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On a Class of Infinite System of Third-Order Differential Equations in ℓ_p via Measure of Noncompactness

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Abstract. In this paper, with the help of measure of noncompactness together with Darbo-type fixed point theorem, we focus on the infinite system of third-order differential equations

$$u_i''' + au_i'' + bu_i' + cu_i = f_i(t, u_1(t), u_2(t), \dots)$$

where $f_i \in C(\mathbb{R} \times \mathbb{R}^\infty, \mathbb{R})$ is ω-periodic with respect to the first coordinate and $a, b, c \in \mathbb{R}$ are constants. The aim of this paper is to obtain the results with respect to the existence of ω-periodic solutions of the aforementioned system in the Banach sequence space ℓ_p $(1 \le p < \infty)$ utilizing the respective Green's function. Furthermore, some examples are provided to support our main results.

1. Introduction

During the last two decades, infinite systems of differential equations in various forms have been investigated in several papers. Recently, the concept of measure of noncompactness has been effectively utilized in sequence spaces (whether classic or non-classic) for various classes of differential equations (see the book [6] and references therein). Lately, via the technique of measure of noncompactness some researchers studied the infinite system of second-order differential equations. Aghajani and Pourhadi [2] considered an infinite system of second-order differential equations in the sequence space ℓ_1 by employing the measure of noncompactness with the help of a Darbo type fixed point theorem. Next, Mohiuddine et al. [12] and Banaś et al. [7] studied the same infinite system in the setting of sequence space ℓ_p.

In 2019, governed by the fundamental solutions, Mursaleen, Pourhadi and Saadati [13] introduced the Green's function of the second-order differential equations in general form with respect to boundary conditions and then, in the Banach sequence space ℓ_p $(1 \le p < \infty)$, they dealt with the solvability of the infinite system of second-order differential equations while the coefficients are real functions.

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The so-called Meir-Keeler condensing operators have recently attracted great attention in numerous existence results and this is because of the imposed conditions are significantly weakened. By this idea, the mentioned system was studied in Banach sequence spaces ℓ_1 and c_0 [14].

The following infinite system of third-order differential equations over the Banach sequence space c_0 was studied in [15]. Inspired by this paper, as the main goal, the authors are interested to study the existence of ω-periodic solutions of the following infinite system over the space ℓ_p $(1 \le p < \infty)$

$$u_i''' + au_i'' + bu_i' + cu_i = f_i(t, u_1(t), u_2(t), \ldots), \quad (i \in \mathbb{N}) \tag{1}$$

such that $f_i \in C(\mathbb{R} \times \mathbb{R}^\infty, \mathbb{R})$ is ω-periodic with respect to the first component t and $a, b, c \in \mathbb{R}$ are constant. To simplify the notation, we denote $f_i(t, u)$ instead of $f_i(t, u_1(t), u_2(t), \ldots)$. Utilizing the main results of Mursaleen and Rizvi [14] together with ones in Aghajani et al. [1] and using the Meir-Keeler condensing operators we establish some existence results for the infinite system (1). The structure of this paper is as follows. In Section 2, we gather some basic definitions and preliminaries concerning with the measure of noncompactness and Meir-Keeler condensing operator. In Section 3, we present some existence results in the sequence space ℓ_p $(1 \le p < \infty)$. We also thank to some useful upper bounds of Green's functions corresponding to our problem recently obtained by Chen et al. [8]. In final section, some illustrative examples are provided to show the usefulness of main results.

2. Preliminaries

Suppose E is a Banach space, $\overline{X}$ and $\operatorname{Conv} X$ stand for the closure and the convex closure of X as a subset of E, respectively. Furthermore, denote by $\mathfrak{M}_E$ the family of all nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

In the following definition we recall the notion of measure of noncompactness which has been initially introduced by Banaś and Goebel [5].

Definition 2.1. *[5, Definition 3.1.3] A mapping $\mu \colon \mathfrak{M}_E \longrightarrow \mathbb{R}^+$ is said to be a measure of noncompactness(MNC, for short) in E if it satisfies the following conditions:*

(i) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker \mu \subset \mathfrak{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

(iii) $\mu(\overline{X}) = \mu(X)$.

(iv) $\mu(\operatorname{Conv} X) = \mu(X)$.

(v) For all $\lambda \in [0, 1]$,

$$\mu(\lambda X + (1 - \lambda) Y) \le \lambda \mu(X) + (1 - \lambda) \mu(Y).$$

(vi) If $(X_n)_{n\in\mathbb{N}}$ *is a sequence of closed sets from* $\mathfrak{M}_E$ *such that*

$$X_{n+1} \subset X_n \quad \text{for all} \quad n = 1, 2, \ldots \quad \text{and} \quad \lim_{n\to\infty} \mu(X_n) = 0,$$

then the intersection set

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \quad \text{is nonempty.}$$

The family $\ker \mu$ *described in (i) is said to be the kernel of the measure of noncompactness* μ.

Here, regarding the facts related to *condensing operator*, *Meir-Keeler contraction* (MKC) and *Meir-Keeler condensing operator* the readers are requested to see Definitions 2.2, 2.4, 2.5 and 2.7, and Theorems 2.3, 2.6 and 2.8 in [15].

We study the solvability of the problem (1) in the sequence space ℓ_p $(1 \le p < \infty)$.

From the theory of ODEs, the associate homogeneous equation of (1) is

$$u_i''' + au_i'' + bu_i' + cu_i = 0, \quad (i \in \mathbb{N}) \tag{2}$$

and the related characteristic equation is given by:

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0. \tag{3}$$

All roots of the third-degree polynomial equation (3) are in form of one of the following four cases:

(i) $\lambda_1 \neq \lambda_2 \neq \lambda_3$,

(ii) $\lambda_1 = \lambda_2 \neq \lambda_3$,

(iii) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

(iv) $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta,\ \lambda_3 = \lambda$, for $\alpha, \beta, \lambda \in \mathbb{R}$.

In our investigation the case $c = 0$ is not included since it would be easy to expand our results to ones corresponding to this special case. Hence, the roots are presumed non-zero.

In the last decades there appeared numerous articles regarding with the concept of measure of noncompactness. There are diverse types of well-known measures of noncompactness introduced in several years ago. One of the most important is the Hausdorff measure of noncompactness (or ball measure of noncompactness) [10] given by:

$$\gamma(X) = \inf\{\epsilon > 0 \mid \text{there exists a finite } \epsilon\text{-net for } X \text{ in } E\}.$$

It is worth mentioning that Hausdorff measure of noncompactness γ possesses the properties of regularity (that is, $\ker\gamma = \mathfrak{N}_E$), semi-additivity, Lipschitzianity, continuity, and some further properties connected with the linear structure.

3. Solvability of the system (1) in ℓ_p

As we know from the literature, the Hausdorff measure of noncompactness of the Banach sequence space $(\ell_p, \|\cdot\|_{\ell_p})$ is formulated by the following relation (cf. [5]):

$$\gamma(X) = \lim_{n\to\infty} \left\{ \sup_{(x_k)\in X} \left(\sum_{k=n}^{\infty} |x_k|^p \right)^{1/p} \right\} \tag{4}$$

for any nonempty and bounded subset X of ℓ_p. We recall that ℓ_p is a Banach space equipped with norm

$$\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}, \quad (1 \le p < \infty)$$

for any real sequence $x = (x_i)$.

Backing to Eq. (1), it is known that $u \in C^3(\mathbb{R}, \mathbb{R}^\infty)$ is a solution of Eq. (1) if and only if $u \in C(\mathbb{R}, \mathbb{R}^\infty)$ is a solution of the following infinite system of integral equations (see also [8]):

$$u_i(t) = \int_t^{t+\omega} G(t,s) f_i(s, u(s)) ds, \quad (i \in \mathbb{N}) \tag{5}$$

where the Green's function $G(t,s)$ would be determined later.

In our considerations we impose the following two hypotheses:

(A_1) The functions $f_i : \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}$ are assumed to be ω-periodic with respect to first coordinate. The operator $f : \mathbb{R} \times \ell_p \to \ell_p$ given by

$$(t,u) \mapsto (fu)(t) = (f_1(t,u), f_2(t,u), \ldots)$$

is such that the class of all functions $\{(fu)(t)\}_{t \in \mathbb{R}}$ is equicontinuous at every point of the space ℓ_p;

(A_2) The following inequality holds:

$$|f_n(t, u_1, u_2, \ldots)| \leq g_n(t) + h_n(t)|u_n(t)|,$$

in which the real functions $g_n(t)$ and $h_n(t)$ are supposed to be continuous on $\mathbb{R}$, such that $\sum_{i=1}^\infty |g_i(t)|^p$ converges uniformly on $\mathbb{R}$ with supremum value $\overline{G}$ and let the family $(h_n(t))_n$ be uniformly bounded on $\mathbb{R}$ by a non-zero constant H.

Now we are prepared to formulate the following results.

Let us first focus the ℓ_p-solvability of the problem for the case (i). Assume M is given as defined in [15, Theorem 3.1], i.e.

$$\begin{aligned} M := &\frac{\exp(\omega|\lambda_1|)}{|(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(1 - \exp(\lambda_1 \omega))|} + \frac{\exp(\omega|\lambda_2|)}{|(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(1 - \exp(\lambda_2 \omega))|} \\ &+ \frac{\exp(\omega|\lambda_3|)}{|(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(1 - \exp(\lambda_3 \omega))|}. \end{aligned}$$

Theorem 3.1. *Suppose the assumptions (A_1)–(A_2) hold and $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Moreover, let $\omega M H < \frac{1}{2}$. Then infinite system* (1) *has at least one ω-periodic solution $u(t) = (u_k(t))$ such that $u(t) \in \ell_p$, $t \in \mathbb{R}$ and in addition, the set of all solutions is compact.*

Proof. The proof is partially similar to one considered for [15, Theorem 3.1]. Here, we only prefer to consider the parts dealing with the norm $\|\cdot\|_{c_0}$ which should be replaced by $\|\cdot\|_p$ with some manipulations. For any arbitrary $t \in \mathbb{R}$ that

$$\begin{aligned} \|u(t)\|_p^p &= \sum_{i=1}^\infty \left| \int_t^{t+\omega} G_1(t,s) f_i(s,u(s)) ds \right|^p \\ &\leq \sum_{i=1}^\infty \left(\int_t^{t+\omega} |G_1(t,s) f_i(s,u(s))|^p ds \right)^{1/p} \left(\int_t^{t+\omega} ds \right)^{1/q} \\ &\leq \omega^{\frac{p}{q}} \sum_{i=1}^\infty \int_t^{t+\omega} |G_1(t,s)|^p |f_i(s,u(s))|^p ds \end{aligned}$$

such that $p > 1$ is the Hölder conjugate of q, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Moreover,

$$\begin{aligned}
\|u(t)\|_p^p &\le \omega^{\frac{p}{q}} M^p \sum_{i=1}^{\infty} \int_t^{t+\omega} (|g_i(s)| + |h_i(s)| \cdot |u_i(s)|)^p ds \\
&\le \omega^{\frac{p}{q}} (2M)^p \int_t^{t+\omega} \sum_{i=1}^{\infty} (|g_i(s)|^p + |h_i(s)|^p \cdot |u_i(s)|^p) ds \\
&\le \omega^{\frac{p}{q}} (2M)^p (\omega\overline{G} + \omega H^p \sum_{i=1}^{\infty} \|u_i\|_p^p) \\
&= (2\omega M)^p (\overline{G} + H^p \|u\|_p^p).
\end{aligned} \tag{6}$$

The case $p = 1$ is easier and one can simply find out an appropriate bound and fortunately it is the same as above:

$$\|u(t)\|_1 \le 2\omega M(\overline{G} + H\|u\|_1).$$

Now, for any case, assuming an arbitrary positive r_0 satisfying

$$0 < r_0 \le \frac{2\omega M \overline{G}^{1/p}}{\sqrt[p]{1 - (2\omega M H)^p}}$$

we see that $\|u\|_p \le r_0$, that is, u belongs to $B_0 := B(\mathbf{0}, r_0)$, the closed ball centered at $\mathbf{0} = (0, 0, \ldots)$ with radius r_0. Let us consider the operator $\mathcal{F} = (\mathcal{F}_i)$ defined on $C(\mathbb{R}, B_0)$ by

$$(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} = \left\{ \int_t^{t+\omega} G_1(t,s) f_i(s, u(s)) ds \right\}, \quad t \in \mathbb{R}, \tag{7}$$

while $u(t) = (u_i(t)) \in B_0$ and $u_i(t) \in C(\mathbb{R}, \mathbb{R})$, $t \in \mathbb{R}$. Bearing the assumption (A_1) in mind, it is evidently understood that $\mathcal{F}$ is continuous on $C(\mathbb{R}, \ell_p)$. Clearly, the function $\mathcal{F}u$ is also continuous, and $(\mathcal{F}u)(t) \in \ell_p$ whenever $u(t) = (u_i(t)) \in \ell_p$ (see also (6)). Indeed, remind that $(f_i(t, u(t))) \in \ell_p$ and

$$\|(f_i(t, u(t)))\|_p^p = \sum_{i=1}^{\infty} |f_i(t, u(t))|^p \le 2^p \sum_{i=1}^{\infty} (|g_i(t)|^p + |h_i(t)|^p \cdot |u_i(t)|^p) \le 2^p (\overline{G} + H^p \|u(t)\|_p^p) < \infty.$$

Further, this shows that

$$\begin{aligned}
\sum_{i=1}^{\infty} |(\mathcal{F}_i u)(t)|^p &\le \sum_{i=1}^{\infty} \int_t^{t+\omega} |G_1(t,s) f_i(s, u(s))|^p ds \\
&\le \omega^{\frac{p}{q}} M^p \sum_{i=1}^{\infty} \int_t^{t+\omega} \|f_i(s, u(s)))\|_p^p ds \\
&\le (2\omega M)^p (\overline{G} + H^p \|u\|_p^p) < \infty.
\end{aligned}$$

Finally, it suffices to show that $\mathcal{F}$ is a Meir-Keeler condensing operator. To do this, let us consider $\epsilon > 0$ is given. We need to find a $\delta > 0$ such that $\epsilon \le \gamma(B_0) < \epsilon + \delta$ implies $\gamma(\mathcal{F}B_0) < \epsilon$. Now, for the case $p > 1$, formulae (4), (6) and the fact that $\sum_{i=1}^{\infty} |g_i(t)|^p$ converges uniformly yield that

$$\begin{aligned}
\gamma(\mathcal{F}B_0) &= \lim_{n\to\infty} \left\{ \sup_{u(t)\in B_0} \left(\sum_{k=n}^{\infty} \left| \int_t^{t+\omega} G_1(t,s) f_k(s,u(s)) ds \right|^p \right)^{1/p} \right\} \\
&\leq \omega^{\frac{1}{q}} M \lim_{n\to\infty} \left\{ \sup_{u(t)\in B_0} \left(\sum_{k=n}^{\infty} \int_t^{t+\omega} (|g_k(s)| + |h_k(s)| \cdot |u_k(s)|)^p ds \right)^{1/p} \right\} \\
&\leq 2\omega^{\frac{1}{q}} M \lim_{n\to\infty} \left\{ \sup_{u(t)\in B_0} \left(\int_t^{t+\omega} \left[(\sum_{k=n}^{\infty} |g_k(s)|^p) + \sum_{k=n}^{\infty} |h_k(s) u_k(s)|^p \right] ds \right)^{1/p} \right\} \\
&\leq 2\omega M H \lim_{n\to\infty} \left\{ \sup_{u(t)\in B_0} \left(\sum_{k=n}^{\infty} |u_k|^p \right)^{1/p} \right\} \\
&= 2\omega M H \gamma(B_0).
\end{aligned} \tag{8}$$

Again, in above, q is the Hölder conjugate of $p > 1$. For the case $p = 1$, an analogous upper bound is derived. Thus, for given $\epsilon > 0$, and taking $\delta := \frac{(1-2\omega MH)\epsilon}{2\omega MH}$ we infer the following implication:

$$\epsilon \leq \gamma(B_0) < \epsilon + \delta \Longrightarrow \gamma(\mathcal{F}B_0) < \epsilon. \tag{9}$$

The Eq. (9) implies $\mathcal{F}$ is a Meir-Keeler condensing operator given on the set $B_0 \subseteq \ell_p$, which means all assumptions of [15, Theorem 2.8] are fulfilled, and following the proof of [15, Theorem 3.1] shows that we are done. □

Similarly, with the preserving all notations and conditions in [15], for the rest of paper we have the following results for the cases (ii)-(iv).

Theorem 3.2 (ℓ_p-Solvability for the case (ii)). *[15, Theorem 3.4] holds by replacing the upper bound 1 and the sequence space c_0 with $\frac{1}{2}$ and ℓ_p, respectively.*

Theorem 3.3 (ℓ_p-Solvability for the case (iii)). *[15, Theorem 3.5] holds by replacing the upper bound 1 and the sequence space c_0 with $\frac{1}{2}$ and ℓ_p, respectively.*

Theorem 3.4 (ℓ_p-Solvability for the case (iv)). *[15, Theorem 3.6] holds by replacing the upper bound 1 and the sequence space c_0 with $\frac{1}{2}$ and ℓ_p, respectively.*

4. Concrete examples

In this section, we provide two examples for the cases (i) and (ii) to illustrate our main results obtained in the former section. Regarding with the examples, the results for the other cases can be similarly structured.

Example 4.1. *Consider the following infinite system of third-order differential equations*

$$u_n''' - 0.601u_n'' + 0.0506u_n' - 5\cdot 10^4 u_n = \frac{1}{n} + \sum_{k=n}^{\infty} \frac{|u_k(t)| \sin\frac{4}{3}t}{k(n+50)(10+(k-n)^2|u_k(t)|)}, \quad (n \in \mathbb{N}). \tag{10}$$

Also, assume the notations p_1, q_1, A_3 and B_3 as given in [15, Remark 3.2], i.e.

$$\begin{aligned}
p_1 &:= (\lambda_2-\lambda_3)\exp(\lambda_3\omega)+2(\lambda_1-\lambda_3)\exp(\lambda_2\omega)\\
&\quad+(\lambda_1-\lambda_2)\exp(\lambda_1\omega)+(\lambda_1-\lambda_3)\exp((\lambda_1+\lambda_2+\lambda_3)\omega),\\
q_1 &:= (\lambda_1-\lambda_3)+(\lambda_1+\lambda_2-2\lambda_3)\exp((\lambda_2+\lambda_3)\omega)+(2\lambda_1-\lambda_2-\lambda_3)\exp((\lambda_1+\lambda_2)\omega),\\
A_3 &:= \frac{\exp(\lambda_1\omega)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))}+\frac{1}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))}\\
&\quad+\frac{\exp(\lambda_3\omega)}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(1-\exp(\lambda_3\omega))},\\
B_3 &:= \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))}+\frac{\exp(\lambda_2\omega)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))}\\
&\quad+\frac{1}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(1-\exp(\lambda_3\omega))}.
\end{aligned}$$

We first note that the function in R.H.S. of (10)*, say $f_n(t,u(t))$, is continuous at each point in $\mathbb{R}$ and for $n\in\mathbb{N}$. Besides, $(f_n(t,u(t)))\in\ell_2$ whenever $u(t)=(u_n(t))\in\ell_2$. Furthermore,*

$$\begin{aligned}
\sum_{n=1}^{\infty}|f_n(t,u(t))|^2 &= \sum_{n=1}^{\infty}\Big(\frac{1}{n}+\sum_{k=n}^{\infty}\frac{|u_k(t)|\sin\frac{4}{3}t}{k(n+50)(10+(k-n)^2|u_k(t)|)}\Big)^2\\
&\le \sum_{n=1}^{\infty}\Big[\frac{1}{n^2}+\sum_{k=n}^{\infty}\frac{2|u_k(t)|}{kn(n+50)(10+(k-n)^2|u_k(t)|)}\\
&\quad+\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{|u_k(t)|^2}{(10+(k-n)^2|u_k(t)|)^2}\Big)\Big(\sum_{k=n}^{\infty}\frac{1}{k^2(n+50)^2}\Big)\Big]\\
&< \sum_{n=1}^{\infty}\Big[\frac{1}{n^2}+\frac{1}{n^3}\Big]+\frac{\pi^2}{6}(\frac{\pi^2}{6}-1-\frac{1}{2^2}-\frac{1}{3^2}-\cdots-\frac{1}{50^2})\\
&\quad\times\sum_{n=1}^{\infty}\Big(0.01|u_n(t)|^2+\sum_{k=n+1}^{\infty}\frac{|u_k(t)|^2}{(10+(k-n)^2|u_k(t)|)^2}\Big)\\
&\le \sum_{n=1}^{\infty}\Big[\frac{1}{n^2}+\frac{1}{n^3}\Big]+\frac{\pi^2}{6}(\frac{\pi^2}{6}-1-\frac{1}{2^2}-\frac{1}{3^2}-\cdots-\frac{1}{50^2})\Big(0.01\|u(t)\|_2^2+\sum_{n=1}^{\infty}\frac{1}{n^4}\Big)\\
&= \frac{\pi^2}{6}+\zeta(3)+\frac{\pi^2}{6}(\frac{\pi^2}{6}-1-\frac{1}{2^2}-\frac{1}{3^2}-\cdots-\frac{1}{50^2})\Big(0.01\|u(t)\|_2^2+\frac{\pi^4}{90}\Big)<\infty
\end{aligned}$$

where ζ is the Riemann zeta function and $\zeta(3)\approx 1.2020569$. Now let us prove that (A_1) holds. To do this, take an arbitrary $\epsilon>0$ and $u(t)=(u_n(t)), v(t)=(v_n(t))\in\ell_2$ such that $\|u(t)-v(t)\|_2<\delta(\epsilon):=\frac{6(10^4)\epsilon^2}{\pi^2}$, then for the operator

$f = (f_n)$ we have

$$\begin{aligned}
\|(fv)(t) - (fu)(t)\|_2^2 &\le \sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{1}{k(n+50)}\cdot\Big|\frac{|u_k(t)|}{(10+(k-n)^2|u_k(t)|)}-\frac{|v_k(t)|}{(10+(k-n)^2|v_k(t)|)}\Big|\Big)^2\\
&\le \sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{1}{k^2}\cdot\sum_{k=n}^{\infty}\frac{100\Big||u_k(t)|-|v_k(t)|\Big|^2}{(n+50)^2(10+(k-n)^2|u_k(t)|)^2(10+(k-n)^2|v_k(t)|)^2}\Big)\\
&\le 0.01\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{1}{k^2}\cdot\sum_{k=n}^{\infty}\frac{|u_k(t)-v_k(t)|^2}{(n+50)^2}\Big)\\
&\le \frac{\pi^2}{600}\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{|u_k(t)-v_k(t)|^2}{(n+50)^2}\Big)\\
&= \frac{\pi^2}{600}\sum_{n=1}^{\infty}\frac{n|u_n(t)-v_n(t)|^2}{(n+50)^2}\\
&\le \frac{\pi^2}{6(10^4)}\|u(t)-v(t)\|_2\\
&< \frac{\pi^2\delta}{6(10^4)} = \epsilon^2
\end{aligned}$$

which establishes the required continuity as supposed in (A_1). In order to verify hypothesis (A_2) we get the following bound

$$\begin{aligned}
|f_n(t,u(t))| &\le \frac{1}{n} + \Big|\sum_{k=n}^{\infty}\frac{|u_k(t)|\sin\frac{4}{3}t}{k(n+50)(10+(k-n)^2|u_k(t)|)}\Big|\\
&\le \frac{1}{n} + \frac{|\sin\frac{4}{3}t|}{n+50}\Big(\frac{1}{10n}|u_n(t)| + \sum_{k=n+1}^{\infty}\frac{1}{k(k-n)^2}\Big)\\
&\le \frac{1}{n} + \frac{|\sin\frac{4}{3}t|}{n+50}\Big(\frac{1}{10n}|u_n(t)| + \zeta(3)\Big).
\end{aligned}$$

Now, taking

$$g_n(t) := \frac{1}{n} + \frac{\zeta(3)|\sin\frac{4}{3}t|}{n+50}, \quad h_n(t) := \frac{|\sin\frac{4}{3}t|}{10n(n+50)}$$

one can observe that (A_2) is satisfied, $H = \frac{1}{510}$ and $\sum_{k=1}^{\infty}|g_n(t)|^2 < \infty$. On the other hand, using the notations in previous section we see that the roots of the associate homogeneous equation of (10) *are* $\lambda_1 = 0.5, \lambda_2 = 0.1, \lambda_3 = 0.01$ *and* $p_1 \approx 15.5096 < q_1 \approx 16.5067$. *Considering the roots as above, and applying the definition of $G_1(t,s)$ we find*

$$A_3 = -46.2250 \le G_1(t,s) \le B_3 = -36.5363 < 0, \quad \text{for } \omega = 1.5\pi.$$

This shows that the assumption (C_1) is satisfied (see [15, Remark 3.2]). Since for any positive integer n the function $f_n(t,u(t))$ is 1.5π-periodic with respect to the first coordinate t, and $\omega H|A_3| \approx 0.4271 < 0.5$, then all the hypotheses of Theorem 3.1 are fulfilled. Therefore, the infinite system (10) *has a 1.5π-periodic solution $u(t) = (u_n(t)) \in \ell_2$.*

We now present another illustrative example in support of our result for the case (ii).

Example 4.2. *Consider the following infinite system of third-order differential equations*

$$u_n''' - 5.2u_n'' + 1.01u_n' - 0.05u_n = \frac{e^{-u_n^2(t)}}{n} - \sum_{k=n}^{\infty}\frac{\sin^2 u_k(t)}{k^2(n+25\pi)(2+\cos t)}, \quad (n \in \mathbb{N}). \tag{11}$$

First note that the characteristic equation of the homogeneous equation corresponding to Eq. (11) *has the roots* $\lambda_1 = \lambda_2 = 0.1, \lambda_3 = 5$. *This shows that the associate Green's function of Eq.* (11) *is as form of* $G_2(t,s)$ *(see [15, Eq. (3.8)]).*

The function $f_n(t,u(t))$ *in R.H.S. of* (11) *is continuous on* $\mathbb{R}$, *and* 2π*-periodic with respect to the first coordinate* t *for* $n \in \mathbb{N}$. *In addition,* $(f_n(t,u(t))) \in \ell_2$ *whenever* $u(t) = (u_n(t)) \in \ell_2$. *Indeed, by the fact that* $\sin^2 u_k(t) \le |u_k(t)|$ *we derive*

$$\begin{aligned}
\sum_{n=1}^{\infty} |f_n(t,u(t))|^2 &\le \sum_{n=1}^{\infty} \frac{e^{-2u_n^2(t)}}{n^2} + \sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty} \frac{\sin^2 u_k(t)}{k^2(n+25\pi)(2+\cos t)}\Big)^2 \\
&\le \frac{\pi^2}{6} + \frac{1}{(2+\cos t)^2}\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{1}{k^4}\Big)\Big(\sum_{k=n}^{\infty}\frac{|u_k(t)|^2}{(n+25\pi)^2}\Big) \\
&\le \frac{\pi^2}{6} + \frac{\pi^4}{90(2+\cos t)^2}\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{|u_k(t)|^2}{(n+25\pi)^2}\Big) \\
&= \frac{\pi^2}{6} + \frac{\pi^4}{90(2+\cos t)^2}\sum_{n=1}^{\infty}\frac{n|u_n(t)|^2}{(n+25\pi)^2} \\
&\le \frac{\pi^2}{6} + \frac{\pi^3}{4500(2+\cos t)^2}\|u(t)\|_2^2 < \infty.
\end{aligned}$$

To verify (A_1), *take an arbitrary* $\epsilon > 0$ *and* $u(t) = (u_n(t)), v(t) = (v_n(t)) \in \ell_2$ *such that* $\|u(t) - v(t)\|_2 < \delta(\epsilon) := (16M^2 + \frac{\pi^3}{1125})^{-1}\epsilon^2$, *then for the operator* $f = (f_n)$ *and utilizing the fact* $|a+b|^p \le 2^p(|a|^p + |b|^p)$ *we have*

$$\begin{aligned}
\|(fu)(t) - (fv)(t)\|_2^2 &\le \sum_{n=1}^{\infty}\Big(\frac{e^{-u_n^2(t)}}{n} - \frac{e^{-v_n^2(t)}}{n} - \sum_{k=n}^{\infty}\Big[\frac{\sin^2 u_k(t)}{k^2(n+25\pi)(2+\cos t)} - \frac{\sin^2 v_k(t)}{k^2(n+25\pi)(2+\cos t)}\Big]\Big)^2 \\
&\le 4\sum_{n=1}^{\infty}\Big|\frac{e^{-u_n^2(t)} - e^{-v_n^2(t)}}{n}\Big|^2 + 4\sum_{n=1}^{\infty}\Big|\sum_{k=n}^{\infty}\frac{\sin^2 u_k(t) - \sin^2 v_k(t)}{k^2(n+25\pi)(2+\cos t)}\Big|^2 \\
&\le 4\sum_{n=1}^{\infty}\Big|\frac{[u_n(t) - v_n(t)](-2\eta_n(t)e^{-\eta_n^2(t)})}{n}\Big|^2 + 4\sum_{n=1}^{\infty}\Big|\sum_{k=n}^{\infty}\frac{\sin^2 u_k(t) - \sin^2 v_k(t)}{k^2(n+25\pi)(2+\cos t)}\Big|^2
\end{aligned}$$

for an appropriate sequence of functions $\eta_n(t)$ *using the mean value theorem. Since* $u_n(t), v_n(t) \in \ell_2$ *one can find an upper bound* M, *sufficiently large, satisfying the following:*

$$\begin{aligned}
\|(fu)(t) - (fv)(t)\|_2^2 &\le 16M^2\|u(t) - v(t)\|_2^2 + 4\sum_{n=1}^{\infty}\Big[\Big(\sum_{k=n}^{\infty}\frac{1}{k^4}\Big)\cdot\Big(\sum_{k=n}^{\infty}\frac{|u_k(t) - v_k(t)|^2}{(n+25\pi)^2}\Big)\Big] \\
&\le 16M^2\|u(t) - v(t)\|_2^2 + \frac{4\pi^4}{90}\sum_{n=1}^{\infty}\Big[\sum_{k=n}^{\infty}\frac{|u_k(t) - v_k(t)|^2}{(n+25\pi)^2}\Big] \\
&= 16M^2\|u(t) - v(t)\|_2^2 + \frac{4\pi^4}{90}\sum_{n=1}^{\infty}\frac{n|u_n(t) - v_n(t)|^2}{(n+25\pi)^2} \\
&\le 16M^2\|u(t) - v(t)\|_2^2 + \frac{\pi^3}{1125}\|u(t) - v(t)\|_2^2 \\
&< \delta(16M^2 + \frac{\pi^3}{1125}) < \epsilon^2.
\end{aligned}$$

which shows (A_1) is satisfied. On the other hand, to verify (A_2) we see

$$\begin{aligned}
|f_n(t,u(t))| &= \left|\frac{e^{-u_n^2(t)}}{n} - \sum_{k=n}^{\infty}\frac{\sin^2 u_k(t)}{k^2(n+25\pi)(2+\cos t)}\right| \\
&\le \frac{1}{n} + \sum_{k=n}^{\infty}\frac{\sin^2 u_k(t)}{k^2(n+25\pi)(2+\cos t)} \\
&= \frac{1}{n} + \frac{\sin^2 u_n(t)}{n^2(n+25\pi)(2+\cos t)} + \sum_{k=n+1}^{\infty}\frac{\sin^2 u_k(t)}{k^2(n+25\pi)(2+\cos t)} \\
&\le g_n(t) + h_n(t)|u_n(t)|
\end{aligned}$$

where

$$g_n(t) := \frac{1}{n} + \frac{\pi^2}{6(n+25\pi)(2+\cos t)}, \quad h_n(t) := \frac{1}{n^2(n+25\pi)(2+\cos t)}.$$

Obviously, $g_n(t) \in \ell_2$ and $H = \frac{1}{1+25\pi}$. Moreover, according to (C_5) (to see this condition and the notations used below we refer [15]):

$$\lambda_3 > \lambda_2 = \lambda_1 > 0,\ \exp(0.2\pi) \approx 1.8745 < 1 + 9.8\pi \approx 31.7876,\ A_5 = -5.8443 \le G_2(t,s) \le B_5 = -1.6293 < 0$$

which together with $\omega H|A_5| = \dfrac{2\pi(5.8443)}{1+25\pi} \approx 0.4616 < \frac{1}{2}$ implies all conditions of Theorem 3.2 are fulfilled and the infinite system (11) *has at least one 2π-periodic solution $u(t) = (u_k(t))$ such that $u(t) \in \ell_2$, $t \in \mathbb{R}$.*

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