sigma_2, \cdots, \sigma_m)$$

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is a bijection, where σ_i $(1 \le i \le m)$ is the i-th symmetric function of $z_1, \cdots, z_m$, i.e.

$$\sigma_i(z_1, \cdots, z_m) = \sum_{1 \le j_1 < j_2 < \ldots < j_i \le m} z_{j_1} \cdot z_{j_2} \cdots z_{j_i}.$$

The mapping φ is also a homeomorphism. In the paper [2] it is proved that $M^{(m)}$ is a complex manifold if M is 1-dimensional complex manifold. This is essential result for the next section. For example, if M is a sphere, i.e. the complex manifold CP^1, then $M^{(m)}$ is the projective complex space CP^m. Using the permutation products it is easy to see how $M^{(m)} = CP^m$ decomposes into disjoint cells $C^0, C^1, \cdots, C^m$. Let $\xi \in M$. Then we define $(x_1, \cdots, x_m)/\approx \in M_i$ if exactly i of the elements $x_1, \cdots, x_m$ are equal to ξ. Thus

$$M^{(m)} = M_0 \cup M_1 \cup \ldots \cup M_m = (M \setminus \xi)^{(m)} \cup (M \setminus \xi)^{(m-1)} \cup \cdots \cup (M \setminus \xi)^{(0)} =$$

$$= C^{(m)} \cup C^{(m-1)} \cup \cdots \cup C^{(0)} = C^m \cup C^{m-1} \cup \cdots \cup C^0.$$

This theory about permutation products has an important role in the theory of the topological commutative vector valued groups [3].

At the end of his Ph.D. thesis, Wagner [4] has proved the following theorem concerning the permutation product $T^{(2)}$ of the torus $T = S^1 \times S^1$.

Theorem 1.1 *The permutation product $T^{(2)}$ is a bundle over T and fibre the sphere S^2.*

In this paper we generalize this theorem, proving that the permutation product $T^{(m)}$ $(m > 1)$ is a bundle over T with fibre CP^{m-1}.

2 Main result

Before we prove the main theorem we give some remarks which naturally yield to the required theorem.

We will consider the set C of complex numbers as pairs of real numbers and the complex zero will be denoted simply by 0. Let us consider the torus T as $C/Z \times Z$, i.e. $T = C/\sim$, where $z \sim w$ if and only if $z - w = (u, v)$ for $u, v \in Z$. Then we define a mapping $\varphi : T^{(m)} \to T$ by $\varphi((z_1, \cdots, z_m)/\approx) = z_1 + \cdots + z_m$. It makes $T^{(m)}$ bundle over T and the fibre we denote by M_{m-1}. The dimension of M_{m-1} is $2(m-1)$. In order to find the fibre, without loss of generality we assume that $\varphi((z_1, \cdots, z_m)/\approx) = 0$. Now the fibre M_{m-1} consists of all m-tuples $(z_1, \cdots, z_m)/\approx$ where $z_1, \cdots, z_m \in C$, such that $z_1 + \cdots + z_m \in Z \times Z$.

Before we consider the properties of M_{m-1}, we consider another two close examples.

1^o. The set of all $(z_1, \cdots, z_m)/\approx$ where $z_1, \cdots, z_m \in C$ and $z_1 + \cdots + z_m = 0$ is the space C^{m-1}. Indeed there is a homeomorphism between such elements

$(z_1, \cdots, z_m)/\approx$ and the $(m-1)$-tuple $(\sigma_2, \sigma_3, \cdots, \sigma_m) \in C^{m-1}$ where

$$\sigma_i(z_1, \cdots, z_m) = \sum_{1 \le a_1 < a_2 \cdots < a_i \le m} z_{a_1} z_{a_2} \cdots z_{a_i},$$

i.e. σ_i is the i-th symmetric function. We denote this space by P_{m-1}.

2^o. The set of all $(z_1, \cdots, z_m)/\approx$ where $z_1, \cdots, z_m \in C \setminus \{0\}$ and $z_1 + \cdots + z_m = 0$ is the space $C^{m-2} \times (C \setminus \{0\})$. Indeed there is a homeomorphism between such elements $(z_1, \cdots, z_m)/\approx$ and the $(m-1)$-tuple $(\sigma_2, \sigma_3, \cdots, \sigma_m) \in C^{m-2} \times (C \setminus \{0\})$ where σ_i is the i-th symmetric function. We denote this space by Q_{m-1}. Thus $Q_{m-1} \cong (C^{m-2}) \times (C \setminus \{0\})$.

Now let us consider some properties of M_{m-1}.

i) M_1 is homeomorphic to the sphere S^2 (theorem 1.1).

ii) The Euler characteristic of M_{m-1} is m, i.e. $\chi(M_{m-1}) = m$.

It is not necessary now to prove it because it is contained in the proof of the main theorem, but we mention a method of its calculation. M_{m-1} can be divided into m^2 disjoint subspaces $M_{m-1}^{(i,j)}$ where $(0 \le i, j \le m-1)$. Indeed without loss of generality we can assume that $z_1, \cdots, z_m \in [0,1) \times [0,1)$ and $z_1 + \cdots + z_m = 0 \pmod{Z \times Z}$. If $\frac{1}{m}(z_1 + \cdots + z_m) = (\frac{i}{m}, \frac{j}{m})$, i.e. if $z_1 + \cdots + z_m = (i, j)$ where $0 \le i, j \le m-1$, then we define that $(z_1, \cdots, z_m)/\approx \in M_{m-1}^{(i,j)}$. The empirical calculation show that $\chi(M_{m-1}^{(i,j)}) = \delta_{ij}$. Hence we obtain

$$\chi(M_{m-1}) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \chi(M_{m-1}^{(i,j)}) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \delta_{ij} = m.$$

Note that the decomposition into cells R^k here is very large and hard for computation. For example the number of cells for M_2 is much bigger than 100.

iii) M_{m-1} is simply connected space.

In order to prove this statement, we consider a closed curve with initial and endpoint in the space $M_{m-1}^{(i,j)}$ described in ii). Without loss of generality we assume that $i, j > 0$. By homotopic transformation this curve is homotopic to a curve where all the points belong to $M_{m-1}^{(i,j)}$. We note here that the topology of $\cup M_{m-1}^{(i,j)}$ is the following. If one point z_i passes through a point $(x, 0)$ $(\equiv (x, 1))$, then $(z_1, \cdots, z_m)/\approx$ passes from $M_{m-1}^{(p,q)}$ into $M_{m-1}^{(p,q+1)}$ or $M_{m-1}^{(p,q-1)}$. Analogously if one point z_i passes through a point $(0, x)$ $(\equiv (1, x))$, then $(z_1, \cdots, z_m)/\approx$ passes from $M_{m-1}^{(p,q)}$ into $M_{m-1}^{(p+1,q)}$ or $M_{m-1}^{(p-1,q)}$. Note that here p, q are considered by modulo m. Further it is homotopic to the trivial curve because locally the space is homeomorphic to $P_{m-1} \cong C^{m-1}$ which is a simply connected. Hence M_{m-1} is a simply connected.

iv) M_{m-1} is a complex manifold.

Note that if $z_1, z_2, \cdots$ are coordinates of M, where M is 1-dimensional complex manifold, then the coordinates of the complex manifold $M^{(m)}$ in [2] were introduced to be the symmetric functions

$$\sigma_1(z_1, \cdots, z_m), \sigma_2(z_1, \cdots, z_m), \cdots, \sigma_m(z_1, \cdots, z_m).$$

Thus if the first coordinate is fixed, then

$$\sigma_2(z_1, \cdots, z_m), \cdots, \sigma_m(z_1, \cdots, z_m)$$

will be coordinates of M_{m-1} and hence M_{m-1} is a complex manifold because $M^{(m)}$ is a complex manifold.

We verify that the complex projective spaces CP^{m-1} satisfy all of the previous properties.

i) S^2 is homeomorphic to CP^1.

ii) $\chi(CP^{m-1}) = m$ because CP^{m-1} decomposes into m disjoint cells $C^0 = R^0, C^1 = R^2, C^2 = R^4, \cdots, C^{m-1} = R^{2m-2}$ (see sect.1).

iii) $\pi_1(CP^{m-1}) = \pi_1((S^2)^{(m-1)}) = H_1(S^2, Z) = \{0\}$, and hence CP^{m-1} is a simply connected manifold.

iv) CP^{m-1} is a complex manifold.

Now we prove the following theorem.

Theorem 2.1 *The permutation product $T^{(m)}$ $(m > 1)$ is a bundle over T with fibre CP^{m-1}.*

Proof. First we divide the set M_{m-1}, i.e. the set of m-tuples $(z_1, \cdots, z_m)/\approx$, such that $\sum z_i$ has integer coordinates, into m disjoint cells $C_0, C_1, \cdots, C_{m-1}$, where C_i consists of those m-tuples $(z_1, \cdots, z_m)/\approx$ where there are exactly $m-1-i$ numbers $z_{\alpha_1}, z_{\alpha_2}, \cdots, z_{\alpha_{m-1-i}}$ equal to 0. Here M_{m-1} is considered as union of $M_{m-1}^{(i,j)}$. We will prove the following two statements.

1^0. C_i is simply connected space for any $i \in \{0, 1, \cdots, m-1\}$.

2^0. C_i is homeomorphic to C^i, $0 \le i \le m-1$.

Without loss of generality we assume in the proofs of 1^0 and 2^0 that $i = m-1$.

The proof of 1^0 is the same as the proof that M_{m-1} is simply connected space. Here we should note that the homotopic set of curves (if they pass through different cells $M_{m-1}^{(i,j)}$) should be chosen such that they do not pass through the point 0.

The second statement will be proved if we show that C_{m-1} is a universal covering of a corresponding space R_{m-1}, whose universal covering is C^{m-1}. For any fixed

$$(z_1^0, \cdots, z_m^0)/\approx \in C_{m-1}, \qquad z_1^0, \cdots, z_m^0 \ne 0,$$

we consider the m-tuples $(z_1, \cdots, z_m)/\approx$, $(z_1, \cdots, z_m \in C)$, such that

a) $z_1 + \cdots + z_m = 0$,

b) there exists a permutation τ such that $z_i - z_{\tau(i)}^0 \in Z \times Z$, for $i = 1, \cdots, m$, and make identification between $(z_1, \cdots, z_m)$ and $(z_1^0, \cdots, z_m^0)$.

Obviously $z_1, \cdots, z_m \notin Z \times Z$. The space R_{m-1} we define as the quotient space U_{m-1}/ρ under the previous identification, where the space U_{m-1} is given by

$$U_{m-1} = \{(z_1, \cdots, z_m)/\approx\colon\ z_1, \cdots, z_m \notin Z \times Z,\ z_1 + \cdots + z_m = 0\},$$

such that the projection $\pi : C_{m-1} \to R_{m-1}$ is well defined. In order to prove that C_{m-1} is homeomorphic to C^{m-1}, it is sufficient to prove that the universal covering of the above space U_{m-1} is homeomorphic to C^{m-1}.

Let

$$U_m^* = \{(z_1, \cdots, z_m)/\approx\colon\ z_1, \cdots, z_m \notin Z \times Z\},$$

and we shall prove that the universal covering of U_m^* is C^m. Indeed, U_m^* is homeomorphic to the permutation product $X^{(m)}$ where X is complex plane without points of integer coordinates. Since the universal covering of X is C, we obtain that the universal covering of $U_m^* = X^{(m)}$ is $C^{(m)} = C^m$.

Now since P_{m-1} (see 1^o) is homeomorphic to C^{m-1}, we obtain that the universal covering of U_{m-1} is homeomorphic to $P_{m-1} \cong C^{m-1}$. Hence, $C_{m-1} \cong C^{m-1}$.

Now we are ready to finish the proof of the theorem. Note that the topology of the union $M_{m-1} = C_0 \cup C_1 \cup \cdots \cup C_{m-1}$ is the following. Let

$$(z_1, \cdots, z_i, 0, \cdots, 0)/\approx \in C_i, \qquad (z_1, \cdots, z_i \neq 0).$$

If s of the nonzero points $z_1, \cdots, z_i$ tend to zero, then $(z_1, \cdots, z_i, 0, \cdots, 0)/\approx$ is close to the cell C_{i-s}. This topology of $M_{m-1} = C_0 \cup C_1 \cup \cdots \cup C_{m-1}$ is just the same as the topology of the decomposition $(S^2)^{(m-1)} = C^0 \cup C^1 \cup \cdots \cup C^{m-1}$ from section 1. Since $(S^2)^{(m-1)} \cong CP^{m-1}$, the proof of the theorem is finished. ∎

Note that the fibre $CP^{m-1} = (S^2)^{(m-1)}$ is also a permutation product. In [4] Wagner has proved also that $T^{(2)}$ is a non-trivial bundle over T, i.e. that $T^{(2)}$ is not homeomorphic to $T \times S^2$, by proving that these two manifolds have different cohomology algebras, although they have the same homology and cohomology modules and the first homotopy groups.

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