Selection principles in topology: New directions

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Abstract

Most mathematical theories start from humble beginnings where a mathematician while examining some mathematical object makes fundamental, but *ad hoc*, observations. When the significance of these observations are realized some organised study begins during which these are explored in detail for some central examples. In this second part of theory building some general themes of investigation are identified. Then, it seems, one enters a third phase in which several mathematicians join in developing the theory. Typically this third phase is marked with much innovation, rapid expansion and application to other areas of mathematics. This brief discussion[1] of the area of *Selection principles in Topology* will be organized according to these three phases.

1 The beginnings

Selection principles appear in diagonalization arguments[2] in mathematics. The early sources for selection principles in topology come from measure theory and basis theory in metric spaces.

Basis theoretic sources

In [26] K. Menger defined the *Menger basis property* for metric spaces: A met/-ric space (X, d) has the Menger basis property if there is for each base $\mathcal{B}$ of X a sequence $(B_n : n \in \mathbb{N})$ of sets from the basis such that $\lim_{n\to\infty} diam_d(B_n) = 0$, and $\{B_n : n \in \mathbb{N}\}$ is a cover for X.

Though at first glance this concept does not seem to be related to diagonalization, evidence that it is came quickly when in [17] W. Hurewicz proved two fundamental theorems about the Menger basis property. The first of the two is that the metric space (X, d) has the Menger basis property if, and only if, for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$

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[2]similar to Cantor's diagonalization proof that the set of real numbers is uncountable

of finite sets such that for each n $\mathcal{V}_n \subseteq \mathcal{U}_n$, and $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is an open cover of X. This statement motivates the following definition:

Let $\mathcal{A}$ and $\mathcal{B}$ be families of collections of subsets of the infinite set S. The symbol $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the statement:

> For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of members of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n $\mathcal{V}_n \subseteq \mathcal{U}_n$, and $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a member of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ is an example of a selection principle.

Throughout this article we will, for a given topological space, let $\mathcal{O}$ denote the collection of all open covers of that space. In this notation, Hurewicz's results is:

Theorem 1 (Hurewicz) *A metric space (X,d) has the Menger basis property if, and only if, $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ holds.*

Measure theoretic sources

In [8] E. Borel defined the notion of a *Borel measure zero*[3] metric space as follows: A metric space (X,d) has Borel measure zero if there is for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers a partition family $(J_n : n \in \mathbb{N})$ of subsets of X such that for each n, $diam_d(J_n) < \epsilon_n$, and $X = \cup_{n\in\mathbb{N}}J_n$. Originally, Borel called this property as property C. Later this property was called to as *strong measure zero*. For practical reasons we will refer to this property as Borel measure zero.

Also this definition at first glance does not seem to be related to diagonalization. For this example evidence came very slowly. First F. Rothberger in his study in [31] of Borel measure zero introduced the following statement: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(U_n : n \in \mathbb{N})$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. Rothberger pointed out that if a metric space has this property, then it has Borel measure zero. In the later paper [32] he proved that the converse is not true.

Rothberger's property motivates the following definition: Again with $\mathcal{A}$ and $\mathcal{B}$ families of collections of subsets of the infinite set S, the symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the statement that

> For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of members of $\mathcal{A}$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n : n \in \mathbb{N}\}$ is a member of $\mathcal{B}$.

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is another example of a selection principle.

In this notation Rothberger's property is denoted by the symbol $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. Later, in [34], the Borel measure zero property of subspaces of a σ-compact metric space were characterized as follows: Let X be a metric space, and let Y be a subspace of X. Let $\mathcal{O}_Y$ denote the covers of Y by sets open in X.

[3] not the same as measure zero Borel set

Theorem 2 (Scheepers) *For a σ-compact metric space X, subspace Y has Borel measure zero in each metrization of X if, and only if, X has property* $\mathsf{S}_1(\mathcal{O}, \mathcal{O}_Y)$.

The two selection principles $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ and $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ are the *classical selection principles*. Actually, there is a third, introduced by Hurewicz in [17], denoted $\mathsf{U}_{fin}(\mathcal{A}, \mathcal{B})$, but it was shown in [24] that Hurewicz's selection principle is equivalent to one of the form $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$, and so we will not define this third one here.

Infinite games

Some of the tools for studying these classical selection principles come from Game Theory. Let $\mathcal{A}$ and $\mathcal{B}$ be families as before. Consider the following infinite game, denoted $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$:

> The players, ONE and TWO, play an inning per positive integer. In the n-th inning ONE first chooses an element O_n of $\mathcal{A}$, and TWO responds with a finite set $T_n \subseteq O_n$. A play
>
> $$O_1,\, T_1,\, \cdots, O_n,\, T_n,\, \cdots$$
>
> is won by TWO if $\cup_{n\in\mathbb{N}} T_n \in \mathcal{B}$; otherwise, ONE wins.

Although he did not explicitly introduce this game, Hurewicz proved

Theorem 3 (Hurewicz) *A topological space has property* $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ *if, and only if, ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$.

Telgársky explitcitly introduced the game $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$ in [46].

Analogously to $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ one defines the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$:

> The players, ONE and TWO, play an inning per positive integer. In the n-th inning ONE first chooses an element O_n of $\mathcal{A}$, and TWO responds with an element $T_n \in O_n$. A play
>
> $$O_1,\, T_1,\, \cdots, O_n,\, T_n,\, \cdots$$
>
> is won by TWO if $\{T_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

Galvin explicitly introduced the game $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ in [10]. By hindsight, and after some reformulation, the games introduced by Telgarsky in [44] are of the form $\mathsf{G}_1(\mathcal{A}, \mathcal{O})$, where $\mathcal{A}$ is a class of open covers of the space in question. In [30] J. Pawlikowski proved:

Theorem 4 (Pawlikowski) *A space has property* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ *if, and only if, ONE has no winning strategy in the game* $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$.

This very brief description of "the beginnings" ignores the appearance of selection principles outside topology, and some of the ad-hoc appearances of such principles in topology during the 1970's and 1980's. A more complete picture of the beginnings can be gleaned from the Selection Principles in Mathematics website (currently, Fall 2001, under development) at the URL

`http:\\iunona.pmf.ukim.edu.mk\~ spm`

2 The second phase: An organized study

During the second phase of work on selection principles an organized study was undertaken. One can divide the types of results that were obtained into at least six themes, as we do now in our discussion.

Theme I: Classes of open covers, monotonicity and equivalences

Several of the popular types of open covers appearing in topological studies were considered. Three of these types of open covers are as follows:

A cover $\mathcal{U}$ of a space X is said to be

1. *large* if for each x in X the set $\{U \in \mathcal{U} : x \in U\}$ is infinite;
2. an ω–*cover* if X is not in $\mathcal{U}$ and for each finite subset F of X, there is a set $U \in \mathcal{U}$ such that $F \subset U$;
3. a γ–*cover* if it is infinite and for each x in X the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

Throughout the following notation will be used:

Λ: The collection of all large covers of the space.

Ω: The collection of all ω-covers of the space.

Γ: The collection of all γ-covers of the space.

It is evident that $\Gamma \subset \Omega \subset \Lambda \subset \mathcal{O}$. Also evident is that for the classical selection principles one has the implication

$$\mathsf{S}_1(\mathcal{A}, \mathcal{B}) \Rightarrow \mathsf{S}_{fin}(\mathcal{A}, \mathcal{B}),$$

and each of these is anti-monotonic in their first variable, and monotonic in the second.

Also for Hurewicz's selection principle $\mathsf{U}_{fin}(\mathcal{A}, \mathcal{B})$ these monotonicity properties hold, and one has

$$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B}) \Rightarrow \mathsf{U}_{fin}(\mathcal{A}, \mathcal{B}).$$

(We did not define Hurewicz's selection principle because it is equivalent to one of the form $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ for appropriate $\mathcal{A}$ and $\mathcal{B}$.)

These implications and monotonicity properties lead to a diagram depicting the basic relationships among the classes when $\mathcal{A}$ and $\mathcal{B}$ range over the four classes of open covers. One shows that some of the classes coincide - for example $\mathsf{S}_1(\Omega, \mathcal{O}) \Leftrightarrow \mathsf{S}_1(\mathcal{O}, \mathcal{O})$ and $\mathsf{S}_{fin}(\Omega, \mathcal{O}) \Leftrightarrow \mathsf{S}_{fin}(\mathcal{O}, \mathcal{O}) \Leftrightarrow \mathsf{U}_{fin}(\Gamma, \mathcal{O})$ - and after elimination of such coincidences one obtains the following diagram, which depicts the relationships among distinct classes:

$$
\begin{tikzcd}
 & & \mathsf{U}_{fin}(\Gamma,\Gamma) \arrow[rr] & & \mathsf{U}_{fin}(\Gamma,\Omega) \arrow[rr] & & \mathsf{U}_{fin}(\Gamma,\mathcal{O}) \\
 & & & \mathsf{S}_{fin}(\Gamma,\Omega) \arrow[ur] & & & \\
\mathsf{S}_1(\Gamma,\Gamma) \arrow[rr] \arrow[uurr] & & \mathsf{S}_1(\Gamma,\Omega) \arrow[rr] \arrow[ur] & & \mathsf{S}_1(\Gamma,\Lambda) \arrow[uurr] & & \\
 & & & \mathsf{S}_{fin}(\Omega,\Omega) \arrow[uu] & & & \\
\mathsf{S}_1(\Omega,\Gamma) \arrow[rr] \arrow[uu] & & \mathsf{S}_1(\Omega,\Omega) \arrow[rr] \arrow[uu] \arrow[ur] & & \mathsf{S}_1(\mathcal{O},\mathcal{O}) \arrow[uu] & &
\end{tikzcd}
$$

Figure 1: The basic classes

In [18] it was shown by examples, for some using the Continuum Hypothesis in their construction, that no two of these classes are equal. The main open problem remaining in connection with this diagram is whether $\mathsf{U}_{fin}(\Gamma,\Gamma) \Rightarrow \mathsf{S}_{fin}(\Gamma,\Omega)$.

Theme II: Closure under operations

It is natural to consider these selection properties also for subspaces of a space, using the inherited relative topology, and for new spaces constructed from given ones. The following summarizes some results regarding closure under finite powers, finite products or finite unions (this is by no means an exhaustive list of results):

Property	powers	products	unions
$\mathsf{S}_1(\Omega,\Gamma)$	Yes	No	No
$\mathsf{S}_1(\Omega,\Omega)$	Yes	No	No
$\mathsf{S}_1(\mathcal{O},\mathcal{O})$	No	No	Yes
$\mathsf{S}_1(\Gamma,\Gamma)$	No	No	Yes
$\mathsf{S}_{fin}(\Omega,\Omega)$	Yes	No	No
$\mathsf{U}_{fin}(\Gamma,\Gamma)$	No	No	Yes
$\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$	No	No	Yes

There are also several results about when a subspace or a continuous image inherits a property.

Theme III: Cardinality

Several cardinal numbers appear naturally in the context of selection principles, and have been considered in the context of subspaces of the real line. For example there is the *minimality number*: $\mathsf{non}(\mathsf{S}_1(\mathcal{A},\mathcal{B}))$ which is defined to be $\min\{|X| : X \subseteq \mathbb{R}$ is infinite and X does not have $\mathsf{S}_1(\mathcal{A},\mathcal{B})\}$. $\mathsf{non}(\mathsf{S}_{fin}(\mathcal{A},\mathcal{B}))$ is defined analogously.

non(Property)	Property
p	$\mathsf{S}_1(\Omega,\Gamma)$
$\mathsf{cov}(\mathcal{M})$	$\mathsf{S}_1(\Omega,\Omega)$, $\mathsf{S}_1(\mathcal{O},\mathcal{O})$
b	$\mathsf{S}_1(\Gamma,\Gamma)$, $\mathsf{U}_{fin}(\Gamma,\Gamma)$
d	The remaining 6 classes

The symbols p, b, d and $\mathsf{cov}(\mathcal{M})$ are defined as follows:

p: The minimal cardinality of an infinite family $\mathcal{X}$ of infinite subsets of $\mathbb{N}$ such that:

1. Each finite nonempty subset of $\mathcal{X}$ has an infinite intersection and
2. There is no infinite set P such that for each $A \in \mathcal{X}$ we have $P \setminus A$ finite.

d: The minimal cardinality of an infinite family $\mathcal{X}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ such that: For each function $f : \mathbb{N} \to \mathbb{N}$ there is a $g \in \mathcal{X}$ with $\lim_{n\to\infty}(g(n) - f(n)) = \infty$.

b: The minimal cardinality of an infinite family $\mathcal{X}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ such that: For each function $f : \mathbb{N} \to \mathbb{N}$ there is a $g \in \mathcal{X}$ with $\{m \in \mathbb{N} : f(m) < g(m)\}$ infinite.

$\mathsf{cov}(\mathcal{M})$: The minimal cardinality of an infinite family $\mathcal{X}$ of first category subsets of $\mathbb{R}$ such that $\cup\mathcal{X} = \mathbb{R}$.

These cardinality results and consistency results regarding relations among these cardinals imply that some of the classes in Figure **??** are not equal. For the remaining inequalities one can use special constructions based for example on consequences of the Continuum Hypothesis.

Theme IV: Game theory

Complementing Pawlikowski's result for $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ it was also shown that:

$$\mathsf{S}_1(\Omega,\Gamma) \quad \Leftrightarrow \quad \text{ONE has no winning strategy in } \mathsf{G}_1(\Omega,\Gamma)$$

$$\mathsf{S}_1(\Omega,\Omega) \quad \Leftrightarrow \quad \text{ONE has no winning strategy in } \mathsf{G}_1(\Omega,\Omega)$$

$$\mathsf{S}_1(\Gamma,\Gamma) \quad \Leftrightarrow \quad \text{ONE has no winning strategy in } \mathsf{G}_1(\Gamma,\Gamma)$$

And complementing Hurewicz's result for $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ it was shown

$$\mathsf{S}_{fin}(\Omega,\Omega) \quad \Leftrightarrow \quad \text{ONE has no winning strategy in } \mathsf{G}_{fin}(\Omega,\Omega)$$

Also, it has been shown that the Hurewicz selection principle $\mathsf{U}_{fin}(\Gamma,\Gamma)$ is equivalent to the statement that ONE has no winning strategy in the corresponding game - but this game was not of the form G_1 or G_{fin}.

Theme V: Ramsey theory

The next development during the second phase was to show a connection between the classical selection principles S_1 and S_{fin}, and a seemingly unrelated field from combinatorial set theory - Ramsey Theory. For $\mathcal{A}$ and $\mathcal{B}$ families of subsets of an infinite set S:

$$\mathcal{A} \to (\mathcal{B})^n_k$$

denotes: For each $A \in \mathcal{A}$, and for each function $f : [A]^n \to \{1, \cdots, k\}$ there is a set $B \subseteq A$ and a $j \in \{1, \cdots, k\}$ such that:

1. $B \in \mathcal{B}$ and
2. for each $F \in [B]^n$ we have $f[F] = \{j\}$.

In [29] Ramsey proved, for $\mathcal{A} = \mathcal{B} = \{A \subset \mathbb{N} : A \text{ infinite}\}$, the theorem that for all n and k, $\mathcal{A} \to (\mathcal{A})^n_k$. This is known as Ramsey's theorem and one of the motivating results for Ramsey Theory, a research area in finite combinatorics, and infinitary set theory.

For selection principles of the form $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ it was shown:

$$\mathsf{S}_1(\Omega,\mathcal{O}) \quad \Leftrightarrow \quad \text{For each } k,\ \Omega \to (\mathcal{O})^2_k$$

$$\mathsf{S}_1(\Omega,\Omega) \quad \Leftrightarrow \quad \text{For all } n \text{ and } k,\ \Omega \to (\Omega)^n_k$$

$$\mathsf{S}_1(\Omega,\Gamma) \quad \Leftrightarrow \quad \text{For all } n \text{ and } k,\ \Omega \to (\Gamma)^n_k$$

In [7] in their study of the theory of ultrafilters on the positive integers, Baumgartner and Taylor introduced a Ramseyan-like partition relation which we will denote with the symbol

$$\mathcal{A} \rightarrow \lceil \mathcal{B} \rceil_k^2$$

and which means: For each $A \in \mathcal{A}$, and for each function $f : [A]^2 \rightarrow \{1, \cdots, k\}$ there is a set $B \subseteq A$, a $j \in \{1, \cdots, k\}$ and a partition $(B_n : n \in \mathbb{N})$ of B into pairwise disjoint finite sets such that:

1. $B \in \mathcal{B}$ and
2. for each $\{x, y\} \in [B]^2$ such that for all n we have $|\{x, y\} \cap B_n| \leq 1$, it is the case that $f[\{x, y\}] = \{j\}$.

For selection principles of the form $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ it was shown:

$$\mathsf{S}_{fin}(\Omega, \mathcal{O}) \quad \Leftrightarrow \quad \text{For each } k,\ \Omega \rightarrow \lceil \mathcal{O} \rceil_k^2$$

$$\mathsf{S}_{fin}(\Omega, \Omega) \quad \Leftrightarrow \quad \text{For each } k,\ \Omega \rightarrow \lceil \Omega \rceil_k^2$$

Theme VI: Hyperspace theory

Often topologists have found ingeneous ways of constructing from a given topological space X a new space $\mathsf{N}(X)$ to give an example illustrating some specific fact. The importance of some of these *ad hoc* examples have often far transcended the specific purpose for which they were invented. How does a selection principle for X manifest itself in the hyperspace $\mathsf{N}(X)$?

One specific example of such a $\mathsf{N}(X)$ is as follows: $\mathsf{C}(X)$ is the set of continuous real-valued functions from X. This is a subset of the Tychonoff power of $|X|$ copies of $\mathbb{R}$ - and endowed with the topology inherited from this power, is denoted by $\mathsf{C}_p(X)$. This is the space of real-valued continuous functions, endowed with the *pointwise topology*.

For a point $f \in \mathsf{C}_p(X)$ we define:

$$\Omega_f = \{A \subset \mathsf{C}_p(X) \setminus \{f\} : f \text{ in the closure of } A\}.$$

A second example is $\mathsf{PR}(X)$, the Pixley-Roy space over X. The elements of $\mathsf{PR}(X)$ are the finite subsets of X, and for $S \subset X$ finite and $U \subset X$ open with $S \subset U$, the symbol $[S, U]$ denotes $\{T \in \mathsf{PR}(X) : S \subseteq T \subset U\}$. Set of the form $[S, U]$ form a basis for a topology on $\mathsf{PR}(X)$.

We define:

$$\mathcal{D} = \{\mathcal{U} : (\forall U \in \mathcal{U})(U \text{ open and } \cup \mathcal{U} \text{ dense in } X)\}$$

Here is a small sample of results that have been proved (a property listed under X holds if, and only if, the corresponding property under $\mathsf{C}_p(X)$ or under $\mathsf{PR}(X)$ holds).

X	$\mathsf{C}_p(X)$	$\mathsf{PR}(X)$
$\mathsf{S}_1(\Omega,\Omega)$	$\mathsf{S}_1(\Omega_f,\Omega_f)$	$\mathsf{S}_1(\mathcal{D},\mathcal{D})$
$\mathsf{S}_{fin}(\Omega,\Omega)$	$\mathsf{S}_{fin}(\Omega_f,\Omega_f)$	$\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$

The result regarding $\mathsf{S}_{fin}(\Omega_f,\Omega_f)$ is due in part to Arhangel'skiǐ - [1], and the result regarding $\mathsf{S}_1(\Omega_f,\Omega_f)$ to Sakai - [33]. Precursors of the results regarding $\mathsf{PR}(X)$ are due to P. Daniels - [9] - and the results mentioned here are from [39].

3 Phase 3: Rapid expansion, innovation, application

We are now in phase 3 of the development of selection principles in topology. This phase is marked by the rapid expansion of the subject, by much innovation and by the appearance of applications to other areas of mathematics. In this article I give only a small glance of this activity. I will organize it along the same topics as that for phase 2.

Theme I: Classes of open covers, monotonicity and equivalences

Several new classes of open covers have been added to the list of four studied during the second phase. Some of these were motivated by the type of combinatorics that was used in phase 2. We define *groupable* and *weakly groupable* versions of covers: An open cover $\mathcal{U}$ of a space is

- groupable if there is a partition $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{U}_n$, and an increasing sequence $(m_n : n \in \mathbb{N})$ of positive integers such that for each element x of the space, for all but finitely many n, $x \in \cup(\cup_{m_n\le j<m_{n+1}}\mathcal{U}_j)$;
- weakly groupable if there is a partition $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{U}_n$, and an increasing sequence $(m_n : n \in \mathbb{N})$ of positive integers such that for each finite subset F of the space there is an n such that $F \subset \cup(\cup_{m_n\le j<m_{n+1}}\mathcal{U}_j)$;

We use the notation $\mathcal{O}^{gp}$ to denote the collection of groupable open covers of a space, and Λ^{gp} to denote the collection of groupable large covers of a space. The analogous notation for weakly groupable covers is $\mathcal{O}^{wgp}$ and Λ^{wgp}.

An ω-cover $\mathcal{U}$ is ω-groupable if there is a partition $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{U}_n$, and an increasing sequence $(m_n : n \in \mathbb{N})$ of positive integers such that for each finite subset F of the space, for all but finitely many n, there is a $U \in \cup_{m_n\le j<m_{n+1}}\mathcal{U}_j$ with $F \subset U$. The symbol Ω^{gp} denotes the collection of ω-groupable ω-covers of a space.

And in [47] Tsaban introduced another new class of open cover, called a τ-cover, and motivated by the type of combinatorics related to a well-known open problem in set theory: A large open cover $\mathcal{U}$ for a space X is said to be a

τ-cover if for each $x, y \in X$:
either for all but finitely many $U \in \mathcal{U}$ we have $x \in U$ implies $y \in U$,
or else for all but finitely many $U \in \mathcal{U}$ we have $y \in U$ implies $x \in U$.

The symbol T is used to denote the collection of τ-covers of X.

Evidently we have the inclusions

$$\Gamma \subset \mathsf{T} \subset \Omega \subset \Lambda \subset \mathcal{O};$$
$$\Gamma \subset \Omega^{gp} \subset \Omega \subset \Lambda^{wgp} \subset \Lambda \subset \mathcal{O};$$
$$\Omega^{gp} \subset \Lambda^{gp} \subset \mathcal{O}^{gp};$$
$$\mathcal{O}^{gp} \subset \mathcal{O}^{wgp} \subset \mathcal{O};$$
$$\Lambda^{gp} \subset \Lambda^{wgp} \subset \Lambda.$$

These inclusions plus the monotonicity properties of the classical selection principles give rise, like before, to a more extensive diagram. Moreover, some of these classes are new and some are not. Most notably the following have been proven:

$$\mathsf{U}_{fin}(\Gamma, \Gamma) \quad \Leftrightarrow \quad \mathsf{S}_{fin}(\Omega, \mathcal{O}^{gp}) \quad \Leftrightarrow \quad \mathsf{U}_{fin}(\mathsf{T}, \Gamma)$$
$$\mathsf{U}_{fin}(\Gamma, \Omega) \quad \Leftrightarrow \quad \mathsf{S}_{fin}(\Omega, \mathcal{O}^{wgp}) \quad \Leftrightarrow \quad \mathsf{U}_{fin}(\mathsf{T}, \Omega)$$

This reduces Hurewicz's selection principle U_{fin} to S_{fin}. With this, Figure 1 becomes Figure 2 (see the next page).

Two more innovations were the introduction of new selection principles, and the idea of relative selection principles.

We can call the new selection principles "Balkan selection principles", since these were invented and initially studied mostly by some mathematicians in the Balkans. Among these we have $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$, introduced and carefully studied in [2]. Also introduced are $\mathsf{S}_{lf}(\mathcal{A}, \mathcal{B})$, which is for example used to characterize paracompactness [6]. And there are several more such selection principles, some motivated by star covering properties of spaces [19].

The idea of relativization is as follows: A space X and subspace Y of X are given. $\mathcal{A}$ is a collection of open covers of X (for example $\mathcal{O}$, Ω, and so on) and $\mathcal{B}$ is a collection of covers of Y by sets open in X. The study of $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ generalizes that for the case when $X = Y$, called the absolute case. There are essentially more delicate techniques needed to generalize the absolute theory to the relative one, as illustrated by some results in [21] and [2].

$$
\begin{tikzcd}
 & & \mathsf{S}_{fin}(\Omega,\mathcal{O}^{gp}) \arrow[rr] & & \mathsf{S}_{fin}(\Omega,\mathcal{O}^{wgp}) \arrow[rr] & & \mathsf{S}_{fin}(\Omega,\mathcal{O}) \\
 & & & \mathsf{S}_{fin}(\Gamma,\Omega) \arrow[ur] & & & \\
\mathsf{S}_1(\Gamma,\Gamma) \arrow[uurr] \arrow[rr] & & \mathsf{S}_1(\Gamma,\Omega) \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\Gamma,\Lambda) \arrow[uurr] & & \\
 & & & \mathsf{S}_{fin}(\Omega,\Omega) \arrow[uu] & & & \\
\mathsf{S}_1(\Omega,\Gamma) \arrow[uu] \arrow[rr] & & \mathsf{S}_1(\Omega,\Omega) \arrow[uu] \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\mathcal{O},\mathcal{O}) \arrow[uu] & &
\end{tikzcd}
$$

Figure 2: The basic classes updated

Theme II: Closure under operations

It is shown that $\mathsf{S}_{fin}(\Omega,\Omega^{gp})$ is preserved by finite powers, and that a space has $\mathsf{S}_{fin}(\Omega,\Omega^{gp})$ if, and only if, each of its finite powers has $\mathsf{S}_{fin}(\Omega,\mathcal{O}^{gp})$. Weiss showed that $\mathsf{S}_1(\Omega,\mathcal{O}^{gp})$ is preserved by finite products, and thus finite powers. The theory of preservation by finite products or powers is more complicated for the Balkan selection principles.

Theme III: Cardinality

Here is a sample of what has been shown:

non(Property)	Property
p	$\mathsf{S}_1(\Omega,\mathsf{T})$
t	$\mathsf{S}_1(\mathsf{T},\Gamma)$
e	$\mathsf{S}_1(\Gamma,\mathsf{T})$, $\mathsf{S}_{fin}(\Gamma,\mathsf{T})$, $\mathsf{U}_{fin}(\Gamma,\mathsf{T})$
$\mathsf{add}(\mathcal{M})$	$\mathsf{S}_1(\Omega,\mathcal{O}^{gp})$
$\mathsf{non}(\mathcal{BMZ})$	$\mathsf{S}_1(\mathcal{O}_{\mathbb{R}},\mathcal{O}_X)$

Since it is known from basic results in Set Theory that it is consistent that $p < \mathsf{add}(\mathcal{M})$, and that $b \neq \mathsf{add}(\mathcal{M})$, and that $\mathsf{cov}(\mathcal{M}) \neq \mathsf{add}(\mathcal{M})$, it follows that $\mathsf{S}_1(\Omega, \mathcal{O}^{gp})$ is a new property.

Similar considerations give that $\mathsf{S}_1(\mathsf{T}, \Gamma)$ is a new property. Indeed, the introduction of τ-covers uncovered eleven new classes of spaces not present in the work of phase 2.

Let $\mathcal{BMZ}$ denote the set of subsets of the real line which have Borel measure zero. For a subset X or $\mathbb{R}$ let $\mathcal{O}_X$ denote the covers of X by sets open in $\mathbb{R}$. Then $X \in \mathcal{BMZ}$ if, and only if, $\mathsf{S}_1(\mathcal{O}_{\mathbb{R}}, \mathcal{O}_X)$ holds. The minimal cardinality of a set of reals without Borel measure zero is denoted $\mathsf{non}(\mathcal{BMZ})$. It was shown in [15] that it is consistent that $\mathsf{cov}(\mathcal{M}) < \mathsf{non}(\mathcal{BMZ})$. Thus it is consistent that there is a set X or reals satisfying $\mathsf{S}_1(\mathcal{O}_{\mathbb{R}}, \mathcal{O}_X)$, but not $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$. In particular, the relative theory subsumes the absolute theory.

X is a set of sets of real numbers and each $A \in X$ has $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ but $\cup X$ does not. $\mathsf{add}(\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B}))$ is defined analogously.

X is a set of sets of real numbers and each $A \in X$ has $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ and $\mathbb{R} = \cup X$. $\mathsf{cov}(\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B}))$ is defined analogously.

Theme IV: Game theory

Additionally it has been shown that $\mathsf{S}_{fin}(\Omega, \mathcal{O}^{gp})$ holds if, and only if, ONE has no winning strategy in $\mathsf{G}_{fin}(\Omega, \mathcal{O}^{gp})$, and $\mathsf{S}_{fin}(\Omega, \mathcal{O}^{wgp})$ holds if, and only if, ONE has no winning strategy in $\mathsf{G}_{fin}(\Omega, \mathcal{O}^{wgp})$. In Figure 2 all classes except those of form $\mathsf{S}_1(\Gamma, \cdot)$ or $\mathsf{S}_{fin}(\Gamma, \cdot)$ have a corresponding Ramsey-theoretic characterization; also all these classes except $\mathsf{S}_1(\Gamma, \Omega)$ and $\mathsf{S}_1(\Gamma, \mathcal{O})$ and $\mathsf{S}_{fin}(\Gamma, \Omega)$ have characterizations in terms of the corresponding G_1 or G_{fin}- games.

Theme V: Ramsey theory

$$\mathsf{S}_{fin}(\Omega, \mathcal{O}^{gp}) \quad \Leftrightarrow \quad \text{For each } k,\ \Omega \to \lceil \mathcal{O}^{gp} \rceil^2_k$$

$$\mathsf{S}_{fin}(\Omega, \mathcal{O}^{wgp}) \quad \Leftrightarrow \quad \text{For each } k,\ \Omega \to \lceil \mathcal{O}^{wgp} \rceil^2_k$$

In Figure 2 all classes except those of form $\mathsf{S}_1(\Gamma, \cdot)$ or $\mathsf{S}_{fin}(\Gamma, \cdot)$ have a corresponding Ramsey-theoretic characterization; also all these classes except $\mathsf{S}_1(\Gamma, \Omega)$ and $\mathsf{S}_1(\Gamma, \mathcal{O})$ and $\mathsf{S}_{fin}(\Gamma, \Omega)$ have characterizations in terms of the corresponding G_1 or G_{fin}- games.

Theme VI: Hyperspace theory

A very nice generalization of the selection principles theory of $\mathsf{C}_p(X)$ has been developed also for the case of the relative selection principles, and a corresponding generalization for $\mathsf{PR}(X)$ is in progress.

4 Final remarks

Due to the pace of innovation much work on these six themes remains for the Balkan selection principles and for the new classes of open covers. In particular, it appears that essentially new ideas are needed to develop the game theory and the Ramsey theory for the Balkan selection principles.

In this brief survey I also did not elaborate on the nice work in connection with selection principles defined using filters on the positive integers. For a taste of this effort, the reader can consult for example [12] and [13].

Besides these usual themes, a beautiful selection principle theory for the case of metrizable spaces has emerged in [2]. In this theory selection principles are characterized by basis properties of the spaces, and by measure-like properties of the spaces. These characterizations lend themselves in particular immediately to applications to topological groups [5].

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