On the Stone-Čech compactification of Ω-spaces

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Abstract

In this paper[1] we study some notions related to the space X for which the remainder $\beta X \setminus \beta(X)$ is one-point set.

1 Introduction and definitions

The closure of a subset A of a space X is denoted by $cl_X(A)$ and the one-point compactification of X is denoted by ωX. In this paper we assume that all spaces are noncompact and T_2. We use the standard definitions for filter-base and filter. For notions and definitions not given here see [3], [6], [8].

Definition 1.1 *Let X be a topological space. Then:*

(a) *The symbol $¶(X)$ denotes the family of all subsets of X.*

(b) $\mathfrak{P}(X) = ¶(X) \setminus \{\emptyset,\}$.

(c) *By $\mathfrak{K}(X)$ we denote the family of all nonempty compact subsets of X.*

(d) $\Omega(X) = \{U : U \subset X \wedge U = X \setminus K,\ K \in \mathfrak{K}(X)\}$. *($\Omega(X) \subset \mathfrak{P}(X)$ if X is noncompact.)*

(e) *$\mathfrak{C}^*(X)$ denotes the ring of all bounded continuous real-valued functions defined on X.*

It is clear that the family $\mathfrak{P}(X) \setminus \{X\}$ is a refinement of $\Omega(X)$. A filter-base in $\mathfrak{P}(X)$ is a non-empty family $\mathfrak{B} \subset \mathfrak{P}(X)$ such that if $A_1,\ A_2 \in \mathfrak{B}$, then there exists an $A_3 \in \mathfrak{B}$ such that $A_3 \subset A_1 \cap A_2$ [2]. By a filter in $\mathfrak{P}(X)$ we mean a non-empty subfamily $\mathfrak{F} \subset \mathfrak{P}(X)$ satisfying the following conditions:

(a) If $A_1,\ A_2 \in \mathfrak{F}$, then $A_1 \cap A_2 \in \mathfrak{F}$.

(b) If $A \in \mathfrak{F}$ and $A \subset A_1 \in \mathfrak{P}(X)$, then $A_1 \in \mathfrak{F}$ [3].

By a filter (filter-base) in a topologigal space X we mean a filter (filter-base) in the family $\mathfrak{P}(X)$.

One readily sees that for any filter-base $\mathfrak{B}$ in $\mathfrak{P}(X)$, the family $\mathfrak{F}_{\mathfrak{B}} = \{A \in \mathfrak{P}(X) : \textit{there exists a } B \in \mathfrak{B} \textit{ such that} B \subset A\}$ is a filter in $\mathfrak{P}(X)$.

Definition 1.2 ([2]) *Filter-bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent if $\mathfrak{F}_{\mathfrak{B}_1} = \mathfrak{F}_{\mathfrak{B}_2}$.*

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Lemma 1.1 ([2]) *Filter-bases* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are equivalent if and only if for every* $B_1 \in \mathfrak{B}_1$ *there exists a* $B_2 \in \mathfrak{B}_2$ *such that* $B_2 \subset B_1$ *and for every* $B_2 \in \mathfrak{B}_2$ *there exists a* $B_1 \in \mathfrak{B}_1$ *such that* $B_1 \subset B_2$.

A point x is called a cluster point of a filter $\mathfrak{F}$(of a filter-base $\mathfrak{B}$) if x belongs to the closure of every member of $\mathfrak{F}$(of $\mathfrak{B}$). A point x is called a limit of filter $\mathfrak{F}$ (of filter-base $\mathfrak{B}$) if every neighbourhood of x is a member of $\mathfrak{F}$ ($\mathfrak{F}_{\mathfrak{B}}$); we then say that the filter (filter-base) converges to x.

Proposition 1.2 ([3]) *The point* x *belongs to* $cl_X(A)$ *if and only if there exists a filter-base consisting of subsets of* A *and converging to* x.

Definition 1.3 *A filter base* $\mathfrak{B} \subset \mathfrak{P}(X)$ *is called a free filter-base if for every point* $x \in X$, *the point* x *is not a cluster point of* $\mathfrak{B}$.

2 Ω-spaces

The following proposition gives an information when the family $\Omega(X)$ satisfies the free filter-base property.

Proposition 2.1 *Let* X *be a noncompact* T_2 *space. Then the family* $\Omega(X)$ *is a free filter-base in if and only if* X *is a locally compact space.*

Proof. $\Rightarrow$: Let $\Omega(X)$ is a free filter-base in X. Then for every point $x \in X$ there exists an open neighbourhood U_x such that for some $A \in \Omega(X)$ the set $A \cap U_x = \emptyset$. Hence $U_x \subset X \setminus A = K \in \mathfrak{K}(X)$. The closure $cl_X(U_x) \in \mathfrak{K}(X)$. This proves that X is locally compact space.

$\Leftarrow$: It suffices to show that if $A_1,\ A_2 \in \Omega(X)$, then there exists an $A_3 \in \Omega(X)$ such that $A_3 \subset A_1 \cap A_2$. Let $A_1 = X \setminus K_1,\ A_2 = X \setminus K_2;\ K_1,\ K_2 \in \mathfrak{K}(X)$. As $K_1 \cup K_2 \in \mathfrak{K}(X)$ the set $A_3 = A_1 \cap A_2 = (X \setminus K_1) \cap (X \setminus K_2) = X \setminus (K_1 \cup K_2) \in \Omega(X)$. Since X is locally compact, for every $x \in X$ there exists a neighbourhood U of the point x such that $cl_X(U) \in \mathfrak{K}(X)$. Then the set $U \cap (X \setminus cl_X(U)) = \emptyset$. Hence, the point x does not belong to the closure of every member of $\Omega(X)$. It follows that x is not a cluster point of the filter base $\Omega(X)$. $\square$

Definition 2.1 *A topological space* X *is called an* Ω*-space if all free filter-bases in* X *are equivalent to the* $\Omega(X)$.

Example 2.1 Let $X = [0, \omega_1)$ be the space of ordinals less than the first uncountable ordinal with the order topology. It is clear (by the order topology and Lemma 1.1.) that every free filter-base in $[0, \omega_1)$ is equivalet to the $\Omega([0, \omega_1))$.

A pair (Y, c), where Y is a compact space and $c : X \longrightarrow Y$ is a homeomorphic embedding of X in Y such that $cl_Y(c(X)) = Y$, is called a compactification of the space X (see [3]). Compactifications c_1X and c_2X of a space X are equivalent if there exists a homeomorphism $f : c_1X \longrightarrow c_2X$ such that $f(c_1x)) = c_2(x)$ for every $x \in X$.

The following theorem shows that when in Ω-spaces the one-point compactification is equivalent to the Stone-Čech compactification.

Theorem 2.2 *Let X be an Ω-space. Then the one-point compactification of X is equivalent to the Stone-Čech compactification of X.*

Proof. It suffices to show that for every compactificatition cX of the space X the remainder $cX \setminus c(X)$ is a one point set. Assume that the remainder $cX \setminus c(X)$ contains two distinct points x_1 and x_2. By Proposition 1.2, there exist filter-bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ consisting of subsets of $c(X)$ such that $\mathfrak{B}_1$ converging to x_1 and $\mathfrak{B}_2$ to x_2. It is known that a space X is a Hausdorff space if and only if every filter in X has at most one limit. This implies, in particular, that filter-bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are free filter-bases in the space $c(X)$ and $\mathfrak{B}_1$ is not equivalent to $\mathfrak{B}_2$. Denote $\mathfrak{A}_1 = c^{-1}(\mathfrak{B}_1) = \{c^{-1}(B_1) : B_1 \in \mathfrak{B}_1\}$, $\mathfrak{A}_2 = c^{-1}(\mathfrak{B}_2) = \{c^{-1}(B_2) : B_2 \in \mathfrak{B}_2\}$. Since the space X is homeomorphic to $c(X)$, we have that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are free filter-bases in the space X and $\mathfrak{A}_1$ is not equivalent to $\mathfrak{A}_2$. By assumption X is an Ω- space, a contradiction. Hence, for every compactification cX of the space X, the remainder $cX \setminus c(X)$ is a one point set. Furthermore, the mapping of cX to $\omega X = X \cup \{\Omega\}$ defined by

$$f(x) = \begin{cases} i \circ c^{-1}(x), & \text{if } x \in c(X), \\ \Omega, & \text{if } x \in cX \setminus c(X) \end{cases}$$

is a homeomorphism. This proves that ωX is equivalent to βX. □

Remark 2.1 The result of Theorem 2.2, can be also described in terms of nets. By Theorem 2.2, every Ω- space has a unique (up to equivalence) compactification. The following example shows that there exists a space X which has a unique compactification and is not a Ω-space.

Example 2.2 Let W be the space of all ordinal numbers $\leq \omega_1$ and W' the subspace consisting of all numbers $\leq \omega_0$; the space $T = W \times W' \setminus \{(\omega_1, \omega_0)\}$ is called the Tychonoff plank. It is known that the Tychonoff plank has a unique compactification. We shall now show that T is not an Ω- space. Consider the subspaces $A = [0, \omega_1) \times \{\omega_0\}$ and $B = \{\omega_1\} \times [0, \omega_0)$. Denote $\mathfrak{A} = \{[\alpha, \omega_1) \times \{\omega_0\} : \alpha \in [0, \omega_1)\}$, $\mathfrak{B} = \{\{\omega_1\} \times [\beta, \omega_0) : \beta \in [0, \omega_0)\}$. It is clear that $\mathfrak{A}$ and $\mathfrak{B}$ are free filter-bases in the space T and $\mathfrak{A}$ is not equivalent to $\mathfrak{B}$.

Proposition 2.3 *If $f : X \longrightarrow Y$ is a continuous mapping of a Tychonoff space X which has a unique compactification, onto a non-compact Tychonoff space Y, then Y has a unique compactification and f is a perfect mapping.*

Proof. It is known that the Tychonoff space X has a unique(up to equivalence) compactification if and only if for any two closed subsets of X which are completely separated, at least one is compact (See [1, IV, 23]). If $P \subset Y,\ Q \subset Y$ are closed completely separated subsets, then there exists a function $g \in \mathfrak{C}^*(Y)$ such that $g(P) = 0$ and $g(Q) = 1$. By continuity,the sets $A = f^{-1}(P)$ and $B = f^{-1}(Q)$ are closed and $A \cap B = \emptyset$. Therefore, $(g \circ f)(A) = g((f \circ f^{-1})(P) = g(P) = 0$ and $(g \circ f)(B) = g((f \circ f^{-1})(Q) = g(Q) = 1$. Hence, the subsets $A,\ B$ are completely separated. By assumption the space X has a unique compactification. If $A = f^{-1}(P)$ is compact in X, then $f(A) = f \circ f^{-1}(P) = P$ is compact in Y. This implies that the space Y has a unique compactification. Hence, $\beta X \approx \omega X = X \cup \{\omega_X\}$ $(\omega_X \notin X)$ and $\beta Y \approx \omega Y = X \cup \{\omega_Y\}$ $(\omega_Y \notin Y)$. The mapping $f : X \longrightarrow \omega Y$ is extendable to a mapping $F : \omega X \longrightarrow \omega Y$. It is clear that $\omega Y \subseteq F(\omega X)$ and $F(\omega X) \subseteq \omega Y$ which implies that $F(\omega X) = \omega Y$. Since the spaces ωX and ωY are compact, $F : \omega X \longrightarrow \omega Y$ is a perfect mapping. Furthermore, $F(\omega X) = F(X \cup \{\omega_X\}) = F(X) \cup F(\{\omega_X\}) = f(X) \cup F(\{\omega_X\}) = Y \cup F(\{\omega_X\}) = \omega Y = Y \cup \{\omega_Y\}$, which implies that $F(\omega_X) = \omega_Y$. For every $y \in Y$ fibers $f^{-1}(y)$ are compact subsets of X. For every closed subset $A \subset X$ the subset $A \cup \{\omega_X\}$ is compact and closed in ωX. By continuity, $F(A \cup \{\omega_X\}) = F(A) \cup F(\{\omega_X\}) = f(A) \cup \{\omega_Y\}$, is a compact and closed subset of $\omega Y = Y \cup \{\omega_Y\}$. This implies that $f(A) \subset Y$ is a closed subset in Y. Hence f is a perfect mapping. □

Proposition 2.4 *If there exists a continuous, open mapping $f : X \longrightarrow Y$ of an Ω-space X onto a Hausdorff space Y, then Y is an Ω-space.*

Proof. Let X be an Ω-space. Hence X is Hausdorff, locally compact and pseudocompact space for which every free filter-base is equivalent to $\Omega(X)$. Since local compactness is an invariant of continuous open mappings we have that Y is a locally compact space. This implies that $\Omega(Y)$ is a free filter-base in Y. Let $\mathfrak{B}_Y \neq \Omega(Y)$ be any free filter-base in Y. By continuity of f, $f^{-1}(\mathfrak{B}_Y)$ and $f^{-1}(\Omega(Y))$ are free filter-bases in X. From the definition of Ω-property it follows that free filter-bases $f^{-1}(\mathfrak{B}_Y)$ and $f^{-1}(\Omega(Y))$ are equivalent to the $\Omega(X)$. By Lemma 1.1, $\mathfrak{B}_Y = f(f^{-1}(\mathfrak{B}_Y))$ is equivalent to $\Omega(Y) = f(f^{-1}(\Omega(Y)))$. By Definition 2.1, the space Y is an Ω-space. □

Proposition 2.5 *Let Y be a closed subspace of an Ω-space X. If for all $U \in \Omega(X)$ the set $U \cap Y \neq \emptyset$, then Y is an Ω-space.*

Proof. Since local compactness is hereditary with respect to closed subsets the subspace Y is locally compact. By Proposition 2.1, the family $\Omega(Y)$ is a free filter-base in Y. Let $\mathfrak{B} \neq \Omega(Y)$ be a free filter base in Y. It is clear that the family $\mathfrak{B}$ is a free filter-base in X. By assumption $\mathfrak{B}$ is equivalent to $\Omega(X)$. For all $U \in \Omega(X)$ the sets $U \cap Y$are nonempty. This implies, in particular, that the free filter-base $\mathfrak{B}$ is equivalent to $\Omega(Y)$. □

3 Equiscalar space

Definition 3.1 *A topological space X is called equiscalar if for each $f \in \mathfrak{C}^*(X)$ there exists a $U \in \Omega(X)$ such that $f/U : U \longrightarrow R$ is constant.*

Remark 3.1 The space X in Example 2.1, is both equiscalar and an Ω-space. By the next theorem, the one-point compactification of X is equivalent to the Stone-Čech compactification.

Theorem 3.1 *Let X be an equiscalar space. Then the one-point compactification of X is equivalent to the Stone-Čech compactification.*

Proof. Let $\omega X = X \cup \{\infty\}$ be the one-point compactification of X. By Corollary 3.6.3, in [3], it suffices to show that every continuous function $f : X \longrightarrow I$ from the space X to the closed interval I is extendable to a function $F : \omega X \longrightarrow I$. Since the space X is equiscalar, we extend f to the corner point ∞ by assigning the value r, where $r = f(U)$; $U \in \Omega(X)$ (see Definition 3.1) at that point and this gives us a continuous extension of f. By Theorem 3.6.3, in [3] the one-point compactification of X is equivalent to the Stone-Čech compactification. □

Proposition 3.2 *If there exists a continuous mapping $f : X \longrightarrow Y$ of an equiscalar space X onto a Tychonoff space Y, then Y is an equiscalar space.*

Proof. Let X be a equiscalar space and $f : X \longrightarrow Y$ a continuous surjection. Let g be any function of $\mathfrak{C}^*(Y)$. Then $g \circ f \in \mathfrak{C}^*(Y)$, hence there exists a $U \in \Omega(X)$ such that $(g \circ f)/U$ is constant. Since $f(X \setminus U) \in \mathfrak{K}(Y)$, set $V = Y \setminus f(X \setminus U) \in \Omega(Y)$. Therefore, g/V is constant. Hence Y is an equiscalar space. □

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