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On the Problem of Discontinuity at Fixed Point

Bharti Joshi[a], Durgesh Pant[b], S. K. Padaliya[c], Govind Pathak[d]

[a]*Byura Chandmari, Kathgodam, Naintal, Uttarakhand (India)-263126*
[b]*Uttarakhand Open University, Dehradun, Uttarakhand (India)*
[c]*Department of Mathematics, S. G. R. R. (P.G.) College Dehradun, Uttarakhand (India)-248001*
[d]*Department of Mathematics, M. B. Govt. P. G. College, Haldwani (Nainital) Uttarakhand (India)-263139*

Abstract. In this paper we obtain two known solutions of the problem of continuity of contractive mappings at fixed point under alternative set of conditions; these known solutions followed the first solution of Rhoades problem by Pant in 1999. We show that these two solutions characterize completeness and we also compare these with some recent solutions of the Rhoades problem.

1. Introduction

If f is a self-mapping of a metric space (X, d), we denote

$$\begin{aligned}
m_1(x,y) &= \max\{d(x,fx), d(y,fy)\}.\\
m_2(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy)\}.\\
m_3(x,y) &= \max\left\{d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}\\
m_4(x,y) &= \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}.
\end{aligned}$$

Consider the following conditions for $i = 1, 2, 4$.
For any $\epsilon > 0$ there exists a $\delta > 0$ such that, for any $x, y \in X$,

$$\epsilon \le m_i(x,y) < \epsilon + \delta \Rightarrow d(fx,fy) < \epsilon. \tag{1}$$

$$\epsilon < m_i(x,y) < \epsilon + \delta \Rightarrow d(fx,fy) \le \epsilon. \tag{2}$$

$$d(fx,fy) < m_i(x,y) \text{ for any } x,y \in \mathrm{X} \text{ with } m_i(x,y) > 0. \tag{3}$$

$$d(fx,fy) \le m_i(x,y) \text{ for any } x,y \in \mathrm{X}. \tag{4}$$

$$d(fx,fy) \le \phi(m_i(x,y)) \text{ for any } x,y \in \mathrm{X}. \tag{5}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ denotes a function such that $\phi(t) < t$ for each $t > 0$. Jachymski [11] studied various Meir-Keeler type conditions and observed that (1) $\Rightarrow$ (2) $\wedge$ (3) but not conversely. The symbol $\wedge$ represents

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Email addresses: bhartijoshi20592@gmail.com (Bharti Joshi), durgesh.pant@gmail.com (Durgesh Pant), spadaliya12@rediffmail.com (S. K. Padaliya), pathakgovind@rediffmail.com (Govind Pathak)

the word *and*.

In 1988, Rhoades [21] examined continuity of a large number of contractive mappings at their fixed points and found that though these contractive definitions do not require the map to be continuous but are strong enough to force the mapping to be continuous at the fixed point. Rhoades [21] proposed an open question whether there exists a contractive definition which is strong enough to generate a fixed point, but which does not force the mapping to be continuous at the fixed point. In 1999, Pant [18] proved the following theorem and obtained the first result as an affirmative answer.

Theorem 1.1 ([18]). *Let f be a self-mapping of a complete metric space (X, d) such that the conditions (2) and (5) hold for $i = 1$. Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x\to z} m(x, z) = 0$.*

In continuation, Pant [19] also obtained another solution of the Rhoades' problem.

Theorem 1.2 ([19]). *Let f be a self-mapping of a complete metric space (X, d) such that the conditions (2) and (5) hold for $i = 2$. Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x\to z} m(x, z) = 0$.*

In 2002, V. Pant [20] gave Matkowski type $\phi-$contractive solution for the Rhoades' problem.

Theorem 1.3 ([20]). *Let f be a self-mapping of a complete metric space (X, d) such that condition (5) holds for $i = 3$ and $\lim_n \phi^n(t) = 0$. Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x_n\to z} M(x, z) = 0$.*

We now recall some weaker forms of continuity.

Definition 1.4 ([16]). *A self-mapping f of a metric space X is called k-continuous, $k = 1, 2, 3, \ldots$, if $f^k x_n \to ft$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1}x_n \to t$.*

It was shown in [16] that continuity of f^k and k-continuity of f are independent conditions when $k > 1$; and k-continuity $\Rightarrow$ $(k + 1)$-continuity, where $k \in \mathbb{N}$, but not conversely. Obviously, 1-continuity is equivalent to continuity.

Definition 1.5 ([5]). *If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i}x$ implies $fu = \lim_i ff^{m_i}x$.*

A continuous mapping is orbitally continuous but not conversely. A k-continuous mapping is obviously orbitally continuous [17].

Definition 1.6 ([17]). *A self-mapping f of a metric space (X, d) is called weakly orbitally continuous if the set $\{y \in X : \lim_i f^{m_i}y = u \Rightarrow \lim_i ff^{m_i}y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i f^{m_i}x = u\}$ is nonempty.*

An orbitally continuous mapping is obviously weakly orbitally continuous but not conversely (see [Example 1.7, [17]]).

In 2017 Bisht and Pant [1] obtained one more solution to the Rhoades' problem under a ϕ-contractive condition and a Meir-Keeler type (2) condition.

Theorem 1.7 ([1]). *Let (X, d) be a complete metric space. Let f be a self-mapping on X such that f^2 is continuous and satisfy the conditions (2) and (5) for $i = 4$. Then f has a unique fixed point, say z, and $f^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z iff $\lim_{x\to z} m_4(x, z) \neq 0$.*

Fixed point theorems for discontinuous mappings have found various applications. Applications of such theorems in the study of neural networks under suitable conditions is a very active area of research (see, [8, 9, 12, 13]). Cromme and Diener [6] and Cromme [7] have proved results on approximate fixed points for discontinuous functions and have given applications of their results to neural nets, economic equilibria and analysis. All the known solutions of the Rhoades' problem (e.g. [1–4, 14–16, 20, 22]) employ condition (2) or some generalized form of (2) together with a ϕ-contractive condition (5) or some weaker form of continuity, e.g., orbital continuity, continuity of f^k or k-continuity for some $k > 1$. Recently, Pant et al [17] obtained a Meir-Keeler type solution for the Rhoades' problem with condition (4) under the weak orbital continuity condition.

Theorem 1.8 ([17]). *Let f be a self-mapping of a complete metric space (X, d) such that conditions (1) and (4) hold for $i = 1$. Then f possesses a fixed point if and only if f is weakly orbitally continuous. Moreover, the fixed point is unique and f is continuous at the fixed point, say z, if and only if $\lim_{x\to z} m_1(x, z) = 0$ or, equivalently, $\lim_{x\to z} \sup d(fz, fx) = 0$.*

In this paper we obtain the solution of the Rhoades' problem with the conditions (2) and (3) under weak orbitally continuity without assuming (5). We show that these solution characterize completeness of the metric space and we also compare these with some recent solutions of Rhoades' problem.

2. Main Results

Theorem 2.1. *Let f be a self-mapping of a complete metric space (X, d) such that for $i = 1, 2$*

(i) $d(fx, fy) < m_i(x, y)$ whenever $m_i(x, y) > 0$

(ii) given $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon < m_i(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \le \epsilon.$$

(iii) f is orbitally continuous or k-continuous or weak orbitally continuous.

Then f has a unique fixed point, say z. Moreover, f is continuous at the fixed point if and only if $\lim_{x\to z} m_i(x, z) = 0$ or, equivalently, $\lim_{x\to z} \sup d(fz, fx) = 0$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. If $x_n = x_{n+1}$ for some n then $x_n = x_{n+1} = x_{n+2} = x_{n+3} \ldots$, that is, $\{x_n\} = \{f^n x_0\}$ is a Cauchy sequence and x_n is a fixed point of f. We can, therefore, assume that $x_n \ne x_{n+1}$ for each n. We first proceed for $i = 1$ and then using (i) we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) &< \max\{d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n). \end{aligned}$$

Thus $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $r \ge 0$. Suppose $r > 0$. Then there exists a positive integer N such that

$$n \ge N \Rightarrow r < d(x_n, x_{n+1}) < r + \delta(r). \tag{6}$$

This yields $r < \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = \max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} < r + \delta(r)$, which by virtue of (iv) yields $d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) < r$. This contradicts (6). Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Similarly it can be shown that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists z in X such that $x_n \to z$. Moreover, for each integer $p \ge 1$, we have $f^p x_n \to z$. Also, using (i) it follows easily that $f^n y \to z$ for any y in X.

Now suppose that f is k-continuous. Since $f^{k-1} x_n \to t$, k continuity of f implies that $f^k x_n \to ft$. Hence $t = ft$ as $f^k x_n \to t$. Therefore, t is fixed point of f.

Next suppose that f^k is continuous for some positive integer k. Then, $\lim_{n\to\infty} f^k x_n = f^k t$. This yields $f^k t = t$ as $f^k x_n \to t$. If $t \ne ft$ we get

$$\begin{aligned} d(t, ft) = d(f^k t, f^{k+1} t) &< \max\{d(f^{k-1} t, f^k t), d(f^k t, f^{k+1} t)\} \\ &= d(f^{k-1} t, f^k t) < d(f^{k-2} t, f^{k-1} t) < \ldots < d(t, ft), \end{aligned}$$

a contradiction. Hence $t = ft$ and t is a fixed point of f.

Suppose that f is orbitally continuous. Since $x_n \to t$, orbital continuity implies that $fx_n \to ft$. This gives $t = ft$ as $fx_n \to t$. Thus t is a fixed point of f.

Finally, suppose that f is weakly orbitally continuous. Since $f^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of f we get $f^n y_0 \to z$ and $f^{n+1} y_0 \to fz$ for some y_0 in X. This implies $z = fz$ since $f^{n+1} y_0 \to z$. Therefore z is a fixed point of f. Uniqueness of the fixed point follows easily.

It is also easy to verify that f is continuous at z if and only if $\lim_{x\to z} \max\{d(x, fx), d(z, fz)\} = 0$ or, equivalently, $\lim_{x\to z} \sup d(fz, fx) = 0$. This can alternatively be stated as:

f is discontinuous at z if and only if $\lim\limits_{x\to z} \sup d(fz, fx) > 0$.

The proof for the case $i = 2$ follows similarly. □

We now show that Theorem 2.1 characterizes metric completeness.

Theorem 2.2. *If every k-continuous or weak orbitally continuous self-mapping of a metric space (X, d) satisfying conditions (i) and (ii) of Theorem 2.1 has a fixed point for $i = 1, 2$; then X is complete.*

Proof. Suppose that every k-continuous or weak orbitally continuous self-mapping of the metric space X satisfying conditions (i) and (ii) of Theorem 2.1 possesses a fixed point for $i = 1$ and 2. We show that X is a complete metric space. If X is not complete, then there exists a Cauchy sequence in X, say $S = \{a_n : n = 1, 2, 3 \ldots\}$, consisting of distinct points which does not converge. Let $x \in X$ be given. Then, since x is not a limit point of the sequence S, we have $d(x, S - \{x\}) > 0$ and there exists a least positive integer, say $N(x)$, such that $x \neq a_{N(x)}$ for each $m \geq N(x)$ we have

$$d(a_{N(x)}, a_m) < \frac{1}{2} d(x, a_{N(x)}). \tag{7}$$

Let us define a mapping $f : X \to X$ by $f(x) = a_{N(x)}$. Then, $f(x) \neq x$ for each x and, using (7), for any x, y in X we get

$$d(fx, fy) = d(a_{N(x)}, a_{N(y)}) < \frac{1}{2} d(x, a_N(x)) = d(x, fx) \text{ if } N(x) \leq N(y) \tag{8}$$

or

$$d(fx, fy) = d(a_{N(x)}, a_{N(y)}) < \frac{1}{2} d(y, a_N(y)) = d(y, fy) \text{ if } N(x) > N(y). \tag{9}$$

This implies that

$$d(fx, fy) < \frac{1}{2} \max\{d(x, fx), d(y, fy)\}. \tag{10}$$

In other words, given $\epsilon > 0$ we can select $\delta(\epsilon) = \epsilon$ such that

$$\epsilon < \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon. \tag{11}$$

It is clear from (10) and (11) that the mapping f satisfies condition (i) and (ii) of Theorem 2.1 for $i = 1$ and 2. Since the range of f is contained in the non-convergent Cauchy sequence $S = \{a_n\}$, there exists no sequence $\{x_n\}$ in X for which the condition $fx_n \to t \Rightarrow f^2 x_n \to ft$ is violated. Therefore, f is a 2-continuous mapping. In a similar manner it follows that f is weak orbitally continuous. Thus, f is a 2-continuous as well as weak orbitally continuous self-mapping of X satisfying (i) and (ii) which does not possess a fixed point. This contradicts our assumption. Hence X is complete. □

The next example illustrate Theorem 2.1.

Example 2.3. *Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \text{if} \quad 1 < x \le 2. \end{cases}$$

Then f satisfies all the conditions of Theorem 2.1 and has a unique fixed point $x = 1$; and f is discontinuous at the fixed point. The mapping f is 2-continuous, f^2 is continuous and f is also orbitally continuous. It can be easily verified that

$$d(fx, fy) = 0, \quad 0 < \max\{d(x, fx), d(y, fy)\} \le 1 \text{ if } x, y < 1,$$

$$d(fx, fy) = 0, \quad 1 < \max\{d(x, fx), d(y, fy)\} \le 2 \text{ if } x, y > 1,$$

$$\text{and } d(fx, fy) = 1, \quad 1 < \max\{d(x, fx), d(y, fy)\} \le 2 \text{ if } x \le 1, y > 1.$$

Therefore, f satisfies condition (ii) with $\delta(\epsilon) = 1 - \epsilon$ if $\epsilon < 1$ and $\delta(\epsilon) = 1$ for $\epsilon \ge 1$. It may also be seen that the function f in this example does not satisfy the Meir and Keeler $(\epsilon - \delta)$ contractive condition, i.e., condition (1).

$$\epsilon \le \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon.$$

Remark 2.4. *The above example also satisfies condition (5) for $i = 1$, i.e.,*

$$d(fx, fy) \le \phi(m_1(x, y)).$$

In a recent paper Pant et al [17] obtained a Meir-Keeler type solution of Rhoades problem but their solution does not satisfy condition (5). This gives rise to following unresolved questions:

Question 1. Does there exist a solution of Rhoades' problem which satisfies (2) and (3) but not (5)?

Question 2. Does there exist a Meir-Keeler type solution (1) of Rhoades' problem which also satisfies (5)?

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