Infinitesimal deformations of a non-symmetric affine connection space *

Svetislav M. Minčić, Ljubica S. Velimirović and Mića S. Stanković

Abstract

In this work[1] we consider infinitesimal deformation $f : x^i \to x^i + \varepsilon z^i(x^j)$, where $z^i(x^j)$ is infinitesimal deformation field, ε-an infinitesimal real magnitude.

We consider basic facts in connection with infinitesimal deformations and Lie derivative at non-symmetric affine connection space. The Lie derivative is expressed with respect to covariant derivatives of four kinds at a space of non-symmetric affine connection L_N, proving tensor character of the Lie derivative.

1 Introduction

The problem of infinitesimal deformations of a space has been treated for years from a lot of authors (for instance see [4] − [7]). We refer to [8], [9] for more details and references.

Let us consider a space L_N of non-symmetric affine connection L^i_{jk} with the torsion tensor

$$T^i_{jk} = L^i_{jk} - L^i_{kj}, \tag{1.1}$$

at local coordinates $x^i \quad (i = 1, \ldots, N)$.

Definition 1.1 A transformation $f : L_N \to L_N : x = (x^1, \ldots, x^N) \equiv (x^i) \to \bar{x} = (\bar{x}^1, \ldots, \bar{x}^N) \equiv (\bar{x}^i)$, where

$$\bar{x} = x + z(x)\varepsilon, \tag{1.2}$$

or in local coordinates

$$\bar{x}^i = x^i + z^i(x^j)\varepsilon, \quad i, j = 1, \ldots, N, \tag{1.2'}$$

*Supported by Grant 04M03C of RFNS trough Math. Inst. SANU.
[1]Presented at the IMC “Filomat 2001”, Niš, August 26–30, 2001
2000 Mathematics Subject Classification: 53C25, 53A45, 53B05
Keywords: Infinitesimal deformation, non-symmetric affine connection, Lie derivative.

where ε is an infinitesimal, is called **infinitesimal deformation of a space** L_N, determined by the vector field $z=(z^i)$, which is called **infinitesimal deformation field**.

We denote with (i) local coordinate system in which the point x is endowed with coordinates x^i, and the point $\bar{x}$ with the coordinates $\bar{x}^i$. We will also introduce **a new coordinate system** (i'), corresponding to the point $x=(x^i)$ new coordinates

$$x^{i'}=\bar{x}^i, \tag{1.3}$$

i.e. as new coordinates $x^{i'}$ of the point $x=(x^i)$ we choose old coordinates (at the system (i)) of the point $\bar{x}=(\bar{x}^i)$. Namely, at the system (i') is $x=(x^{i'}) \underset{(1.3)}{=} (\bar{x}^i)$, where $\underset{(1.3)}{=}$ denotes "equal according to (1.3)".

Definition 1.2 Coordinate transformation which we get based on punctual transformation $f:x\to\bar{x}$, getting for the new coordinates of the point x the old coordinates of its transform $\bar{x}$, is called **dragging along by point transformation**. New coordinates $x^{i'}=\bar{x}^i$ of the point $\bar{x}$ are called **dragged along coordinates.**

In the case of infinitesimal deformation $(1.2')$ coordinate transformation

$$x^{i'}=\bar{x}^i=x^i+z^i(x^1,\dots,x^N)\varepsilon \tag{1.4}$$

is called **dragging along** by $z^i\varepsilon$.

Let us consider a geometric object $\mathcal{A}$ with respect to the system (i) at the point $x=(x^i)\in L_N$, denoting this with $\mathcal{A}(i,x)$.

Definition 1.3 The point $\bar{x}$ is said to be **deformed point** of the point x, if (1.2) holds. Geometric object $\bar{\mathcal{A}}(i,x)$ is **deformed object** $\mathcal{A}(i,x)$ with respect to deformation (1.2), if its value at system (i'), at the point x is equal to the value of the object $\mathcal{A}$ at the system (i) at the point $\bar{x}$, i.e. if

$$\bar{\mathcal{A}}(i',x)=\mathcal{A}(i,\bar{x}). \tag{1.5}$$

Remark 1.1. In this study of infinitesimal deformations according to $(1.2')$ quantities of an order higher then the first with respect to ε are neglected.

We will now define some important notions of the theory of infinitesimal deformations, following from (1.2): Lie differential and Lie derivative, and in further considerations we will find them for some geometric objects.

Definition 1.4 The magnitude $\mathcal{DA}$, the difference between deformed object $\bar{\mathcal{A}}$ and initial object $\mathcal{A}$ at the same coordinate system and at the same point with respect to $(1.2')$, i.e.

$$\mathcal{DA}=\bar{\mathcal{A}}(i,x)-\mathcal{A}(i,x), \tag{1.6}$$

is called **Lie difference (Lie differential)**, and the magnitude

$$(1.6') \qquad \mathcal{L}_z \mathcal{A} = \lim_{\varepsilon \to 0} \frac{\mathcal{D}\mathcal{A}}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\bar{\mathcal{A}}(i,x) - \mathcal{A}(i,x)}{\varepsilon}$$

is **Lie derivative** of geometric object $\mathcal{A}(i,x)$ with respect to the vector field $z = (z^i(x^j))$.

Using the relation (1.6) for deformed object $\bar{\mathcal{A}}(i,x)$ we have

$$(1.6'') \qquad \bar{\mathcal{A}}(i,x) = \mathcal{A}(i,x) + \mathcal{D}\mathcal{A},$$

and thus we can express $\bar{\mathcal{A}}$, finding previously $\mathcal{D}\mathcal{A}$. We will consider the main cases.

1.1. According to (1.6) we have $\mathcal{D}x^i = \bar{x}^i - x^i$, i.e. for the **coordinates** we have

$$(1.7) \qquad \mathcal{D}x^i = z^i(x^j)\varepsilon,$$

from where

$$(1.7') \qquad \mathcal{L}_z x^i = z^i(x^j).$$

Although x^i is not a vector, we see that $\mathcal{L}_z x^i$ is a vector.

The next cases were considered at [4] – [7].

1.2. For the **scalar function** $\varphi(x) \equiv \varphi(x^1, \dots, x^N)$ we have

$$(1.8) \qquad \mathcal{D}\varphi(x) = \varphi_{,p} z^p(x)\varepsilon = \mathcal{L}_z\varphi(x)\varepsilon, \quad (\varphi_{,p} = \partial\varphi/\partial x^p),$$

i.e. Lie derivative of the scalar function is derivative of this function in the direction of the vector field z.

1.3. For the **covariant vector** $v_i(x)$ we have

$$(1.9) \qquad \mathcal{D}v_i = (v_{i,p}z^p + z^p_{,i}v_p)\varepsilon = \mathcal{L}_z v_i \varepsilon \quad (v_{i,p} = \partial v_i/\partial x^p),$$

1.4. Let us consider **contravariant vector** $u^i(x)$. According to (1.6) we have

$$(1.10) \qquad \mathcal{D}u^i = \bar{u}^i(x) - u^i(x),$$

and we have to find $\bar{u}^i(x)$. According to the coordinate transformation low

$$(1.11) \qquad \bar{u}^i(x) = \frac{\partial x^i}{\partial x^{j'}} u^{j'}(x),$$

where the right side is to be determined. Based on (1.4) we have

$$(1.12) \qquad \frac{\partial x^i}{\partial x^{j'}} = \frac{\partial x^{i'}}{\partial x^{j'}} - \frac{\partial z^i(x)}{\partial x^{j'}}\varepsilon = \delta^i_j - \frac{\partial z^i}{\partial x^{j'}}\varepsilon.$$

Taking account of $z^i(x) = z^i(x^1, \dots, x^N)$, we have

$$\frac{\partial z^i}{\partial x^{j'}} = \frac{\partial z^i}{\partial x^k}\frac{\partial x^k}{\partial x^{j'}} \underset{(1.12)}{=} \frac{\partial z^i}{\partial x^k}(\delta^k_j - \frac{\partial z^k}{\partial x^{j'}}\varepsilon).$$

Substituting at (1.12) we get

$$\frac{\partial x^i}{\partial x^{j'}} = \delta^i_j - \frac{\partial z^i}{\partial x^k}\varepsilon(\delta^k_j - \frac{\partial z^k}{\partial x^{j'}}\varepsilon),$$

and neglecting the member with $(\varepsilon)^2$:

$$\frac{\partial x_i}{\partial x^{j'}} = \delta^i_j - \frac{\partial z_i}{\partial x^j}\varepsilon. \tag{1.13}$$

For the second member at the right side at (1.11), using Taylor's formula, we have:

$$\bar{u}^{j'}(x) \underset{(1.5)}{=} u^j(\bar{x}) = u^j(x^i + z^i\varepsilon) = u^j(x) + \frac{\partial u^j}{\partial x^k}z^k\varepsilon + \dots \tag{1.14}$$

Substituting (1.13, 14) into (1.11):

$$\bar{u}^i(x) = u^i(x) + \frac{\partial u^i}{\partial x^k}z^k\varepsilon - \frac{\partial z^i}{\partial x^j}u^j\varepsilon,$$

and substituting this value into (1.10) we get:

$$\mathcal{D}u^i = (u^i_{,p}z^p - z^i_{,p}u^p)\varepsilon = \mathcal{L}_z u^i\varepsilon. \tag{1.15}$$

1.5. In the same manner for a **tensor of the kind** (u, v) we get

$$\begin{aligned}\mathcal{D}t^{i_1\dots i_u}_{j_1\dots j_v} &= [t^{i_1\dots i_u}_{j_1\dots j_v,p}z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{,p}\binom{p}{i_\alpha}t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^p_{,j_\beta}\binom{j_\beta}{p}t^{i_1\dots i_u}_{j_1\dots j_v}]\varepsilon \\ &= \mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v}\varepsilon,\end{aligned} \tag{1.16}$$

where we denoted

$$\binom{p}{i_\alpha}t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_{\alpha-1}pi_{\alpha+1}\dots i_u}_{j_1\dots j_v}, \quad \binom{j_\beta}{p}t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_{\beta-1}pj_{\beta+1}\dots j_v}. \tag{1.17}$$

Remark 1.2. We can also see that the equations (1.8, 9, 15) are the special cases of the equation (1.16).

1.6. For the **vector** dx^i we have

$$\mathcal{D}(dx^i) = \mathcal{L}_z(dx^i) = 0. \tag{1.18}$$

1.7. In the same way, as for the tensors, for the **connection coefficients** we have

$$\mathcal{D}L^i_{jk} = (L^i_{jk,p}z^p + z^i_{,jk} - z^i_{,p}L^p_{jk} + z^p_{,j}L^i_{pk} + z^p_{,k}L^i_{jp})\varepsilon = \mathcal{L}_z L^i_{jk}\varepsilon. \tag{1.19}$$

1.8. For Lie differential (derivative) of a sum, product, contraction, composition of geometric objects the same rules hold as in the case of covariant derivative.

2 Tensor character of the Lie derivative

2.1 Tensor character of the Lie derivative of a tensor

In the previous considerations we expressed the Lie derivative with respect to partial derivatives. We will now express it by covariant derivatives and prove that the Lie derivative of a tensor is a tensor too.

Because of non-symmetry of the connection, at L_N we can consider two types of covariant derivatives for a vector and four types for general tensor. So, denoting by $\underset{\theta}{|}(\theta = 1, \dots, 4)$ a derivative of the type θ, we have ([1]-[3]):

(2.1)
$$z^i_{\underset{\substack{1\\2}}{|}m} = z^i_{,m} + \underset{mp}{L^i_{pm}} z^p = z^i_{\underset{\substack{3\\4}}{|}m}$$

(2.2)
$$t^{ij}_{k\underset{\substack{1\\2\\3\\4}}{|}m} = t^{ij}_{k,m} + \underset{\substack{mp\\pm\\mp}}{L^i_{pm}} t^{pj}_k + \underset{\substack{mp\\pm\\mp}}{L^j_{pm}} t^{ip}_k - \underset{\substack{mk\\mk\\km}}{L^p_{km}} t^{ij}_p .$$

According to (1.16) for the tensor t^{ij}_k we have

(2.3)
$$\mathcal{L}_z t^{ij}_k = t^{ij}_{k,p} z^p - z^i_{,p} t^{pj}_k - z^j_{,p} t^{ip}_k + z^p_{,k} t^{ij}_p .$$

Based on (2.1, 2), we can express partial derivatives with respect to covariant derivatives, and we get

(2.4a)
$$\begin{aligned}\mathcal{L}_z t^{ij}_k = \underset{1}{\mathcal{L}}_z t^{ij}_k \equiv\ & t^{ij}_{k\underset{1}{|}p} z^p - z^i_{\underset{1}{|}p} t^{pj}_k - z^j_{\underset{1}{|}p} t^{ip}_k \\ & + z^p_{\underset{1}{|}k} t^{ij}_p + T^i_{ps} t^{sj}_k z^p + T^j_{ps} t^{is}_k z^p + T^s_{kp} t^{ij}_s z^p ,\end{aligned}$$

(2.4b)
$$\begin{aligned}\mathcal{L}_z t^{ij}_k = \underset{2}{\mathcal{L}}_z t^{ij}_k \equiv\ & t^{ij}_{k\underset{2}{|}p} z^p - z^i_{\underset{2}{|}p} t^{pj}_k - z^j_{\underset{2}{|}p} t^{ip}_k \\ & + z^p_{\underset{2}{|}k} t^{ij}_p + T^i_{sp} t^{sj}_k z^p + T^j_{sp} t^{is}_k z^p + T^s_{pk} t^{ij}_s z^p ,\end{aligned}$$

(2.4c)
$$\begin{aligned}\mathcal{L}_z t^{ij}_k = \underset{3}{\mathcal{L}}_z t^{ij}_k \equiv\ & t^{ij}_{k\underset{3}{|}p} z^p - z^i_{\underset{3}{|}p} t^{pj}_k - z^j_{\underset{3}{|}p} t^{ip}_k \\ & + z^p_{\underset{3}{|}k} t^{ij}_p + T^i_{ps} t^{sj}_k z^p + T^j_{ps} t^{is}_k z^p ,\end{aligned}$$

(2.4d)
$$\begin{aligned}\mathcal{L}_z t^{ij}_k = \underset{4}{\mathcal{L}}_z t^{ij}_k \equiv\ & t^{ij}_{k\underset{4}{|}p} z^p - z^i_{\underset{4}{|}p} t^{pj}_k - z^j_{\underset{4}{|}p} t^{ip}_k \\ & + z^p_{\underset{4}{|}k} t^{ij}_p + T^i_{sp} t^{sj}_k z^p + T^j_{sp} t^{is}_k z^p ,\end{aligned}$$

where $\underset{\theta}{\mathcal{L}}_z$ denotes that the Lie derivative $\mathcal{L}_z$ is expressed by covariant derivatives of the type θ $\quad (\underset{\theta}{|}), \quad \theta = 1, \dots, 4.$

Naturally, as the same magnitude at the right side at (2.3) was expressed in different ways we have

$$\underset{\theta}{\mathcal{L}}_z t^{ij}_k = \mathcal{L}_z t^{ij}_k, \quad \theta = 1, \dots, 4. \tag{2.5}$$

We will prove only (2.4c). The other cases can be proved in similar way. According to (2.1) and (2.2) we have

$$z^i_{,p} = z^i_{\underset{3}{|}p} - L^i_{sp} z^s,$$

$$t^{ij}_{k,p} = t^{ij}_{k\underset{3}{|}p} - L^i_{sp} t^{sj}_k - L^j_{sp} t^{is}_k + L^s_{pk} t^{ij}_s,$$

which we substitute at (2.3) and using, for example,

$$-L^i_{sp} t^{sj}_k z^p + L^i_{sp} t^{pj}_k z^s = -L^i_{sp} t^{sj}_k z^p + L^i_{ps} t^{sj}_k z^p = T^i_{ps} t^{sj}_k z^p,$$

we get (2.4c).

2.2 Lie derivative of the connection as a tensor

On the base of (1.19) for the Lie derivative of the connection we have

$$\mathcal{L}_z L^i_{jk} = z^i_{,jk} + L^i_{jk,p} z^p - z^i_{,p} L^p_{jk} + z^p_{,j} L^i_{pk} + z^p_{,k} L^i_{jp}. \tag{2.6}$$

In order to express $z^i_{,jk} = \partial^2 z^i / \partial x^j \partial x^k$ with respect to $z^i_{\underset{1}{|}jk} = z^i_{\underset{1}{|}j\underset{1}{|}k}$ we find

$$z^i_{\underset{1}{|}j} = z^i_{,j} + L^i_{pj} z^p \tag{2.7}$$

$$\begin{aligned} z^i_{\underset{1}{|}jk} &= z^i_{\underset{1}{|}j\underset{1}{|}k} = (z^i_{\underset{1}{|}j})_{,k} + L^i_{pk} z^p_{\underset{1}{|}j} - L^p_{jk} z^i_{\underset{1}{|}p} \\ &\underset{(2.7)}{=} (z^i_{,j} + L^i_{pj} z^p)_{,k} + L^i_{pk}(z^p_{,j} + L^p_{sj} z^s) - L^p_{jk}(z^i_{,p} + L^i_{sp} z^s). \end{aligned} \tag{2.8}$$

Finding from here $z^i_{,jk}$ and substituting it at (2.6), we obtain

$$\mathcal{L}_z L^i_{jk} = z^i_{\underset{1}{|}jk} + \underset{1}{R}{}^i_{jkp} z^p + T^i_{jp,k} z^p + L^s_{jk} T^i_{ps} z^p + L^i_{sk} T^s_{jp} z^p + T^i_{jp} z^p_{,k},$$

where ([1] – [3])

$$\underset{1}{R}{}^i_{jkp} = L^i_{jk,p} - L^i_{jp,k} + L^s_{jk} L^i_{sp} - L^s_{jp} L^i_{sk} \tag{2.9}$$

is **curvature tensor of the first kind** of the space L_N. The last four summands at the previous equation for $\mathcal{L}_z L^i_{jk}$ give $(T^i_{jp}z^p)_{\underset{1}{|}k}$, and finally we have

$$\mathcal{L}_z L^i_{jk} = \underset{1}{\mathcal{L}}_z L^i_{jk} \equiv z^i_{\underset{1}{|}jk} + \underset{1}{R}{}^i_{jkp} z^p + (T^i_{jp}z^p)_{\underset{1}{|}k}. \tag{2.10}$$

Using different types of covariant derivatives, we have

$$\begin{aligned}
z^i_{\underset{2}{|}j} &= z^i_{,j} + L^i_{jp}z^p = (z^i_{,j} + L^i_{pj}z^p) - L^i_{pj}z^p + L^i_{jp}z^p = z^i_{\underset{1}{|}j} + T^i_{jp}z^p,\\
z^i_{\underset{2}{|}j\underset{1}{|}k} &= (z^i_{\underset{1}{|}j} + T^i_{jp}z^p)_{\underset{1}{|}k} = z^i_{\underset{1}{|}jk} + (T^i_{jp}z^p)_{\underset{1}{|}k},
\end{aligned}$$

and (2.10) becomes

$$\mathcal{L}_z L^i_{jk} = z^i_{\underset{2}{|}j\underset{1}{|}k} + \underset{1}{R}{}^i_{jkp} z^p. \tag{2.10$'$}$$

In the similar way we obtain

$$\begin{aligned}
\mathcal{L}_z L^i_{jk} = \underset{2}{\mathcal{L}}_z L^i_{jk} &\equiv z^i_{\underset{2}{|}jk} + \underset{2}{R}{}^i_{jkp} z^p + T^i_{pj\underset{2}{|}k} z^p + T^i_{pk} z^p_{\underset{2}{|}j}\\
&+ T^p_{kj} z^i_{\underset{2}{|}p} + T^i_{jk\underset{2}{|}p} z^p + (T^i_{sj}T^s_{kp} + T^i_{sk}T^s_{pj} + T^i_{sp}T^s_{jk}) z^p.
\end{aligned} \tag{2.11}$$

$$\mathcal{L}_z L^i_{jk} = \underset{3}{\mathcal{L}}_z L^i_{jk} \equiv z^i_{\underset{3}{|}jk} + \underset{3}{R}{}^i_{jkp} z^p - T^p_{jk} z^i_{\underset{3}{|}p} + T^i_{jp} z^p_{\underset{3}{|}k}, \tag{2.12}$$

$$\mathcal{L}_z L^i_{jk} = \underset{4}{\mathcal{L}}_z L^i_{jk} \equiv z^i_{\underset{4}{|}jk} + \underset{4}{R}{}^i_{jkp} z^p + (T^i_{pj\underset{4}{|}k} + T^i_{sj}T^s_{pk} + T^i_{sk}T^s_{pj}) z^p + T^i_{pk} z^p_{\underset{4}{|}j}, \tag{2.13}$$

where

$$\underset{2}{R}{}^i_{jkp} = L^i_{kj,p} - L^i_{pj,k} + L^s_{kj}L^i_{ps} - L^s_{pj}L^i_{ks} \tag{2.14}$$

$$\underset{3}{R}{}^i_{jkp} = L^i_{jk,p} - L^i_{pj,k} + L^s_{jk}L^i_{ps} - L^s_{pj}L^i_{sk} + L^s_{pk}T^i_{sj} \tag{2.15}$$

$$\underset{4}{R}{}^i_{jkp} = L^i_{jk,p} - L^i_{pj,k} + L^s_{jk}L^i_{ps} - L^s_{pj}L^i_{sk} + L^s_{kp}T^i_{sj} \tag{2.16}$$

are **curvature tensors of the second, the third and the fourth kind** of the space L_N respectively (see [1]-[3]).

References

[1] S.M. Minčić, *Ricci identities in the space of non-symmetric affine connexion*, Matematički vesnik, 10(25)Sv. 2,(1973), 161-172.

[2] S.M. Minčić, *New commutation formulas in the non-symetric affine connexion space*, Publ. Inst. Math. (Beograd)(N.S), 22(36) (1977), 189-199.

[3] S.M. Minčić, *Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion*, Coll. math. soc. János Bolyai, 31. Dif. geom., Budapest (Hungary), (1979), 445-460.

[4] R. Stojanović, Osnovi diferencijalne geometrije, Gradjevinska knjiga, Beograd, (1963),

[5] J.A. Schouten, Ricci calculus, Springer Verlag, Berlin-Gotingen-Heildelberg, (1954).

[6] K. Yano, Sur la theorie des deformations infinitesimales, Journal of Fac. of Sci. Univ. of Tokyo, 6, (1949), 1-75.

[7] K. Yano, The theory of Lie derivatives and its applications, N-Holland Publ.Co.Amsterdam, (1957).

[8] I. Ivanova-Karatopraklieva, Sabitov, I. Kh., *Surface deformation,* J. Math. Sci., New York 70, N^o2, (1994), 1685–1716.

[9] I. Ivanova-Karatopraklieva, I. Kh. Sabitov, *Bending of surfaces II* , J. Math. Sci., New York 74, N^o3 (1995), 997–1043.

Faculty of Science and Mathematics
Višegradska 33, 18000 Niš
smica@pmf.pmf.ni.ac.yu,
vljubica@pmf.pmf.ni.ac.yu