Selective versions of screenability

Liljana Babinkostova

Abstract

In this paper[1] we introduce a new selection principle for topological spaces and consider its connection with the classical selection principles.

1 Introduction

In 1973 in [4] W. Haver introduced a covering property, called property C, for metric spaces. Addis and Gresham reformulated the original definition of C as follows in [1] :

Definition 1 *A topological space X has property C if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ of families of open subsets of X satisfying the following three conditions:*
(1) *For each n $\mathcal{V}_n$ is a disjoint family;*
(2) *For each n $\mathcal{V}_n$ refines $\mathcal{U}_n$ (we write $\mathcal{V}_n < \mathcal{U}_n$);*
(3) *$\bigcup_{n\in\mathbf{N}} \mathcal{V}_n$ is an open cover of X.*

The case when for each n we have $\mathcal{U}_n = \mathcal{U}$, some fixed open cover of X, was considered in [3] by Bing, and the corresponding property was called *screenability* by Bing. Addis and Gresham's reformulation of Haver's property C thus is a selective version of screenability.

We now introduce the following new selection hypothesis.

Definition 2 *Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of topological space X. The symbol $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ of families such that*
(i) each $\mathcal{V}_n$ is a pairwise disjoint collection of open sets,
(ii) for each n, if $V \in \mathcal{V}_n$, then $V \subset G$ for some member $G \in \mathcal{U}_n$ and
(iii) the family $\bigcup_{n\in\mathbf{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$.

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be any of the following classes of open covers:

[1] Presented at the IMC "Filomat 2001", Niš, August 26–30, 2001

2000 Mathematics Subject Classification: 54D20, 54E99
Keywords: Selection principles, screenability, property C, ω-cover, γ-cover

$\mathcal{O}$ – the collection of all open covers of the space;

Ω – the collection of ω-covers of the space. An open cover $\mathcal{U}$ of X is an *ω-cover* if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$;

Γ – the collection of γ-covers of the space. A cover $\mathcal{U}$ of X is said to be a *γ-cover* if it is infinite and for each $x \in X$ the set $\{U \in \mathcal{U} : x \notin X\}$ is finite.

The property of anti-monotonicity in the first variable, and monotonicity for the second is true for the selection principle $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$. This means that if $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are nonempty subsets of $\mathcal{O}$ such that $\mathcal{A} \subset \mathcal{B}$, then we have: $\mathsf{S}_c(\mathcal{D}, \mathcal{A}) \subseteq \mathsf{S}_c(\mathcal{D}, \mathcal{B})$ and $\mathsf{S}_c(\mathcal{A}, \mathcal{D}) \supseteq \mathsf{S}_c(\mathcal{B}, \mathcal{D})$. Directly from the inclusions $\Gamma \subseteq \Omega \subseteq \Lambda \subseteq \mathcal{O}$ and the monotonicity properties of the selection principle S_c, we have the following diagram:

$$
\begin{tikzcd}
\mathsf{S}_c(\Gamma,\Gamma) \arrow[r] & \mathsf{S}_c(\Gamma,\Omega) \arrow[r] & \mathsf{S}_c(\Gamma,\Lambda) \arrow[r] & \mathsf{S}_c(\Gamma,\mathcal{O}) \\
\mathsf{S}_c(\Omega,\Gamma) \arrow[r] \arrow[u] & \mathsf{S}_c(\Omega,\Omega) \arrow[r] \arrow[u] & \mathsf{S}_c(\Omega,\Lambda) \arrow[r] \arrow[u] & \mathsf{S}_c(\Omega,\mathcal{O}) \arrow[u] \\
\mathsf{S}_c(\Lambda,\Gamma) \arrow[r] \arrow[u] & \mathsf{S}_c(\Lambda,\Omega) \arrow[r] \arrow[u] & \mathsf{S}_c(\Lambda,\Lambda) \arrow[r] \arrow[u] & \mathsf{S}_c(\Lambda,\mathcal{O}) \arrow[u] \\
\mathsf{S}_c(\mathcal{O},\Gamma) \arrow[r] \arrow[u] & \mathsf{S}_c(\mathcal{O},\Omega) \arrow[r] \arrow[u] & \mathsf{S}_c(\mathcal{O},\Lambda) \arrow[r] \arrow[u] & \mathsf{S}_c(\mathcal{O},\mathcal{O}) \arrow[u]
\end{tikzcd}
$$

Figure 1: S_c monotonicity diagram

2 Some equivalences for $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$

Theorem 1 *For a topological space X the following statements are equivalent:*
(i) X has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$,
(ii) X has property $\mathsf{S}_c(\Omega, \mathcal{O})$.

Proof. We must prove that (ii) implies (i). Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of open covers of X.

Let $(Y_n : n \in \mathbf{N})$ be a pairwise disjoint sequence of infinite sets of positive integers whose union is the set $\mathbf{N}$. For each n define $\mathcal{W}_n$ to be the set whose elements are of the form $U_{n_1} \cup U_{n_2} \cup ... \cup U_{n_k}$, where k is any positive integer and $n_1 < n_2 < ... < n_k$ are elements of Y_n and each U_{n_j} is an element of $\mathcal{U}_{n_j}$.

If there is an n for which $X \in \mathcal{W}_n$, then $X = U_{n_1} \cup U_{n_2} \cup ... \cup U_{n_k}$ for k a positive integer and $n_1 < n_2 < ... < n_k$ elements of Y_n.

For n_i put $\mathcal{V}_{n_i} = \{U_{n_i}\}$, $i \leq k$, and for $j \in \mathbf{N} \setminus \{n_1, \cdots, n_k\}$ pick $U_j \in \mathcal{U}_j$ arbitrarily, and set $\mathcal{V}_j = \{U_j\}$. Then the sequence $(\mathcal{V}_j : j \in \mathbf{N})$ witnesses $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ for X for the sequence of $\mathcal{U}_n$'s.

So, we may assume that X is in no $\mathcal{W}_n$. Then each $\mathcal{W}_n$ is an ω-cover of X. Now apply the fact that X is in $\mathsf{S}_c(\Omega, \mathcal{O})$ and choose for each n a W_n such that W_n refines $\mathcal{W}_n$ and W_n is pairwise disjoint collection of open sets, and $\cup\{W_n : n \in \mathbf{N}\}$ is an open cover of X. For each n and for each $U \in W_n$ we take $F(U) \in \mathcal{W}_n$ such that $U \subset F(U)$. Fix for each such U a representation

$$F(U) = U_{i_1} \cup U_{i_2} \cup ... \cup U_{i_k},\ i_1 < i_2 < ... < i_k \in Y_n$$

and define $Ind(U) = \{i_1, i_2, ..., i_k\}$. Then W_n is used as follows to define a refinement for each $\mathcal{U}_m$ with $m \in Y_n$: For each $i \in Y_n$, we define $\mathcal{V}_i$ in the following way: If $U \in W_n$ has $i \in Ind(U)$ we put $S_i(U) = U \cap U_i$. Then we define $\mathcal{V}_i = \{S_i(U) : U \in W_n \text{ and } i \in Ind(U)\} \setminus \{\emptyset\}$. Then $\mathcal{V}_i$ is a pairwise disjoint collection of open sets and refines $\mathcal{U}_i$ and the union $\cup\{\mathcal{V}_i : i \in \mathbf{N}\}$ is an open cover of X. $\diamondsuit$

Theorem 2 $\mathsf{S}_c(\mathcal{O}, \mathcal{O}) = \mathsf{S}_c(\Lambda, \Lambda)$

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of λ-covers for a space X. From each cover $\mathcal{U}_n$ we choose λ-cover $\mathcal{V}_n \subset \mathcal{U}_n$, such that for each $m \neq n$, $\mathcal{V}_n \cap\ \mathcal{V}_m = \emptyset$. Let $(Y_n : n \in \mathbf{N})$ be a pairwise disjoint sequence of infinite sets of positive numbers. We use the fact that X belongs to the class $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ for each sequence $(\mathcal{V}_m : m \in\ Y_n)$ of open covers for the space X. For each m, we choose sequence $(U_n : n \in\ Y_m)$, such that for each $n \in Y_m$, the family U_n refines $\mathcal{V}_n$ and $\bigcup_{n\in\mathbf{N}} U_n$ is open cover for X. Then the family $\bigcup_{n\in\mathbf{N}} U_n$ is λ-cover for X, such that:

1. for each sequence U_n is pairwise disjoint family of open sets;

2. for each n, if $G \in U_n$, then $G \subset V$ and V is an element of the sequence $\mathcal{V}_n$;

From the fact that $\Omega \subseteq \Lambda$ we have that $\mathsf{S}_c(\Lambda, \Lambda) \Rightarrow \mathsf{S}_c(\Omega, \mathcal{O})$, i.e. $\mathsf{S}_c(\mathcal{O}, \mathcal{O}) = \mathsf{S}_c(\Omega, \mathcal{O}) = \mathsf{S}_c(\Lambda, \Lambda)$. $\diamondsuit$

3 Relations between S_c and the classical selection principles

About the relations among the selection principles $\mathsf{S}_c(\mathcal{A},\mathcal{B})$ and $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ and $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$, we show the following results:

Theorem 3 (1) $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ *does not imply* $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$.
(2) $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ *does not imply* $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.

Proof. (1) Addis and Gresham showed that the Hilbert cube is not a C-space. From the fact that the Hilbert cube is a compact space and thus has $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$, we obtain that $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ does not imply $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$.
(2) Consider N^N with the usual Tychonoff product topology (N is considered discrete and N^N is the countable power of N). This topology is metrizable and in fact the metric $d(g,h) = 1/n$ where n is least with $g(n) \neq h(n)$ generates this topology. In the literature (N^N, d) is often called "the Baire space". Let $\mathcal{R}$ denote the set of rational numbers in $[0,1]$ and let $\mathcal{N}$ denote the set of irrational numbers in [0,1]. The map from N^N to $\mathcal{N}$ which maps a sequence to the irrational number having that sequence as continued fraction expansion, is a homeomorphism. Thus, for topological purposes, we may replace the set of irrational numbers in $[0,1]$ with N^N. Addis and Gresham showed that zerodimensional paracompact spaces have property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Thus, as N^N is zerodimensional and paracompact, it has property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. According to an old result of Hurewicz, N^N does not have property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$. $\diamondsuit$

Theorem 4 *For each* $\mathcal{A},\mathcal{B} \subseteq \mathcal{O}$*: If the topological space* X *has selection property* $\mathsf{S}_1(\mathcal{A},\mathcal{B})$*, then* X *has selection property* $\mathsf{S}_c(\mathcal{A},\mathcal{B})$.

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be any sequence of covers from $\mathcal{A}$ for the space X. Since the space X satisfies $\mathsf{S}_1(\mathcal{A},\mathcal{B})$, choose a sequence $(U_n : n \in \mathbf{N})$ such that each $U_n \in \mathcal{U}_n$ and the set $\{U_n : n \in \mathbf{N}\}$ is a cover for X and belongs to $\mathcal{B}$.

Consider the sequence $(\mathcal{V}_n : n \in \mathbf{N})$ where $\mathcal{V}_n$ is the one-point $\{U_n\}$. It is clear that this sequence satisfies the conditions for $\mathsf{S}_c(\mathcal{A},\mathcal{B})$ of the space X. $\diamondsuit$

With the next results we show that only for some classes of covers for which X has the S_c-property, X has the S_1-property.

Theorem 5 $\mathsf{S}_c(\mathcal{A},\Gamma) \Rightarrow \mathsf{S}_{fin}(\mathcal{A},\Gamma)$.

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of $\mathcal{A}$-covers of X. We apply that X has the $\mathsf{S}_c(\mathcal{A},\Gamma)$ property: there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ of families such that
(i) each $\mathcal{V}_n$ is pairwise disjoint collection of open sets,
(ii) for each n, if $V \in \mathcal{V}_n$, then $V \subset G$ for some member $G \in \mathcal{U}_n$ and
(iii) the family $\mathcal{G} = \bigcup_{n\in\mathbf{N}}\mathcal{V}_n$ is a γ-cover of X.
Consider an $x \in X$. Then $\{G \in \mathcal{G} : x \notin G\}$ is finite, but $\mathcal{G}$ is infinite. Thus, by (i), each $\mathcal{V}_n$ is finite. $\diamondsuit$

Corollary 6 *If* $\mathcal{A} \in \{\Omega, \Gamma\}$ *then* $\mathsf{S}_c(\mathcal{A}, \Gamma) \Rightarrow \mathsf{S}_1(\mathcal{A}, \Gamma)$.

Proof. It was shown in [5] that $\mathsf{S}_{fin}(\Omega, \Gamma) = \mathsf{S}_1(\Omega, \Gamma)$ and that $\mathsf{S}_{fin}(\Gamma, \Gamma) = \mathsf{S}_1(\Gamma, \Gamma)$. Apply Theorem 5. $\diamondsuit$

Theorem 7 $\mathsf{S}_c(\mathcal{A}, \Omega) \Rightarrow \mathsf{S}_1(\mathcal{A}, \Omega)$.

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of covers of X from $\mathcal{A}$. The space X belongs to the class $\mathsf{S}_c(\mathcal{A}, \Omega)$, so choose a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ of families, such that:

1. $\mathcal{V}_n$ is pairwise disjoint collection of open sets;
2. For each n, if $V \in \mathcal{V}_n$, then $V \subset G$ where G is an element of the sequence $\mathcal{U}_n$;
3. The family $\bigcup_{n \in \mathbf{N}} \mathcal{V}_n$ is an ω-cover for X.

Consider $x \in X$. For each n, let

$$S_n(x) = \{V \in \mathcal{V}_n : x \in V\}$$

Because $\mathcal{V}_n$ is pairwise disjoint collection of open sets we have that $\mid S_n(x) \mid \leq 1$. For a subset F of X and for each n, let

$$S_n(F) = \{V \in \mathcal{V}_n : F \subset V\}$$

Also, because $\mathcal{V}_n$ is a pairwise disjoint collection of open sets, we have that $\mid S_n(F) \mid \leq 1$. For each finite set $F \subset X$, we have

$$S_n(F \cup \{x\}) \subseteq S_n(x)$$
$$\text{and } S_n(F \cup \{x\}) \subseteq S_n(F)$$

If $S_n(F \cup \{x\}) \neq \emptyset$, then $S_n(F) = S_n(x)$ and because of condition 3) there exists $n \in \mathbf{N}$ such that $S_n(F \cup \{x\}) \neq \emptyset$. $\diamondsuit$

Let X be a non-compact space. Then, the following statements are true:

Lemma 8 *For each ω-cover $\mathcal{U}$ there exists a refinement $\mathcal{V}$, such that:*

1. *$\mathcal{V}$ is an ω-cover for X and*
2. *There is no finite subcover of X from the cover $\mathcal{V}$.*

Proof. Let $\mathcal{U}$ be an ω-cover for the space X. It means that for each finite subset $F \subset X$ there exists an element $U \in \mathcal{U}$, such that $F \subset U$. We choose the cover $\mathcal{V}$ for the space X in the following way: for each finite subset $F \subset X$ there exists $V \in \mathcal{V}$, such that $F \subseteq V \subseteq U$, and U is an element of the cover $\mathcal{U}$. Then, the cover $\mathcal{V}$ is an ω-cover for X which refines the cover $\mathcal{U}$.
The condition 2) is true, because if not, the topological space X will be a compact space which is contradiction. $\diamondsuit$

Definition 3 ([2]) *An open cover* $\mathcal{U}$ *is* weakly groupable *if there exists a partition* $\mathcal{U} = \cup_{n\in\mathbf{N}}\mathcal{V}_n$*, such that each* $\mathcal{V}_n$ *is finite,* $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$ *for* $m \neq n$ *and for each finite subset* $F \subset X$ *there exists* n *such that* $F \subset \cup\mathcal{U}_n$*.*

The symbol $\mathcal{O}^{wgp}$ denote the family of all weakly groupable covers for the space X.

Theorem 9 *If* X *is not a compact space, then* $\mathsf{S}_c(\Gamma,\Omega) \Rightarrow \mathsf{S}_c(\Omega,\mathcal{O}^{wgp})$

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of ω-covers for the space X. Because X is not a compact space we apply the previous Lemma to each element of the sequence $(\mathcal{U}_n : n \in \mathbf{N})$: *i.e.*, there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ of ω-covers for X such that each $\mathcal{V}_n$ refines $\mathcal{U}_n$ and no finite subset $F \subset \mathcal{V}_n$ is a cover of X.

For each m define:

$$\mathcal{W}_m = \{\cup_{m<j\leq n} V_j : n \in \mathbf{N} \text{ and } V_j \in \mathcal{V}_j\}$$

Then for each $m \in \mathbf{N}$, $\mathcal{W}_m$ is a γ-cover for X. Because X belongs to the class $\mathsf{S}_c(\Gamma,\Omega)$ there exists a sequence $(\mathcal{K}_m : m \in \mathbf{N})$ such that:

1. Each $\mathcal{K}_n$ is a pairwise disjoint family of open sets;
2. for each m, if $K \in \mathcal{K}_m$, then there exists an element $V \in \mathcal{W}_m$ such that $K \subset V$ and
3. the family $\bigcup_{n\in\mathbf{N}} \mathcal{K}_n$ is an ω-cover for X.

Because $(\mathcal{K}_m : m \in \mathbf{N})$ refines the sequence $(\mathcal{W}_m : m \in \mathbf{N})$ we have that the sequence $(\mathcal{K}_m : m \in \mathbf{N})$ refines the sequence $(\mathcal{U}_m : m \in \mathbf{N})$, and that $\mathcal{K}_m$ is a pairwise disjoint family of open sets for each $m \in \mathbf{N}$.
From the condition 3) it follows that $\bigcup_{m\in\mathbf{N}} \mathcal{K}_m \in \mathcal{O}^{wgp}$. $\diamondsuit$

We have Figure 2 on the next page.

4 Preservation under finite powers

For S_1 and S_{fin} the following statements are true for any space X:

1. X has $\mathsf{S}_1(\Omega,\Omega)$ if, and only if, each finite power of X has $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ - [6].
2. X has $\mathsf{S}_{fin}(\Omega,\Omega)$ if, and only if, each finite power of X has $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ - [5].

In the corresponding statement:

X has $\mathsf{S}_c(\Omega,\Omega)$ if, and only if, each finite power of X has $\mathsf{S}_c(\mathcal{O},\mathcal{O})$

the implication

$$X \text{ has } \mathsf{S}_c(\Omega,\Omega) \Rightarrow \text{ each finite power of } X \text{ has } \mathsf{S}_c(\mathcal{O},\mathcal{O})$$

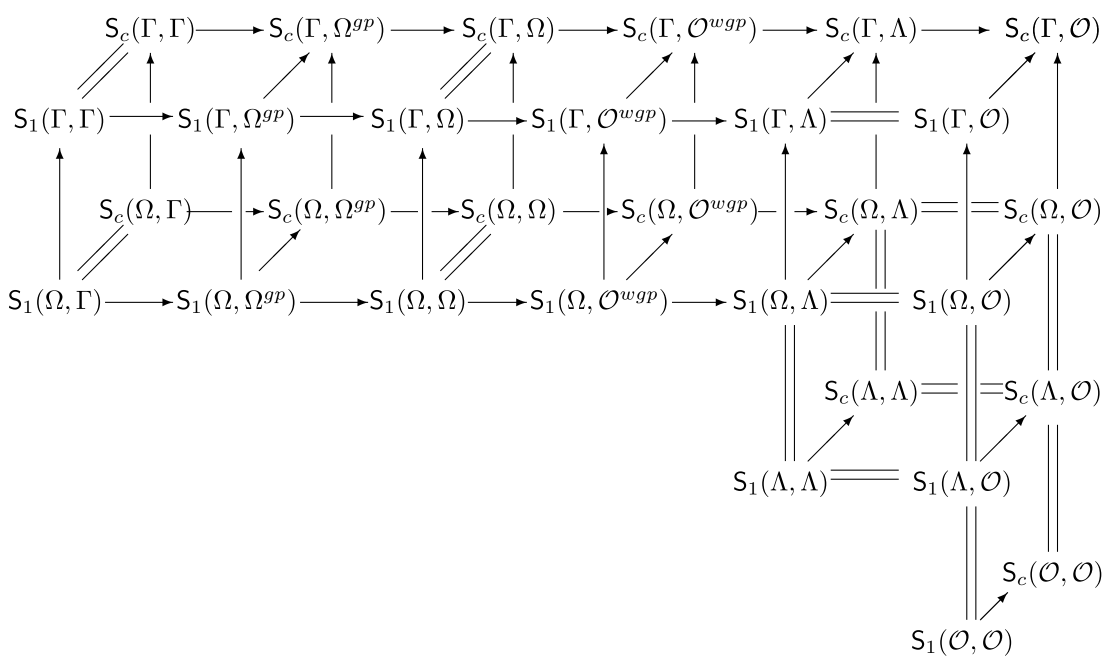

Figure 2: $\mathsf{S}_1 - \mathsf{S}_c$ - diagram

is true, but the implication

$$\text{each finite power of } X \text{ has } \mathsf{S}_c(\mathcal{O},\mathcal{O}) \Rightarrow X \text{ has } \mathsf{S}_c(\Omega,\Omega)$$

is false.

To see this, recall from Theorem 7 that if X has property $\mathsf{S}_c(\Omega,\Omega)$, then it has property $\mathsf{S}_1(\Omega,\Omega)$. Then, by Sakai's theorem [6], each finite power of X has $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. Then by Theorem 4, each finite power of X has $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.

To see that the other implication is false, recall that Addis and Gresham showed that paracompact zerodimensional spaces have property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Thus, with X the Baire space, all finite powers of X are paracompact and zerodimensional, and so have property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. But X does not even have property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$, and so does not have property $\mathsf{S}_1(\Omega,\Omega)$ (which by Theorems 4 and 7 is $\mathsf{S}_c(\Omega,\Omega)$).

Acknowledgements

I would like to thank L. Rubin for bringing [1] to my attention.

References

[1] D.F. Addis and J.H. Gresham, *Dimension theory and Aleksandroff's problem*, Fund. Math. **101:3** (1978), 195–205.

[2] L. Babinkostova, Lj.D.R. Kočinac and M. Scheepers, *Combinatorics of open covers* (VIII), preprint.

[3] R.H. Bing, *Metrization of topological spaces*, Can. J. Math. **3** (1951), 175–186.

[4] W. Haver, *A covering property for metric space*, Lecture Notes in Mathematics 375, Springer-Verlag, New York (1974), 108–113.

[5] W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki, *The combinatorics of open covers (II)*, Topology Appl. **73** (1996), 241–266.

[6] M. Sakai, *Property C'' and function spaces*, Proc. Amer. Math. Soc. **104:3** (1988), 917–919.

Institute of Mathematics
Faculty of Natural Sciences and Mathematics
PO Box 162
1000 Skopje
Republic of Macedonia
liljanab@iunona.pmf.ukim.edu.mk