Semigroup cohomology as a derived functor

A. A. Kostin and B. V. Novikov

Abstract

In this work[1] we construct an extension for the category of 0-modules by analogy with [5]. The 0-cohomology functor becomes a derived functor in the extended category. As an application of this construction we calculate the cohomological dimension of so-called 0-free monoids.

1. 0-cohomology of semigroups appeared in research of projective representations of semigroups [1]. Besides, it was useful in studying of matrix algebras [3] and Brauer monoids [4] (see also survey [2] and references there).

However the further study of its properties is complicated. One of the reasons is that the semigroup 0-cohomology is not a derived functor in the category where it is built (so-called category of 0-modules).

The purpose of this paper is to describe the extension of 0-cohomology on a larger category where it becomes a derived functor. Our construction is similar to Baues theory for cohomology of small categories [5]. Therefore we omit some proofs replacing them by references to [5].

As an example of application of our construction we prove that a cohomological dimension of a so-called 0-free semigroup equals one. In particular, it follows that all projective representations of a free semigroup are linearizable.

2. We begin with definitions. Let S be a monoid. We may assume that S has a zero element (if not, let us join it to S). By analogy with [5] *the category of factorizations in* S is given as follows. The objects are all nonzero elements of S and the set of morphisms $\mathrm{Mor}(a,b)$ consists of all triples $(\alpha,a,\beta)\quad(\alpha,\beta\in S)$ such that $\alpha a\beta=b$. We will denote (α,a,β) by (α,β) if this cannot lead to confusion. The composition is defined by the rule: $(\alpha',\beta')(\alpha,\beta)=(\alpha'\alpha,\beta\beta')$; hence we have $(\alpha,\beta)=(\alpha,1)(1,\beta)=(1,\beta)(\alpha,1)$. Denote this category by $\mathcal{F}acS$.

A natural system on S is a functor $\mathbf{D}:\mathcal{F}acS\longrightarrow\mathcal{A}b$. The category $\mathcal{N}atS=\mathcal{A}b^{\mathcal{F}acS}$ is an Abelian category with enough projectives and injectives [6]. Denote the value of $\mathbf{D}$ at the object $a\in\mathrm{Ob}\mathcal{F}acS$ by $\mathbf{D}_a$. By α_* and β^* denote values of $\mathbf{D}$ at morphisms $(\alpha,1)$ and $(1,\beta)$ respectively. We have $\mathbf{D}(\alpha,\beta)=\alpha_*\beta^*$ for all morphisms (α,β).

For given natural number n denote by Ner_nS the set of all n-tuples

$$(a_1,\dots,a_n),\ a_i\in S,$$

[1]Presented at the IMC “Filomat 2001”, Niš, August 26–30, 2001

2000 Mathematics Subject Classification: 20J, 20M50
Keywords: 0-cohomolgy, 0-free monoid, cohomological dimension, natural system

such that $a_1 \cdots a_n \neq 0$. By definition $Ner_0 S = \{1\}$. A *n-cochain* assigns to each point $a = (a_1, \ldots, a_n)$ of $Ner_n S$ an element on $\mathbf{D}_{a_1 \cdots a_n}$. The set of all n-cochains is an Abelian group $C^n(S, \mathbf{D})$ with respect to the pointwise addition. Set $C^0(S, \mathbf{D}) = \mathbf{D}_1$.

The *coboundary* $\delta = \delta^n : C^n(S, \mathbf{D}) \longrightarrow C^{n+1}(S, \mathbf{D})$ is given by the formula $(n \geq 1)$

$$(\delta f)(a_1, \ldots, a_{n+1}) = a_{1*} f(a_2, \ldots, a_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} a_{n+1}^* f(a_1, \ldots, a_n).$$

For $n = 0$ let $\delta^0 : C^0(S, \mathbf{D}) \longrightarrow C^1(S, \mathbf{D})$ be defined by

$$\delta f(x) = x_* f - x^* f \quad (f \in D_1, \quad x \in S \setminus 0).$$

One can check directly that $\delta^n \delta^{n-1} = 0$. By $H^n(S, \mathbf{D})$ denote the cohomology groups of the complex $\{C^n(S, \mathbf{D}), \delta^n\}_{n \geq 0}$.

3. Now we define a *trivial natural system* $\mathbf{Z}$. To each object $a \in S \setminus 0$ it assigns the infinite cyclic group $\mathbf{Z}_a$ generated by a symbol $[a]$; and to each morphism $(\alpha, \beta) : a \longrightarrow b$ it assigns a homomorphism of the groups $\mathbf{Z}(\alpha, \beta) : \mathbf{Z}_a \longrightarrow \mathbf{Z}_b$ which takes $[a]$ to $[b]$.

Since $\mathcal{N}atS$ has enough projective and injective, hence there exists the derived functor $\mathrm{Ext}^n_{\mathcal{N}atS}(\mathbf{Z}, -)$. This functor is isomorphic to the cohomology functor $H^n(S, -)$ which is defined in Section **2.** To prove this statement we construct a suitable projective resolution of $\mathbf{Z}$.

For every $n \geq 0$ we denote by $\mathbf{B}_n : \mathcal{F}acS \longrightarrow \mathcal{A}b$ the following natural system. For an object $a \in S \setminus 0$ the group $\mathbf{B}_n(a)$ is a free Abelian group generated by the set of symbols $[a_0, \ldots, a_{n+1}]$ such that $a_0 \cdots a_{n+1} = a$. To each morphism (α, β) we assign a homomorphism of groups by the formula

$$\mathbf{B}_n(\alpha, \beta) : \ [a_0, \ldots, a_{n+1}] \longmapsto [\alpha a_0, \ldots, a_{n+1} \beta].$$

The functors $\mathbf{B}_n$ $(n \geq 0)$ constitute a chain complex $\{\mathbf{B}_n, \partial_n\}_{n \geq 0}$, where $\partial_n : \mathbf{B}_n \overset{\cdot}{\longrightarrow} \mathbf{B}_{n-1}$ $(n \geq 1)$ is a natural transformation with the set of its components

$$(\partial_n)_a : \mathbf{B}_n(a) \longrightarrow \mathbf{B}_{n-1}(a),$$

$$(\partial_n)_a [a_0, \ldots, a_{n+1}] = \sum_{i=0}^{n} (-1)^i [a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}].$$

4. Lemma. *The natural system $\mathbf{B}_n$ is a projective object in $\mathcal{N}atS$.*

Proof. Consider the following diagram with the exact row

$$\begin{array}{ccccc} & & \mathbf{B}_n & & \\ & & \downarrow{\scriptstyle \nu} & & \\ \mathbf{D} & \xrightarrow{\mu} & \mathbf{E} & \longrightarrow & 0 \end{array}$$

and construct a natural transformation $\tau : \mathbf{B}_n \stackrel{\cdot}{\longrightarrow} \mathbf{D}$ which turns this diagram into commutative.

Let $s = s_0 \cdots s_{n+1}$, $\hat{s} = s_1 \cdots s_n$. Choose $a_{(s_1,\ldots,s_n)} \in \mathbf{D}(\hat{s})$ such that $\mu_{\hat{s}} a_{(s_1,\ldots,s_n)} = \nu_{\hat{s}}[1, s_1, \ldots, s_n, 1]$, and put

$$\tau_s[s_0, \ldots, s_{n+1}] = \mathbf{D}(s_0, s_{n+1}) a_{(s_1,\ldots,s_n)}.$$

The natural transformation is well defined. Indeed,

$$\tau_{\alpha s \beta} \mathbf{B}_n(\alpha, \beta)[s_0, \ldots, s_{n+1}] = \mathbf{D}(\alpha s_0, s_{n+1}\beta) a_{(s_1,\ldots,s_n)} =$$
$$\mathbf{D}(\alpha, \beta)\mathbf{D}(s_0, s_{n+1}) a_{(s_1,\ldots,s_n)} = \mathbf{D}(\alpha, \beta)\tau_s[s_0, \ldots, s_{n+1}].$$

■

5. Lemma. *The chain complex $\{\mathbf{B}_n, \partial_n\}_{n\geq 0}$ is a projective resolution of the natural system $\mathbf{Z}$.*

The proof is similar to [5].

6. Now we are ready to prove the main result of this paper.

Theorem. *For any monoid S with a zero element there is an isomorphism of the functors:*

$$H^n(S, -) \cong \mathrm{Ext}^n_{\mathcal{N}atS}(\mathbf{Z}, -).$$

Proof. Define an isomorphism of complexes

$$\Psi^*_{\mathbf{D}} : \{\mathrm{Hom}_{\mathcal{N}atS}(\mathbf{B}_n, \mathbf{D}), \partial^n\}_{n\geq 0} \longrightarrow \{C^n(S, \mathbf{D}), \delta^n\}_{n\geq 0}$$

(here we denote $\partial^n = \mathrm{Hom}_{\mathcal{N}atS}(\partial_{n-1}, \mathbf{D})$) as follows. Let the homomorphism of Abelian group

$$\Psi^n_{\mathbf{D}} : \mathrm{Hom}_{\mathcal{N}atS}(\mathbf{B}_n, \mathbf{D}) \longrightarrow C^n(S, \mathbf{D})$$

be given by

$$(\Psi^n_{\mathbf{D}}\tau)(a_1, \ldots, a_n) = \tau_{a_1\cdots a_n}[1, a_1, \ldots, a_n, 1] \in \mathbf{D}_{a_1\cdots a_n} \text{ for } a_1 \cdots a_n \neq 0.$$

Let $a = a_0 \cdots a_{n+1}$, i.e. $[a_0, \ldots, a_{n+1}] \in \mathbf{B}_n(a)$. Since the diagram

$$\begin{array}{ccc}
\mathbf{B}_n(a_1 \cdots a_n) & \xrightarrow{\tau_{a_1\cdots a_n}} & \mathbf{D}_n(a_1 \cdots a_n) \\
{\scriptstyle \mathbf{B}_n(a_0, a_{n+1})}\downarrow & & \downarrow{\scriptstyle \mathbf{D}_n(a_0, a_{n+1})} \\
\mathbf{B}_n(a) & \xrightarrow{\tau_a} & \mathbf{D}_n(a)
\end{array}$$

is commutative we have

$$\tau_a[a_0, \ldots, a_{n+1}] = \mathbf{D}(a_0, a_{n+1})\tau_{a_1\cdots a_n}[1, a_1, \ldots, a_n, 1].$$

Therefore $\Psi^n_{\mathbf{D}}\tau = 0$ implies that τ_a vanishes on all generators of the group $\mathbf{B}_n(a)$. Hence $\Psi^n_{\mathbf{D}}$ is injective.

Further, for any $f \in C^n(S, \mathbf{D})$ define a natural transformation $\varphi : \mathbf{B}_n \stackrel{\cdot}{\longrightarrow} \mathbf{D}$:

$$\varphi_a[a_0, \dots, a_{n+1}] = \mathbf{D}(a_0, a_{n+1}) f(a_1, \dots, a_n)$$

It is clear that $\Psi^n_{\mathbf{D}}\varphi = f$ and hence Ψ^n is surjective. The commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{N}atS}(\mathbf{B}_n, \mathbf{D}) \arrow[r, "\partial^n"] \arrow[d, "\Psi^n_{\mathbf{D}}"'] & \mathrm{Hom}_{\mathcal{N}atS}(\mathbf{B}_{n+1}, \mathbf{D}) \arrow[d, "\Psi^{n+1}_{\mathbf{D}}"] \\
C^n(S, \mathbf{D}) \arrow[r, "\delta^n"] & C^{n+1}(S, \mathbf{D})
\end{tikzcd}$$

is established immediately.

It can easily be checked that the family $\Psi^n = \{\Psi^n_{\mathbf{D}} | \mathbf{D} \in \mathcal{N}atS\}$ is a natural transformation. From above we see that Ψ^n induces an isomorphism of functors H^n and Ext^n. ■

7. Let us discuss the relation between cohomology which is defined above and cohomology groups of other kinds. In Section **1** we note that the 0-cohomology is a particular case of our construction. This can be shown in the following way. Let A be an Abelian group and $\mathbf{A}$ be a natural system given by

$$\mathbf{A}(s) = A \quad \text{and} \quad \alpha_* \beta^* a = \alpha a$$

for all $s \in \mathcal{F}acS$, $\quad (\alpha, \beta) \in \mathrm{Mor}\mathcal{F}acS$. In other words, $\mathbf{A}$ is so-called 0-module over S [1]: an action $(S \setminus \{0\}) \times A \longrightarrow A$ is given, which satisfies the following conditions:

$$s(a + b) = sa + sb,$$

$$st \neq 0 \Rightarrow s(ta) = (st)a,$$

where $s, t \in S \setminus 0$ and $a, b \in A$. 0-Cohomology groups are denoted by $H^n_0(S, A)$.

Note that Eilenberg-MacLane cohomology of semigroups [8] can be considered as a particular case of the 0-cohomology. Namely, if S is a semigroup (possibly without a zero), then $H^n(S, -) \cong H^n_0(S^0, -)$, where S^0 is the semigroup S with an adjoint zero.

The category of 0-modules arises naturally in applications of 0-cohomology theory [4]. However it is easily shown that the second 0-cohomology group of the commutative semigroup $S = \{u, v, w, 0\}$ with $u^2 = v^2 = uv = w, \quad uw = vw = 0$ (see [1]) is not trivial for all nonzero 0-module over S. Therefore the 0-cohomology is not a derived functor on the category of 0-modules. This is the reason for introducing the category $\mathcal{N}atS$.

Our construction differs from Baues' cohomology theory for monoids [5] in the first step only. Actually in [5] a monoid S is regarded as a category with a single object. At the same time the Baues' category of factorizations in S is equal to $\mathcal{F}acS^0$ out of Section **2**. Therefore the Baues' cohomology groups of monoid S and cohomology grops of S^0 in our sense are the same. However if S possesses a zero element then the category $\mathcal{F}acS$ and Baues' one are not equivalent and we obtain the different cohomology groups.

8. Let us consider an application of the obtained results. *Cohomological dimension* c.d.S of monoid S is the greatest natural number such that $H^n(S, \mathbf{D}) \neq 0$ for some $\mathbf{D} \in \mathcal{N}atS$. The Theorem from Section **6** allows us to use a projective resolution for calculation of the dimension.

It is well-known that in many cohomological theories c.d. of free objects equals 1. Free objects in the class of monoids with zero are free monoids with adjoint zero element. Nevertheless in our case the family of monoids having c.d.1 is larger.

A monoid is called a *0-free monoid* if it is isomorphic to a Rees factor monoid of a free monoid. Free monoids with adjoint zero will be regarded as 0-free monoids too.

9. We shall need the following

Lemma. *Let $\mathcal{A}, \mathcal{B}$ be categories, $\mathbf{F} : \mathcal{A} \longrightarrow \mathcal{B}$, $\mathbf{G} : \mathcal{B} \longrightarrow \mathcal{A}$ be adjoint functors ($\mathbf{F} \dashv \mathbf{G}$), functor $\mathbf{G}$ preserves epimorphisms and the counit $\varepsilon : \mathbf{FG} \stackrel{\cdot}{\longrightarrow} \mathrm{Id}_{\mathcal{B}}$ is identical. If an object $a \in \mathcal{A}$ is projective then $\mathbf{F}(a)$ is projective too.*

Proof. Let $a \in \mathcal{A}$ be a projective object. Consider a diagram

$$\begin{tikzcd} & \mathbf{F}(a) \arrow[d, "\alpha"] \\ c \arrow[r, "\beta"] & b \end{tikzcd}$$

with the exact row ($c, b \in \mathcal{B}$). Since functor $\mathbf{G}$ preserves epimorphisms we obtain the diagram:

$$\begin{tikzcd} & a \arrow[d, "\mathbf{G}(\alpha)\eta_a"] \\ \mathbf{G}(c) \arrow[r, "\mathbf{G}(\beta)"] & \mathbf{G}(b) \end{tikzcd} \tag{1}$$

where $\eta : \mathrm{Id}_{\mathcal{A}} \stackrel{\cdot}{\longrightarrow} \mathbf{GF}$ is the unit of the adjunction $\mathbf{F} \dashv \mathbf{G}$. Since a is projective, there is a homomorphism $\gamma : a \longrightarrow \mathbf{G}(c)$ which makes diagram (1) commutative. This means that $\mathbf{G}(\beta)\gamma = \mathbf{G}(\alpha)\eta_a$ and $\beta\mathbf{F}\gamma = \alpha\mathbf{F}(\eta_a)$. Using the equalities $\mathbf{F}(\eta_a) = \mathrm{Id}_{\mathbf{F}(a)}$ and $\mathbf{FG} = \mathrm{Id}_{\mathcal{B}}$ we get $\beta\mathbf{F}\gamma = \alpha$. ■

10. Theorem. *c.d.$M \leq 1$ for all 0-free monoids M.*

Proof. For a given monoid M consider the exact sequence

$$0 \longrightarrow \mathbf{P}_M \stackrel{\cdot}{\longrightarrow} \mathbf{B}_M \stackrel{\cdot}{\longrightarrow} \mathbf{Z}_M \longrightarrow 0$$

where $\mathbf{Z}_M, \mathbf{B}_M$ are natural systems defined in Section **3**, $\mathbf{P}_M = \mathrm{Ker}(\mathbf{B}_M \stackrel{\cdot}{\longrightarrow} \mathbf{Z}_M)$. We need to prove that $\mathbf{P}_M$ is a projective functor.

It follows from Section 7 that $\mathbf{P}_M$ is a free functor whenever M is a free monoid with adjoint zero (see [5], Lemma 6.7).

Now let M be a 0-free monoid, $M \cong W/I$ where W is a free monoid and I is an ideal in W. Consider the category of factorizations $\mathbf{F}W$ which was defined in [5], i.e. $\mathbf{F}W = \mathcal{F}ac(W^0)$. Define the functor $\mathbf{K} : \mathcal{F}acM \longrightarrow \mathbf{F}W$ which takes each nonzero element from M to its preimage under the canonic homomorphism $W \longrightarrow W/I$. Functor $\mathbf{K}$ is well defined and induces the functor $\mathbf{K}^* : \mathbf{Nat}W \longrightarrow \mathcal{N}atM$, where $\mathbf{Nat}W = \mathcal{A}b^{\mathbf{F}W}$.

Consider the exact sequence which is defined in [5], Sec.5:

$$0 \longrightarrow \tilde{\mathbf{P}}_W \xrightarrow{\tilde{\delta}_W} \tilde{\mathbf{B}}_W \xrightarrow{\tilde{\varepsilon}_W} \tilde{\mathbf{Z}}_W \longrightarrow 0,$$

where $\tilde{\mathbf{P}}_W, \tilde{\mathbf{B}}_W, \tilde{\mathbf{Z}}_W : \mathbf{F}W \longrightarrow \mathcal{A}b$ are natural systems on W. We have

$$\mathbf{K}^*(\tilde{\mathbf{Z}}_W) = \mathbf{Z}_M,\ \mathbf{K}^*(\tilde{\mathbf{B}}_W) = \mathbf{B}_M,\ \mathbf{K}^*(\tilde{\varepsilon}_W) = \varepsilon_M$$

hence $\mathbf{K}^*(\tilde{\mathbf{P}}_W) = \mathbf{P}_M$.

Consider the functor $\mathbf{L} : \mathcal{N}atM \longrightarrow \mathbf{Nat}W$ which is given by

$$\mathbf{L}(\mathbf{G})_a = \begin{cases} \mathbf{G}_a, & \text{if } a \notin I \\ 0, & \text{if } a \in I \end{cases}$$

where $\mathbf{G} \in \mathcal{N}atM$, and

$$\mathbf{L}(\mathbf{G})(x, a, y) = \begin{cases} \mathbf{G}(x, a, y), & \text{if } xay \notin I \\ 0, & \text{if } xay \in I \end{cases}$$

Evidently $\mathbf{K}^*\mathbf{L} = \mathrm{Id}_{\mathcal{N}atM}$ and there is a natural transformation $\mathrm{Id}_{\mathbf{Nat}W} \stackrel{\cdot}{\longrightarrow} \mathbf{L}\mathbf{K}^*$. It implies that $\mathbf{L}$ is right adjoint to $\mathbf{K}^*$. Besides, $\mathbf{L}$ preserves epimorphisms and by [5] $\tilde{\mathbf{P}}_W$ is a free object. Using Lemma 9 we get $\mathbf{P}_M$ is a projective object. ■

11. The semigroup is called *0-cancellative* if

$$ax = bx \neq 0 \Rightarrow a = b \text{ and } xa = xb \neq 0 \Rightarrow a = b$$

for all elements a, b, x. In view of Theorem 10 the following question arises: is a 0-cancellative monoid of cohomological dimension one a 0-free monoid?

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andreykostin@mail.com
boris.v.novikov@univer.kharkov.ua