Axiomatically established order geometry in the EH-geometry

Miroljub Đ. Milojević

Abstract

By EH-geometry we understood the two-dimensional geometry which is dual to (Lobachevsky's) hyperbolic geometry. Without employing the duality principle, we have attempted[1)] a foundation of an order geometry, a geometry founded on incidence and order axioms.

Basic to this geometry is the postulation of a non-empty set $\mathcal{J}$, the C_a, $C_{\underset{a}{\sim}}$ and $C_{\overline{a}}$-class which are subsets of $\mathcal{J}$, as well as two basic relations. The set $\mathcal{J}$ is then labeled as an EH-plane, and its elements are the points of the plane. The elements of the class C_a are labeled as EH-plane lines, $C_{\underset{a}{\sim}}$-class elements are isotropic lines and $C_{\overline{a}}$-class elements are ideal line.

The basic relation is two-member incidence relation $i \subset \mathcal{J}xC_a$, $i \subset \mathcal{J}xC_{\underset{a}{\sim}}$, $i \subset \mathcal{J}xC_{\overline{a}}$, which defines the point-line set relation, the point-isotropic line set relation and the point-ideal line set relation. Another basic relation is a four-member relation of separation of two point pairs which are incidence to one line.

The present order geometry is founded on 8 incident axioms and 10 ordering axioms, its consistency being proven by the projective model.

0 Basic terms

In the construction of this geometry we start with a non-empty set $\mathcal{J}$, classes C_a, $C_{\underset{a}{\sim}}$, $C_{\overline{a}}$ subsets of $\mathcal{J}$ and two basic relations over the set $\mathcal{J}$. The set $\mathcal{J}$ is referred to as the *EH-plane*, and its elements as *points* of the EH-plane and they are marked with capital letters $A, B, C, \ldots$. The elements of the class C_a are called *lines* of the EH-plane and they are marked as $a, b, c, \ldots$; the elements of the class $C_{\underset{a}{\sim}}$ are called *isotropic lines* of the EH-plane and they are marked as $\widetilde{a}, \widetilde{b}, \widetilde{c}, \ldots$. The elements of the class $C_{\overline{a}}$ are called *ideal plane* of the EH-plane and they are marked as $\overline{a}, \overline{b}, \overline{c}, \ldots$. So, the *basic objects* are points, lines, isotropic lines and ideal lines.

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The basic relation is a two-member (binary) relation of *incidence* $i \subset \mathcal{J}xC_a$, $i \subset \mathcal{J}xC_{\widetilde{a}}$, $i \subset \mathcal{J}xC_{\overline{a}}$, which represents a set relation between points and lines, points and isotropic lines, points and ideal lines. For example, the formula $i(A, \widetilde{a})$ is to be read as follows: The point is incident to the isotropic line $\widetilde{a}$ or the isotropic line a is incident to the point A.

The basic relation is a two-member relation *with a distance between two pairs of points* incident to a line. The formula $s(A, B; C, D)$ is to be read as follows: The pair of points A, B is separated by the pair of points A, C.

So, the *basic terms* of the order geometry in the EH-geometry are points, lines, isotropic lines, ideal lines, relation of incidence and relation of separation of two pairs of points.

The basic terms in the order geometry of EH-plane have been described by *incidence axioms* and *order axioms*

1 Incidence axioms and their consequences

The basic characteristics of the incidence relation i can be represented by 8 incidence axioms:

Axiom $\mathbf{I_1}$: *For every two different points A and B there is a common line a, or a common ideal line $\overline{a}$, or a common isotropic line $\widetilde{a}$, incident to each of them.*

Definition 1.1 *Line, isotropic line and ideal line are referred to as generalised lines.*

Generalised lines are marked as $\alpha, \beta, \gamma, \delta, \ldots$.

Axiom $\mathbf{I_2}$: *For each line, ideal line, isotropic line there are at least two different points incident to it.*

Axiom $\mathbf{I_3}$: *There is a point and a line which are not incident.*

Axiom $\mathbf{I_4}$: *For every two different lines a and b (isotropic lines $\widetilde{a}$ and $\widetilde{b}$, for line a and isotropic line $\widetilde{a}$, for line a and ideal line $\overline{a}$) there is a common point S incident to each of them.*

Axiom $\mathbf{I_5}$: *If $\widetilde{p}$ and $\widetilde{q}$ are isotropic lines incident to the point S, all the points of the EH-plane which are not incident to $\widetilde{p}$ and $\widetilde{q}$ belong into two disjunctive classes, so that for each point M of the first class and the point S there is a common line m incident to the points, and for each point N of the second class and the point S there is a common line $\overline{n}$ incident to the points.*

Axiom $\mathbf{I_6}$: *For every point A there are two and only two isotropic lines $\widetilde{a}'$ and $\widetilde{a}''$ incident to the point A and a common ideal line $\overline{a}$ which is not incident to A, so that $\widetilde{a}'$ and $\overline{a}$, namely $\widetilde{a}''$ and $\overline{a}$ do not have any common points.*

From Axiom I_6 we can deduce:

Theorem 1.1 *There is a point and an ideal line (isotropic line) which are not incident.*

On the basis of Axioms I_3, I_6 and I_4 we can establish:

Theorem 1.2 *Every line contains at least three different points.*

Axiom $\mathbf{I_7}$: *If the point A is incident to isotropic lines $\widetilde{a}'$ and $\widetilde{a}''$, and if $\overline{a}$ is an ideal line which has no common points with $\widetilde{a}'$ and $\widetilde{a}''$, then for every point M which is incident to $\overline{a}$ and the point A there is a common line m incident to the points A and M.*

Axiom $\mathbf{I_8}$: *If $\overline{a}$ is an ideal line, then there are two and only two isotropic lines $\widetilde{p}$ and $\widetilde{q}$ which have no common points with $\overline{a}$.*

On the basis of Axiom I_8 and Axiom I_4, we can establish:

Theorem 1.3 *For every ideal line $\overline{a}$ there is a point M which is not incident to $\overline{a}$, so that to isotropic lines $\widetilde{m}'$ and $\widetilde{m}''$ incident to M we can apply $\overline{a} \cap \widetilde{m}' = \vartheta$ and $\overline{a} \cap \widetilde{m}'' = \vartheta$.*

2 Axioms of order and their consequences

First, we cite 10 axioms which represent the initial relation s of the separation of two pairs of points.

Axiom $\mathbf{II_1}$: *If $s(A, B; C, D)$, then A, B, C and D are different points incident to a common line.*

Definition 2.1 *Points incident with a common line are called collinear points.*

Axiom $\mathbf{II_2}$: *If $s(A, B; C, D)$, then $s(A, B; D, C)$, $s(B, A; C, D)$, $s(B, A; D, C)$, $s(C, D; A, B)$, $s(C, D; B, A)$, $s(D, C; A, B)$ and $s(D, C; B, A)$.*

Besides the relation s its negation is also used. The formula $\neg s(A, B; C, D)$ is to be read as follows: The pair of points A, C is *not separated* by the pair of points B, D.

Axiom $\mathbf{II_3}$: *If $s(A, B; C, D)$, then $\neg s(A, C; B, D)$, $\neg s(A, C; D, B)$.*

Axiom $\mathbf{II_4}$: *If A, B and C are three different collinear points, then there is a point D so that $s(A, B; C, D)$.*

Axiom $\mathbf{II_5}$: *If A, B, C and D are four different collinear points, then or*

$$s(A, B; C, D), \text{ or } s(A, C; B, D), \text{ or } s(A, D; B, C).$$

Axiom $\mathbf{II_6}$: *If A, B, C, D and E are five collinear points, so that $s(A, B; C, D)$ and $s(A, B; C, E)$ then $\neg s(A, B; D, E)$.*

Axiom $\mathbf{II_7}$: *If A, B, C, D and E are five collinear points, so that $\neg s(A, B; C, D)$ and $\neg s(A, B; C, E)$ then $\neg s(A, B; D, E)$.*

Axiom $\mathbf{II_8}$: *If the point A is incident to isotropic lines $\widetilde{a}'$ and $\widetilde{a}''$, $\overline{a}$ is an ideal line which has common points with $\widetilde{a}'$ and $\widetilde{a}''$, p is a line which is not incident to the point A where $p \cap \widetilde{a}' = \{A'\}$, $p \cap \widetilde{a}'' = \{A''\}$, $p \cap \overline{a} = \{P\}$, and $s(A', A''; P, Q)$, then there is a common ideal line $\overline{q}$ which incident to the points A and Q.*

Axiom $\mathbf{II_9}$: *If m is a line and points M', M'', X and Y incident to m so that $\neg s(M', M''; X, Y)$, then there is a point M which is not incident to the line m, so that the pairs M, M' and M, M'' are incident respectively to isotropic lines $\widetilde{m}'$ and $\widetilde{m}''$, and pairs of points M, X and M, Y incident respectively to lines x and y, or to ideal lines $\overline{x}$ and $\overline{y}$.*

Axiom $\mathbf{II_{10}}$: *If p and q are two different lines and P_i and Q_i, $i = 1, 2, 3, 4$, are points such that $i(P_i, p)$ and $i(Q_i, q)$ and at the same time the pairs of points P_i and Q_i are incident to generalised lines $\mu_i, i = 1, 2, 3, 4$, where $\mu_1 \cap \mu_2 \cap \mu_3 \cap \mu_4 = \{M\}$, then if $s(P_1, P_2; P_3, P_4)$ then $s(Q_1, Q_2; Q_3, Q_4)$.*

On the basis of Axioms II_2 and II_3 we can establish:

Theorem 2.1 *If $s(A, B; C, D)$ then*

$$\begin{array}{llll}
\neg s(A,C;B,D), & \neg s(A,C;D,B), & \neg s(A,D;B,C), & \neg s(A,D;C,B), \\
\neg s(B,C;A,D), & \neg s(B,C;D,A), & \neg s(B,D;A,C), & \neg s(B,D;C,A), \\
\neg s(C,A;B,D), & \neg s(C,A;D,B), & \neg s(C,B;A,D), & \neg s(C,B;D,A), \\
\neg s(D,A;B,C), & \neg s(D,A;C,B), & \neg s(D,B;A,C), & \neg s(D,B;C,A).
\end{array}$$

On the basis of Axioms II_5 and II_3 and Theorem 2.1 we can establish:

Theorem 2.2 *If A, B, C and D are four different collinear points, then one and only one relations holds: $s(A, B; C, D)$, $s(A, C; B, D)$, $s(A, D; B, C)$.*

On the basis of Theorem 2.2 and Axiom II_4 we can establish:

Theorem 2.3 *Suppose that A, B abd C are three different points incident to a line a. The point X is incident to this line if and only if it is identical to one of the points or if one of the relations hold $s(A, B; C, X)$, $s(A, C; B, X)$, $s(A, X; B, C)$.*

On the basis of Theorem 2.2 and Axioms II_2 and II_3 we can establish:

Theorem 2.4 *If A, B, C and D are four different collinear points and $\neg s(A, C; B, D)$, then $\neg s(A, C; D, B)$, $\neg s(B, D; A, C)$, $\neg s(D, B; C, A)$, $\neg s(C, A; B, D)$, $\neg s(C, A; D, B)$, $\neg s(D, B; A, C)$, $\neg s(D, B; C, A)$.*

On the basis of Theorem 2.4 and Axioms II_4, II_3 and II_7 we can establish:

Theorem 2.5 *If A, B and C are three different points incident to the line a, then there is a point X incident to this line, so that $\neg s(A, B; C, X)$.*

Theorem 2.6 *Every point of the EH-plane is incident to two isotropic lines, with an infinite number of lines and an infinite number of ideal lines.*

Proof. Let M be an arbitrary point of EH-plane. According to Axiom I_6 there are isotropic lines $\widetilde{m}'$ and $\widetilde{m}''$ incident to the point M and an ideal line $\overline{m}$ which is not incident to the point M so that $\widetilde{m}'$ and $\overline{m}$, i.e. $\widetilde{m}''$ and $\overline{m}$ do not have any common points. According to Axiom I_3 there is a line p and a point M which are not incident, and according to Axiom I_4 there are points P, M' and M'' incident to the line p so that $p \cap \overline{m} = \{P\}$, $p \cap \widetilde{m}' = \{M'\}$, $p \cap \widetilde{m}'' = \{M''\}$. For the points M', M'' and P incident to the line p, according to Axiom II_4 there is a point Q so that $s(M', M''; P, Q)$.

For the points M and P, according to Axiom I_7, there is a line m incident to these points, and for the points M and Q, according to Axiom II_8, there is an ideal line $\overline{q}$ incident to M and Q.

According to Theorem 2.5, for the points M', M'' and P, i.e. M', M'' and Q incident to the line p, there is a point X or Y such that $\neg s(M', M''; P, X)$, i.e. $\neg s(M', M''; Q, Y)$. From this and according to Axiom II_9 it follows that the points M and X are incident to the line x, and the points M and Y are incident to the ideal line $\overline{y}$.

Continuing this procedure it can be concluded that the point M of the EH-plane is incident to an infinite number of lines and an infinite number of ideal lines. ■

Theorem 2.6 enables the definition of pencils in the EH-plane.

Definition 2.2 *Let M be a point of the EH-plane. A pencil of lines with the centre M, designed as $M(a, b, c, \ldots)$, is a set of lines $a, b, c, \ldots$ incident to the point M.*

Definition 2.3 *Let M be a point of the EH-plane. A pencil of ideal lines with the centre M, designated as $M(\overline{a}, \overline{b}, \overline{c}, \ldots)$ is a set of lines $\overline{a}, \overline{b}, \overline{c}, \ldots$ incident to the point M.*

Definition 2.4 *Let M be a point of the EH-plane. A generalised pencil of lines with centre M, designated as $M(\widetilde{m}', \widetilde{m}'', m_i, \overline{m}_i)$, $i = 1, 2, 3, \ldots$, is a set of isotropic lines $\widetilde{m}'$ and $\widetilde{m}''$, lines m_i and ideal lines $\overline{m}_i$ incident to the point M.*

Using the initial relation of the separation of pairs of points on a line we define the relation of separation of pairs of generalised lines in a generalised pencil of lines.

Definition 2.5 *Let α, β, γ and δ be a four different generalised lines of a generalised pencil of lines with the centre M, and let A, B, C and D be respectively the points of intersections of these lines and a line p which is not incident to M. The pair α, β is separated by the pair γ, δ, designated as $s(\alpha, \beta; \gamma, \delta)$, if $s(A, B; C, D)$.*

Regarding Axiom II_{10} the separation of generalised lines of a generalised pencil does not depend on the line p.

On the basis of Axioms I_3, I_4, II_4, I_1 and Definition 2.5 we can establish:

Theorem 2.7 *If α, β and γ are three generalised lines of a generalised pencil of lines with the centre M, then there is a generalised line δ of that pencil so that $s(\alpha, \beta; \gamma, \delta)$.*

On the basis of Definition 2.5 and Axiom II_2 indirectly follows:

Theorem 2.8 *Let α, β, γ and δ be a four different generalised lines of a generalised pencil of lines with centre M. If $s(\alpha, \beta; \gamma, \delta)$, then $s(\alpha, \beta; \delta, \gamma)$, $s(\beta, \alpha; \gamma, \delta)$, $s(\beta, \alpha; \delta, \gamma)$, $s(\gamma, \delta; \alpha, \beta)$, $s(\gamma, \delta; \beta, \alpha)$, $s(\delta, \gamma; \alpha, \beta)$, $s(\delta, \gamma; \beta, \alpha)$.*

On the basis of Definition 2.5 and Theorem 2.1 we can establish:

Theorem 2.9 *Let α, β, γ and δ be a four different generalised lines of a generalised pencil of lines with centre M. If $s(\alpha, \beta; \gamma, \delta)$, then*

$$\begin{array}{cccc}
\neg s(\alpha, \gamma; \beta, \delta), & \neg s(\alpha, \gamma; \delta, \beta), & \neg s(\alpha, \delta; \beta, \gamma), & \neg s(\alpha, \delta; \gamma, \beta), \\
\neg s(\beta, \gamma; \alpha, \delta), & \neg s(\beta, \gamma; \delta, \alpha), & \neg s(\beta, \delta; \alpha, \gamma), & \neg s(\beta, \delta; \gamma, \alpha), \\
\neg s(\gamma, \alpha; \beta, \delta), & \neg s(\gamma, \alpha; \delta, \beta), & \neg s(\gamma, \beta; \alpha, \delta), & \neg s(\gamma, \beta; \delta, \alpha), \\
\neg s(\delta, \alpha; \beta, \gamma), & \neg s(\delta, \alpha; \gamma, \beta), & \neg s(\delta, \beta; \alpha, \gamma), & \neg s(\delta, \beta; \gamma, \alpha).
\end{array}$$

On the basis of Definition 2.5 and Theorem 2.2 we can establish:

Theorem 2.10 *If α, β, γ and δ are four different generalised lines of a generalised pencil of lines with the centre M, then exactly one of the relations hold: $s(\alpha, \beta; \gamma, \delta)$, $s(\alpha, \gamma; \beta, \delta)$, $s(\alpha, \delta; \beta, \gamma)$.*

If we notice three generalised lines α, β, γ and an ideal line $\overline{m}$ of a generalised pencil of lines with the centre M, according to Theorem 2.10 it follows that two pairs of that pencil which separate one another can be formed in a unique manner.

Let $\widetilde{m}$ be an arbitrary isotropic line and let P be na arbitrary point incident to it. According to Axiom I_6 for the point P besides the isotropic line $\widetilde{m}$ there exists an isotropic line $\widetilde{p}$ incident to P and an ideal line $\overline{m}$ such that $\widetilde{m} \cap \overline{m} = \vartheta$ and $\widetilde{p} \cap \overline{m} = \vartheta$. Let an arbitrary point M is incident to $\overline{m}$ chosen to be the centre of a generalised pencil of lines while α, β and γ are three generalised lines of this pencil.

Therefore, there is a generalised pencil of lines with centre M so that to the ideal line $\overline{m}$ of that pencil and a given isotropic line $\widetilde{m}$ holds $\overline{m} \cap \widetilde{m} = \vartheta$.

The relation of separation of pairs of lines in a generaslised pencil of lines enables the definition of the relation *between* on an isotropic line.

Definition 2.6 *Let $\widetilde{m}$ be an arbitrary line and let M is the centre of a generalised pencil of lines which is not incident to $\widetilde{m}$, so that to generalised lines α, β and γ and ideal line $\overline{m}$ of this pencil and $\widetilde{m}$ holds*

$$\alpha \cap \widetilde{m} = \{A\},\ \beta \cap \widetilde{m} = \{B\},\ \gamma \cap \widetilde{m} = \{C\},\ \overline{m} \cap \widetilde{m} = \vartheta.$$

The point B incident with the isotropic line $\widetilde{m}$ is between points A and C of that line, designated as $b(A, B, C)$, if $s(\alpha, \gamma; \beta, \overline{m})$.

According to Theorem 1.3 it follows that for a given ideal line $\overline{p}$ there is a point M which is not incident to $\overline{p}$ such that to isotropic lines $\widetilde{m}'$ and $\widetilde{m}''$ incident to M holds

$$\overline{p} \cap \widetilde{m}' = \vartheta \text{ and } \overline{p} \cap \widetilde{m}'' = \vartheta.$$

The relation of separation of pairs of lines in a generalised pencil of lines enables the definition of the relation *between* as well as on an ideal line.

Definition 2.7 *Let $\overline{p}$ bi an arbitrary ideal line and let M be the centre of a generalised pencil of lines which is not incident to $\overline{p}$, so that to lines a, b, c and isotropic line $\widetilde{m}'$ (isotropic line $\widetilde{m}''$) of this pencil holds:*

$$\overline{p} \cap a = \{A\},\ \overline{p} \cap b = \{B\},\ \overline{p} \cap c = \{C\},\ \overline{p} \cap \widetilde{m}' = \vartheta\ (\overline{p} \cap \widetilde{m}'' = \vartheta).$$

The point B incident to the ideal line $\overline{p}$ is between the points A and C of that line, designated as $b(A, B, C)$ if $s(a, c; b, \widetilde{m}')$.

2.1 Line segment, ideal line segment, isotropic line segment

Theorem 2.5 enables the introduction of a line segment in the EH-geometry. A line segment is defined in the same manner as in the elliptic geometry.

We cite theorems which characterize the relation between on an ideal line introduced by Definition 4.2.7.

Theorem 2.11 *If A and B are two different points incident to an ideal line $\overline{p}$, then there is a point C incident to $\overline{p}$ such that $b(A, C, B)$.*

Proof. For an ideal line $\overline{p}$, according to Theorem 1.3, there is a point M which is not incident to $\overline{p}$ and isotropic lines $\widetilde{m}'$ and $\widetilde{m}''$ incident to M such that $\widetilde{m}' \cap \overline{p} = \vartheta$ and $\widetilde{m}'' \cap \overline{p} = \vartheta$. According to Axiom I_7 there are lines a and b incident respectively to pairs of points A, M and B, M. Notice a generalised pencil of lines to the centre M, i.e. $M(\widetilde{m}', \widetilde{m}'', a, b, \ldots)$. According to Theorem 2.7 for lines a, b and $\widetilde{m}'$, i.e. a, b, $\widetilde{m}''$, there is a generalised line χ of a pencil of lines $M(\widetilde{m}', \widetilde{m}'', a, b, \chi, \ldots)$ such that $s(a, b; \widetilde{m}', \chi)$ and $s(a, b; \widetilde{m}'', \chi)$, and therefore, according to Theorem 2.8 we obtain

$$s(a, b; \chi, \widetilde{m}') \text{ and } s(a, b; \chi, \widetilde{m}''). \tag{1}$$

Let l be an arbitrary line which is not incident to M. On the basis Axiom I_4 we obtain

$$l \cap a = \{A'\},\ l \cap b = \{B'\},\ l \cap \chi = \{C'\},\ l \cap \widetilde{m}' = \{M'\},\ l \cap \widetilde{m}'' = \{M''\}.$$

Regarding (1), on the basis of Definition 2.5, we conclude that $s(A', B'; C', M')$ and $s(A', B'; C', M'')$, and according to Axiom II_3 it follows that $\neg s(A', C'; B', M')$ and $\neg s(A', C'; B', M'')$, and according to Axiom II_7 we conclude that $\neg s(A', C';\ M', M'')$, i.e. $\neg s(M', M''; A', C')$ (Theorem 2.4).

As a result, to the points M', M'', A and C incident to the line l holds $\neg s(M', M''; A', C')$, the pairs of points M', M and M'', M are incident respectively to isotropic lines $\widetilde{m}'$ and $\widetilde{m}''$, and the points A' and M are incident to line a, so that according to Axiom II_9 it follows that the points C' and M are incident to line. If we mark this line as c then it is identical with the generalised line χ of the pencil $M(\widetilde{m}', \widetilde{m}'', a, b, \chi, \ldots)$, i.e. $c = \chi$. According to Axiom I_4 for the line c and ideal line $\overline{p}$ there is a point C incident to each of them. Regarding (1), it follows that $s(a, b, c, \widetilde{m}')$. Since A, B and C are points incident to the ideal line $\overline{p}$ and at the same time intersection points of this line respectively to the lines a, b and c of the pencil $M(\widetilde{m}', \widetilde{m}'', a, b, c, \ldots)$, and also $\widetilde{m}' \cap \overline{p} = \vartheta$, from $s(a, b; c, \widetilde{m}')$, according Definition 2.7, is $b(A, C, B)$. ■

On the basis Theorem 2.7, Theorem 2.8, Axiom II_6, Theorem 2.4, we can establish:

Theorem 2.12 *If A and B are two different points incident to an ideal line $\overline{p}$, then there is a point C incident to $\overline{p}$ such that $b(A,B,C)$.*

On the basis of Definition 2.7, Definition 2.5 and Axiom II_1, we can establish:

Theorem 2.13 *If to the points A, B and C incident to an ideal line $\overline{a}$ holds $b(A,B,C)$ then A, B and C are three different points.*

On the basis of Definition 2.7 and Theorem 2.8 we can establish:

Theorem 2.14 *If to the points A, B and C incident to an ideal line $\overline{a}$ holds $b(A,B,C)$, then $b(C,B,A)$.*

On the basis of Definition 2.7 and Theorem 2.9 we can establish:

Theorem 2.15 *If to the points A, B and C incident to an ideal line $\overline{a}$ holds $b(A,B,C)$, then $\neg b(A,C,B)$.*

On the basis of Definition 2.7 and Theorem 2.10 we can establish:

Theorem 2.16 *If A, B and C are three different points with the same ideal line, then only one of the relations is valid: $b(A,B,C)$, $b(B,C,A)$, $b(C,A,B)$.*

Definition 2.8 *For a finite set of points $A_1, A_2, \ldots, A_n, n > 3$, incident to an ideal line, we can say that it is linearly arranged, designated as $b(A_1, A_2, \ldots, A_n)$, if $b(A_i, A_j, A_k)$ in the case when $1 \leq i < j < k \leq n$.*

If $b(A_1, A_2, \ldots, A_i, A_{i+1}, \ldots, A_n)$ then according to Definition 2.8, Theorem 2.14 and Theorem 2.15, it follows that

$$b(A_n, A_{n-1}, \ldots, A_{i+1}, A_i, \ldots, A_1) \text{ and } \neg b(A_1, A_2, \ldots, A_i, A_{i+1}, \ldots, A_n).$$

On the basis of Theorems 1.3, 2.1, 2.4, 2.2, Axiom II_7, Definitions 2.7, 2.5 and 2.8, we can establish:

Theorem 2.17 *If the points A, B, C and D are incident to an ideal line $\overline{p}$ and at the same time holds $b(A,B,C)$ and $b(B,C,D)$, then $b(A,B,C,D)$.*

In the same manner by referring to Theorem 2.17, we can establish:

Theorem 2.18 *If the points A, B, C and D are incident to an ideal line $\overline{p}$ and at the same time hold $b(A,B,C)$ and $b(A,C,D)$, then $b(A,B,C,D)$.*

On the basis of Theorem 1.3, Axioms II_7, I_3, Theorems 2.2, 2.18, Axiom II_2 and Theorem 2.14, we can establish:

Theorem 2.19 *If the points A, B, C and D are incident to the same ideal line $\overline{p}$ and at the same time hold $b(A,B,C)$ and $b(A,B,D)$, where C and D are different points, then either $b(A,B,C,D)$ or $b(A,B,D,C)$.*

On the basis if Theorems 2.18, 2.2, 2.14 and 2.17, we can establish:

Theorem 2.20 *If the points A, B, C and D are incident to the same ideal line $\overline{a}$ and at the same time hold $b(A,C,B)$ and $b(A,D,B)$, where C and D are different points, then either $b(A,D,C,B)$ or $b(A,C,D,B)$.*

At this point an ideal line segment can be defined as:

Definition 2.9 *Let A and B be two different points incident to an ideal line $\overline{a}$. An open ideal line segment, designated as $(\overline{AB})$, is the set of all points of an ideal line $\overline{a}$ which are between A and B.*

We can establish theorems analogous to Theorems 2.11 to 2.20, which characterize the relation between on an isotropic line introduced by Definition 2.6, and then an *isotropic line segment* can be defined (similarly to Definition 2.9).

2.2 Ideal ray and isotropic ray

Using the relation between on an ideal line, introduced by Definition 2.7, we can define a new relation *on the same side* of a point on an ideal line.

Definition 2.10 *Let $\overline{a}$ be an ideal line and let A, X and Y be three different points incident to it. The points X and Y are on the same side of the point A, designated as $\eta_A(X,Y)$, if $\neg b(X,A,Y)$. If $b(X,A,Y)$ then we conclude that the points X and Y are on the different sides of the point A.*

On the basis of Theorems 2.14, 2.16, 2.18, 2.17, 2.12, 2.19 and Definitions 2.10 and 2.8, we can establish:

Theorem 2.21 *Let a point A is incident to an ideal line $\overline{a}$. The relation η_A on the ideal line $\overline{a}$ is the equivalence relation which separates the set of points $\overline{a} \backslash \{A\}$ into two classes of equivalence.*

Definition 2.11 *Let a point A is incident to an ideal line $\overline{a}$. An open ideal ray of the ideal line $\overline{a}$ is a class of equivalence of the relation η_A on $\overline{a}$.*

Using the relation between on an isotropic line (Definition 2.6) in the same manner as in Definition 2.10, we can define a new relation *on the same side* of a point on this line. A theorem analogous to Theorem 2.21 can be established, and then in the analogous manner as in Definition 2.11 *an isotropic ray* can be defined.

2.3 Angular line and angle

According to Axiom I_4 for any two lines of the EH-plane there is a point O incident to each of them which is called a point of their intersection.

Definition 2.12 *Let a and b are different lines intersecting in a point O. An angular line ab, designated as $\angle ab$, is a set of points incident to the lines a and b. The lines a and b are called sides, and the point O of the intersection is the vertex of the angular line ab.*

Theorem 2.6 enabled the introduction of the pencil of lines with the center M by Definition 2.2, i.e. $M(a, b, c, \ldots)$. The relation between on an ideal line introduced by Definition 2.7 enables the definition of the relation between in the pencil of lines $M(a, b, c, \ldots)$.

Definition 2.13 *Let $M(a, b, c, \ldots)$ be a pencil of lines with a center M, let $\overline{p}$ be an ideal line which is not incident to M, such that $\overline{p} \cap \widetilde{m}' = \vartheta$, $\overline{p} \cap \widetilde{m}'' = \vartheta$, where $\widetilde{m}'$ and $\widetilde{m}''$ are isotropic lines incident to the point M, and at the same time hold $\overline{p} \cap a = \{A\}, \overline{p} \cap b = \{B\}, \overline{p} \cap c = \{C\}$. The line b of the pencil of lines $M(a, b, c, \ldots)$ is between the lines a and c of the pencil, designated as $b(a, b, c)$, if $b(A, B, C)$.*

On the basis of Axioms I_6, I_4, Theorem 2.11, and Axiom I_7, we can establish:

Theorem 2.22 *If a and b are two different lines of the pencil of lines with the center M, then there is a line c of that pencil such that $b(a, c, b)$.*

On the basis of Axioms I_6, I_4, Theorem 2.12, and Axiom I_7, we can establish:

Theorem 2.23 *If a and b are two different lines of the pencil of lines with the center M, then there is a line c of that pencil such that $b(a, b, c)$.*

On the basis of Axioms I_6, I_4, Theorem 2.14, and Definition 2.12, we can establish:

Theorem 2.24 *If* a, b *and* c *are three different lines of the pencil of lines* $M(a, b, c, \ldots)$*, where* $b(a, b, c)$*, then* $b(c, b, a)$*.*

On the basis of Axioms I_6, I_4, Theorem 2.15 and Definition 2.12, we can establish:

Theorem 2.25 *If* a, b *and* c *are three lines of the pencil of lines* $M(a, b, c, \ldots)$*, where* $b(a, b, c)$*, then* $\neg b(a, c, b)$*.*

On the basis of Axioms I_6, I_4, Theorem 2.16 and Definition 2.12, we can establish:

Theorem 2.26 *If* a, b *and* c *are three lines of the pencil of lines* $M(a, b, c, \ldots)$*, then only one of the relations is valid* $b(a, b, c)$*,* $b(b, c, a)$*,* $b(c, a, b)$*.*

At this point an angle can be defined.

Definition 2.14 *If* ab *is an angular line with vertex* O*, then the set of points incident to the lines of the pencil with the center* O *which are between this angular line, without the point* O*, is called an open angle* ab *and marked as* $\angle(ab)$*.*

2.4 A triangle figure and triangle

On the basis of Axioms I_3, I_6, I_4, I_2, and I_7, we can establish:

Theorem 2.27 *There are three different points* A*,* B *and* C *which are not at the same time incident to the same line and each pair is at the same time a pair of collinear points.*

Definition 2.15 *Let* A, B *and* C *are three different points where each pair is at the same time a pair of collinear points which are not at the same time incident to the same line. The set of the points* A, B *and* C *is called a triangle figure* ABC*.*

On the basis of Axioms I_4, II_{10}, II_7, Theorem 2.4 and Definition 2.5, we can establish:

Theorem 2.28 *Let* ABC *is a triangle figure and* μ *and* μ' *are two generalised lines (two lines, or a line and isotropic line, or a line and ideal line, or an isotropic and ideal line) which are not incident to any of the points* A, B *and* C*. If* P, Q *and* R *(*P', Q' *and* R"*) are the points obtained in the intersection of the generalised line* μ *(*μ'*) respectively with the lines* $p(AB)$*,* $p(BC)$ *and* $p(AC)$ *where* $R \neq R'$*,* $\neg s(P, P'; A, B)$ *and* $\neg s(Q, Q'; B, C)$*, then* $\neg s(R, R'; A, C)$*.*

In order to define a triangle in the EH-geometry, similarly to elliptic geometry, of six possible line segments determined by the triangle figure ABC, we should decide which three have the role of the sides of triangle. This decision will be made in such a manner to satisfy Pashov's axiom.

Definition 2.16 *Let ABC be a triabgle figure. The union of the line segment $[AB]$, $[BC]$ and $[AC]$ is called a triangle, if there is a generalised line μ (aline m, or an isotropic line $\widetilde{m}$, or an ideal line $\overline{m}$), and if μ intersects one of the line segments, i.e. (AB), then it intersects one and only one of the remaining two line segments (BC) and (AC).*

Theorem 2.29 *Let ABC be a triangle figure and $\widetilde{a}'$, $\widetilde{a}''$, $\widetilde{b}'$, $\widetilde{b}''$ and $\widetilde{c}'$, $\widetilde{c}''$ are isotropic lines incident respectively to the vertexes A, B and C. If*

$$\widetilde{a}' \cap p(BC) = \{A'\},\ \widetilde{a}'' \cap p(BC) = \{A''\},\ \widetilde{b}' \cap p(AC) = \{B'\},$$

$$\widetilde{b}'' \cap p(AC) = \{B''\},\ \widetilde{c}' \cap p(AB) = \{C'\},\ \widetilde{c}'' \cap p(AB) = \{C''\},$$

then the points A' and A'' are incident to the same line segment determined by the points B and C on the line $p(BC)$; the points B' and B'' are incident to the same line segment determined by the points A and C on the line $p(AC)$; and the points C' and C'' are incident to the same line segment determined by the points A and B on the line $p(AB)$.

On the basis of Axioms I_6, Theorem 2.29 and Axioms II_4, II_8, I_7, II_3 and II_7, we can establish:

Theorem 2.30 *In any triangle there is one and only one vertex which is incident to at least one ideal line which intersects the opposite side of the triangle.*

Definition 2.17 *If in the triangle $\triangle ABC$ the ideal line $\overline{m}$ is incident to the vertex B where $\overline{m}$ intersects the opposite side (AC) in the point M, then the point B is the middle vertex, the points A and C are lateral vertexes. The line segment AC_M is the basic side of the triangle, and the sides AB and BC are lateral sides. The angles at the vertexes A and C are called the inner angles, and the angle at the vertex B is the outer angle of the triangle.*

Regarding the above analysis it is not difficult to conclude that the triangle figure ABC determines three triangles with common vertexes A, B and C.

3 Non-contradictoriness of the order geometry in the EH-geometry

The realization of this geometry will be given in a projective plane. Suppose that a given real projective plane ω has been determined axiomatically (see [6]). In that plane an undegenerated hyperbolic polarity π has been given as well as a real undegenerated curve of the second order k as a set of autoconjugating points in relation to this polarity. The curve k is called *absolute*.

The basic objects and relations in the *projective model* will be defined.

Definition 3.1 *The points of the projective plane ω which at the same time represent outer points in relation to the absolute k, not including the points on the absolute, are points of the EH-geometry or briefly EH-points.*

The set of points dealt with in Definition 3.1 are marked as S_k.

Definition 3.2 *The set S_k makes the EH-plane.*

Definition 3.3 *A projective line a which has no common points with the absolute k is referred to as EH-line a.*

Definition 3.4 *Let the projective line a is a tangent line of the absolute k, i.e. $a \cap k = \{P\}$. The line a without the point P, $a \backslash \{P\}$, is called EH-isotropic line $\widetilde{a}$.*

Definition 3.5 *Let a be a projective line which intersects the absolute k, i.e. $a \cap k = \{P, Q\}$. The open projective line segment (PQ) of the line a which represents a subset of the set S_k, i.e. $a \cap S_k$, is EH-ideal line $\overline{a}$.*

Definition 3.6 *An EH-point A is EH-incident to the EH-line a, designated as EH-$i(A, a)$, if A in the projective sense is incident to the projective line a. Analogously, the EH-point A is EH-incident to EH-isotropic line $\widetilde{a}$ (EH-ideal line $\overline{a}$), if A in the projective sense is incident to $a \backslash \{P\}$ (with $a \cap S_k$).*

Definition 3.7 *Let EH-points A, B, C and D are EH-incident to the EH-line a. The pair A, B EH-separates the pair C, D designated as EH-$s(A, B; C, D)$, if those pairs separate each other in the projective sense.*

It has been established that the described *projective model* meets all axioms of incidence and order suggested in the first and second chapter. This has proved that the suggested system of axioms is non-contradictory, namely that order geometry in the EH-geometry is non-contradictory if projective geometry is non-contradictory.

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Faculty of Economics, University of Niš
P. O. Box 121, 18000 Niš, Yugoslavia