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Expressions for the g-Drazin inverse in a Banach algebra

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Abstract. We explore the generalized Drazin inverse in a Banach algebra. Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $ab = \lambda a^\pi bab^\pi$ for a nonzero complex number λ, then $a + b \in \mathcal{A}^d$. The explicit representation of $(a + b)^d$ is presented. As applications of our results, we present new representations for the generalized Drazin inverse of a block matrix in a Banach algebra. The main results of Liu and Qin [Representations for the generalized Drazin inverse of the sum in a Banach algebra and its application for some operator matrices, Sci. World J., **2015**, 156934.8] are extended.

1. Introduction

Throughout the paper, $\mathcal{A}$ is a complex Banach algebra with an identity and λ is a nonzero complex number. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in \mathcal{A} \mid xa = ax\}$. An element a in $\mathcal{A}$ has g-Drazin inverse (that is, generalized Drazin inverse) provided that there exists $b \in comm(a)$ such that $b = bab$ and $a - a^2b \in \mathcal{A}^{qnil}$. Here, $\mathcal{A}^{qnil}$ is the set of all quasinilpotents in $\mathcal{A}$, i.e.,

$$\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + ax \in \mathcal{A} \text{ is invertible for every } x \in comm(a)\}.$$

For a Banach algebra $\mathcal{A}$ we have

$$a \in \mathcal{A}^{qnil} \Leftrightarrow 1 + \mu a \in \mathcal{A} \text{ is invertible for any } \mu \in \mathbb{C}.$$

We use $\mathcal{A}^d$ to denote the set of all g-Drazin invertible elements in $\mathcal{A}$. As is well known, $a \in \mathcal{A}^d$ if and only if there exists an idempotent $p \in comm(a)$ such that $a + p$ is invertible and $ap \in \mathcal{A}^{qnil}$ (see [10, Theorem 4.2]). The objective of this paper is to further explore the generalized Drazin inverse in a Banach algebra.

The g-Drazin invertibility of the sum of two elements in a Banach algebra is attractive. Many authors have studied such problems from many different views, e.g., [3, 4, 6, 7, 11, 13, 15, 17]. In Section 2, we investigate when the sum of two g-Drazin invertible elements in a Banach algebra has g-Drazin inverse. Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $ab = \lambda a^\pi bab^\pi$, we prove that $a + b \in \mathcal{A}^d$. The explicit representation of $(a + b)^d$ is presented. This extends [11, Theorem 4] to more general setting.

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It is a hard problem to find a formula for the g-Drazin inverse of a block matrix. There have been many papers on this subject under different conditions, e.g., [5, 6, 9, 14, 16]. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathcal{A})$, A and D have g-Drazin inverses. If $a \in \mathcal{A}$ has g-Drazin inverse a^d, the element $a^{\pi} = 1 - aa^d$ is called the spectral idempotent of a. In Section 3, we concern new conditions on spectral idempotent matrices under which M has g-Drazin inverse. If $BD = \lambda(BC)^{\pi}ABD^{\pi}$ and $CA = \lambda(CB)^{\pi}DCA^{\pi}$, we prove that $M \in M_2(\mathcal{A})^d$. The formula for M^d is given. This extends [11, Theorem 10] to the wider case.

Finally, in the last section, we present certain simpler representations of the g-Drazin inverse of the block matrix M. If $BC = 0$ and $BD = \lambda A^{\pi}AB, DC = \lambda^{-1}D^{\pi}CAA^{\pi}$, then $M \in M_2(\mathcal{A})^d$ and

$$M^d = \begin{pmatrix} A^d & (A^d)^2B + \sum\limits_{n=0}^{\infty} A^nB(D^d)^{n+2} \\ C(A^d)^2 & D^d + C(A^d)^3B + \sum\limits_{n=0}^{\infty}\sum\limits_{k=1}^{n} D^{k-1}CA^{n-k}B(D^d)^{n+2} \end{pmatrix}.$$

2. Additive results

In this section we establish some additive properties of g-Drazin inverse in Banach algebras. Let $p \in \mathcal{A}$ be an idempotent, and let $x \in \mathcal{A}$. Then we write

$$x = pxp + px(1-p) + (1-p)xp + (1-p)x(1-p),$$

and induce a Pierce representation given by the matrix

$$x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{pmatrix}_p.$$

We begin with

Lemma 2.1. *Let $a, b \in \mathcal{A}^d$ and $c \in \mathcal{A}$, and let*

$$x = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}_p \text{ or } \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}_p.$$

Then

$$x^d = \begin{pmatrix} a^d & 0 \\ z & b^d \end{pmatrix}_p, \text{ or } \begin{pmatrix} b^d & z \\ 0 & a^d \end{pmatrix}_p,$$

where

$$z = (b^d)^2\Big(\sum_{i=0}^{\infty}(b^d)^i ca^i\Big)a^{\pi} + b^{\pi}\Big(\sum_{i=0}^{\infty} b^i c(a^d)^i\Big)(a^d)^2 - b^d ca^d.$$

Proof. See [3, Lemma 2.1]. □

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^{qnil}$. If $ab = \lambda ba$, then $a + b \in \mathcal{A}^{qnil}$.*

Proof. See [2, Lemma 2.1] and [8, Lemma 2.1]. □

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}^{qnil}$ and $b \in \mathcal{A}^d$. If*

$$ab = \lambda bab^{\pi},$$

then $a + b \in \mathcal{A}^d$ and

$$(a+b)^d = b^d + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n.$$

Proof. Let $p = bb^d$. Then we have

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p, a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_p.$$

Hence,

$$b^d = \begin{pmatrix} b_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}_p \text{ and } b^\pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 - bb^d \end{pmatrix}_p.$$

Since $ab = \lambda bab^\pi$,, we get

$$\begin{pmatrix} a_1b_1 & a_2b_2 \\ a_3b_1 & a_4b_2 \end{pmatrix}_p = ab = \lambda bab^\pi = \begin{pmatrix} 0 & \lambda b_1a_2 \\ 0 & \lambda b_2a_4 \end{pmatrix}_p.$$

Thus $a_1b_1 = 0$ and $a_3b_1 = 0$, and then $a_1 = 0$ and $a_3 = 0$. Obviously, $b_2 = b - b^2b^d \in ((1-p)\mathcal{A}(1-p))^{qnil}$. Since $ab = \lambda bab^\pi$, we have $abb^d = \lambda bab^\pi b^d = 0$. Hence $a(1 - bb^d) = a \in \mathcal{A}^{qnil}$. In view of Cline's formula (see [12, Theorem 2.1]), we prove that $a_4 = b^\pi ab^\pi \in \mathcal{A}^{qnil}$. As $a_4b_2 = \lambda b_2a_4$, by Lemma 2.2, we show that $a_4 + b_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$, i.e., $(a_4 + b_2)^d = 0$.

Since

$$a + b = \begin{pmatrix} b_1 & a_2 \\ 0 & a_4 + b_2 \end{pmatrix}_p,$$

it follows by Lemma 2.1 that

$$(a+b)^d = \begin{pmatrix} b_1 & a_2 \\ 0 & a_4 + b_2 \end{pmatrix}^d = \begin{pmatrix} b_1^{-1} & z \\ 0 & 0 \end{pmatrix}_p,$$

where $z = (b^d)^2\Big(\sum\limits_{i=0}^{\infty}(b^d)^i a(a_4 + b_2)^i\Big)$. Since $abb^d = 0$, we derive

$$(a+b)^d = b^d + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n,$$

the result follows. □

Now we state one of our main results.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If*

$$ab = \lambda a^\pi bab^\pi,$$

then $a + b \in \mathcal{A}^d$ and

$$\begin{aligned}(a+b)^d &= b^\pi a^d + b^d a^\pi + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n a^\pi \\ &+ b^\pi \sum_{n=0}^{\infty}(a+b)^n b(a^d)^{n+2} \\ &- \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^d)^{k+1}a(a+b)^{n+k}b(a^d)^{n+2} \\ &- \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n ba^d.\end{aligned}$$

Proof. Let $p = aa^d$. Then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_{11} & b_{12} \\ b_1 & b_2 \end{pmatrix}_p.$$

Since $ab = \lambda a^{\pi}bab^{\pi}$, we get $aa^db = \lambda a^da^{\pi}bab^{\pi} = 0$; hence, $b_{11} = b_{12} = 0$. Thus,

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}_p,$$

So we get

$$a^d = \begin{pmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}_p, b^d = \begin{pmatrix} 0 & 0 \\ (b_2^d)^2b_1 & b_2^d \end{pmatrix}_p.$$

Hence,

$$a^{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p, b^{\pi} = \begin{pmatrix} p & 0 \\ -b_2^db_1 & b_2^{\pi} \end{pmatrix}_p.$$

Clearly, $a_2 = (1-p)a(1-p) = a - a^2a^d \in \mathcal{A}^{qnil}$. Since $(1-aa^d)b = b \in \mathcal{A}^d$, it follows by Cline's formula that $b_2 = a^{\pi}ba^{\pi} \in ((1-p)\mathcal{A}(1-p))^d$. As $ab = \lambda a^{\pi}bab^{\pi}$, we infer that

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ a_2b_1 & a_2b_2 \end{pmatrix} &= ab = \lambda a^{\pi}bab^{\pi} \\ &= \lambda \begin{pmatrix} 0 & 0 \\ b_1a_1 - b_2a_2b_2^db_1 & b_2a_2b_2^{\pi} \end{pmatrix}, \end{aligned}$$

and then

$$a_2b_2 = \lambda b_2a_2b_2^{\pi}.$$

In view of Lemma 2.3,

$$(a_2+b_2)^d = b_2^d + \sum_{n=0}^{\infty}(b_2^d)^{n+2}a_2(a_2+b_2)^n.$$

By virtue of Lemma 2.1, we have

$$(a+b)^d = \begin{pmatrix} a_1^{-1} & 0 \\ z & (a_2+b_2)^d \end{pmatrix} = \begin{pmatrix} a^d & 0 \\ z & (a_2+b_2)^d \end{pmatrix},$$

where

$$z = (a_2+b_2)^{\pi}\Big(\sum_{i=0}^{\infty}(a_2+b_2)^ib(a^d)^i\Big)(a^d)^2 - (a_2+b_2)^dba^d.$$

We easily see that $a_2b_2^d = (\lambda b_2a_2b_2^{\pi})(b_2^d)^2 = 0$; hence,

$$\begin{aligned} (a_2+b_2)^{\pi} &= (1-aa^d) - b_2b_2^d - \sum_{n=0}^{\infty}(b_2^d)^{n+1}a_2(a_2+b_2)^n \\ &= b_2^{\pi} - \sum_{n=0}^{\infty}(b_2^d)^{n+1}a_2(a_2+b_2)^n. \end{aligned}$$

Moreover, we have

$$\begin{aligned} z &= \sum_{i=0}^{\infty} b_2^{\pi}(a_2+b_2)^ib(a^d)^{i+2} \\ &- \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}(b_2^d)^{n+1}a_2(a_2+b_2)^{n+i}b(a^d)^{i+2} \\ &- b_2^dba^d - \sum_{n=0}^{\infty}(b_2^d)^{n+2}a_2(a_2+b_2)^nba^d. \end{aligned}$$

Clearly, $a^d b = ab^d = (a^d)^2(\lambda a^\pi bab^\pi) = (\lambda a^\pi bab^\pi)(b^d)^2 = 0$, one easily verifies that

$$\begin{pmatrix} a^d & 0 \\ -b_2^d b_1 a^d & 0 \end{pmatrix} = b^\pi a^d,$$
$$\begin{pmatrix} 0 & 0 \\ b_2^\pi (a_2+b_2)^i b(a^d)^{i+2} & 0 \end{pmatrix} = b^\pi (a+b)^i b(a^d)^{i+2},$$
$$\begin{pmatrix} 0 & 0 \\ (b_2^d)^{n+2} a_2(a_2+b_2)^n ba^d & 0 \end{pmatrix} = (b^d)^{n+2} a(a+b)^n ba^d,$$
$$\begin{pmatrix} 0 & 0 \\ (b_2^d)^{n+1} a_2(a_2+b_2)^{n+i} b(a^d)^{i+2} & 0 \end{pmatrix} = (b^d)^{n+1} a(a+b)^{n+i} b(a^d)^{i+2}.$$

Also we have

$$\begin{pmatrix} 0 & 0 \\ 0 & b_2^d \end{pmatrix} = b^d a^\pi,$$
$$\begin{pmatrix} 0 & 0 \\ 0 & (b_2^d)^{n+2} a_2(a_2+b_2)^n \end{pmatrix} = (b^d)^{n+2} a(a+b)^n a^\pi,$$

hence the result. □

Example 2.5. *Let $\mathcal{A} = M_3(\mathbb{C})$ and let*

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \in \mathcal{A}^d.$$

Then $ab = \frac{1}{2} a^\pi bab^\pi$, while $ab \neq a^\pi bab^\pi$.

Proof. It is clear that $a^3 = b^3 = 0$, then $a^d = b^d = 0$ which implies that $a^\pi = b^\pi = I_3$. Obviously,

$$ab = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{1}{2} a^\pi bab^\pi,$$

while $a^\pi bab^\pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \neq ab.$ □

3. Block operator matrices

In this section, we turn to study the g-Drazin inverse of the block matrix M by applying Theorem 2.4. We now derive

Theorem 3.1. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathcal{A})$, A and D have g-Drazin inverses. If $BD = \lambda(BC)^\pi ABD^\pi$ and $CA = \lambda(CB)^\pi DCA^\pi$, then $M \in M_2(\mathcal{A})^d$ and*

$$\begin{aligned} M^d = & \begin{pmatrix} A^d(BC)^\pi & A^\pi B(CB)^d \\ D^\pi C(BC)^d & D^d(CB)^\pi \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} 0 & (A^d)^{n+2}B \\ (D^d)^{n+2}C & 0 \end{pmatrix} \\ & M^n \begin{pmatrix} (BC)^\pi & 0 \\ 0 & (CB)^\pi \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} A^\pi & 0 \\ 0 & D^\pi \end{pmatrix} M^n \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \\ & \begin{pmatrix} 0 & B(CB)^d \\ C(BC)^d & 0 \end{pmatrix}^{n+2} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \begin{pmatrix} 0 & (A^d)^{k+1}B \\ (D^d)^{k+1}C & 0 \end{pmatrix} M^{n+k} \\ & \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & B(CB)^d \\ C(BC)^d & 0 \end{pmatrix}^{n+2} - \sum_{n=0}^{\infty} \begin{pmatrix} 0 & (A^d)^{n+2}B \\ (D^d)^{n+2}C & 0 \end{pmatrix} \\ & M^n \begin{pmatrix} 0 & AB(CB)^d \\ DC(BC)^d & 0 \end{pmatrix}. \end{aligned}$$

Proof. Clearly, we have $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Then

$$P^d = \begin{pmatrix} A^d & 0 \\ 0 & D^d \end{pmatrix}, P^\pi = \begin{pmatrix} A^\pi & 0 \\ 0 & D^\pi \end{pmatrix};$$
$$Q^2 = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix}, (Q^2)^d = \begin{pmatrix} (BC)^d & 0 \\ 0 & (CB)^d \end{pmatrix}.$$

By using Cline's formula, we get

$$Q^d = Q(Q^2)^d = \begin{pmatrix} 0 & B(CB)^d \\ C(BC)^d & 0 \end{pmatrix}.$$

Hence,

$$Q^\pi = \begin{pmatrix} (BC)^\pi & 0 \\ 0 & (CB)^\pi \end{pmatrix}.$$

Clearly,

$$PQ = \begin{pmatrix} 0 & AB \\ DC & 0 \end{pmatrix},$$
$$QP = \begin{pmatrix} 0 & BD \\ CA & 0 \end{pmatrix},$$

and so

$$Q^\pi PQP^\pi = \begin{pmatrix} 0 & (BC)^\pi ABD^\pi \\ (CB)^\pi DCA^\pi & D \end{pmatrix}.$$

By hypothesis, we have

$$QP = \lambda Q^\pi PQP^\pi.$$

According to Theorem 2.4, M has g-Drain inverse. The representation of M^d is easily obtained by Theorem 2.4. □

Corollary 3.2. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathcal{A})$, A and D have g-Drazin inverses. If $BD = \lambda ABD^\pi, CA = \lambda DCA^\pi$ and $BC = 0$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} A^d & 0 \\ 0 & D^d \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} 0 & (A^d)^{n+2}B \\ (D^d)^{n+2}C & 0 \end{pmatrix} M^n.$$

Proof. Since $BC = 0$, we have $(BC)^\pi = I = (CB)^\pi$. Moreover, $(BC)^d = 0, B(CB)^d = B(CB)((BC)^d)^2 = 0$. Construct Q as in the proof of Theorem 3.1, we have $Q^d = 0$, and so $Q^\pi = I$. This completes the proof by Theorem 3.1. □

In a similar way as it was done in Theorem 3.1, using the another splitting, we have

Theorem 3.3. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathcal{A})$, A and D have g-Drazin inverses. If $AB = \lambda A^\pi BD(CB)^\pi$ and*

$DC = \lambda D^{\pi}CA(BC)^{\pi}$, *then* $M \in M_2(\mathcal{A})^d$ *and*

$$M^d = \begin{pmatrix} (BC)^{\pi}A^d & B(CB)^dD^{\pi} \\ C(BC)^dA^{\pi} & (CB)^{\pi}D^d \end{pmatrix} + \sum_{n=0}^{\infty}\begin{pmatrix} 0 & B(CB)^d \\ C(BC)^d & 0 \end{pmatrix}^{n+2}$$
$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} M^n \begin{pmatrix} A^{\pi} & 0 \\ 0 & D^{\pi} \end{pmatrix} + \sum_{n=0}^{\infty}\begin{pmatrix} (BC)^{\pi} & 0 \\ 0 & (CB)^{\pi} \end{pmatrix} M^n$$
$$\begin{pmatrix} 0 & B(D^d)^{n+2} \\ C(A^d)^{n+2} & 0 \end{pmatrix} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix} 0 & B(CB)^d \\ C(BC)^d & 0 \end{pmatrix}^{k+1}$$
$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} M^{n+k} \begin{pmatrix} 0 & B(D^d)^{n+2} \\ C(A^d)^{n+2} & 0 \end{pmatrix}$$
$$- \sum_{n=0}^{\infty}\begin{pmatrix} 0 & B(CB)^d \\ C(BC)^d & 0 \end{pmatrix}^{n+2}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} M^n \begin{pmatrix} 0 & BD^d \\ CA^d & 0 \end{pmatrix}.$$

Proof. Construct P and Q as in the proof of Theorem 3.1, we have

$$PQ = \begin{pmatrix} 0 & AB \\ DC & 0 \end{pmatrix},$$
$$P^{\pi}QPQ^{\pi} = \begin{pmatrix} 0 & A^{\pi}BD(CB)^{\pi} \\ D^{\pi}CA(BC)^{\pi} & 0 \end{pmatrix}.$$

By hypothesis, we see that $PQ = \lambda P^{\pi}QPQ^{\pi}$. The theorem is therefore established by Theorem 2.4. □

As a consequence of the above, we now derive

Corollary 3.4. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathcal{A})$, *A and D have g-Drazin inverses. If* $AB = \lambda A^{\pi}BD, DC = \lambda D^{\pi}CA$ *and* $BC = 0$, *then* $M \in M_2(\mathcal{A})^d$ *and*

$$M^d = \begin{pmatrix} A^d & 0 \\ 0 & D^d \end{pmatrix} + \sum_{n=0}^{\infty} M^n \begin{pmatrix} 0 & B(D^d)^{n+2} \\ C(A^d)^{n+2} & 0 \end{pmatrix}.$$

4. Certain simpler expressions

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathcal{A})$. The aim of this section is to present certain simpler representations of the g-Drazin inverse of the block matrix M in the case $BC = 0$ or $CB = 0$. We now come to the main result of this section.

Theorem 4.1. *Let A and D have g-Drazin inverses. If* $BD = \lambda A^{\pi}AB, DC = \lambda^{-1}D^{\pi}CAA^{\pi}$ *and* $BC = 0$, *then* $M \in M_2(\mathcal{A})^d$ *and*

$$M^d = \begin{pmatrix} A^d & (A^d)^2B + \sum_{n=0}^{\infty} A^nB(D^d)^{n+2} \\ C(A^d)^2 & D^d + C(A^d)^3B + \sum_{n=0}^{\infty}\sum_{k=1}^{n} D^{k-1}CA^{n-k}B(D^d)^{n+2} \end{pmatrix}.$$

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} AA^{\pi} & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} A^2A^d & B \\ C & 0 \end{pmatrix}.$$

Then

$$P^d = \begin{pmatrix} 0 & 0 \\ 0 & D^d \end{pmatrix}, P^{\pi} = \begin{pmatrix} I & 0 \\ 0 & D^{\pi} \end{pmatrix}.$$

As $BC = 0$, we have

$$Q^d = \begin{pmatrix} A^d & (A^d)^2B \\ C(A^d)^2 & C(A^d)^3B \end{pmatrix}, Q^\pi = \begin{pmatrix} A^\pi & -A^dB \\ -CA^d & I - C(A^d)^2B \end{pmatrix}.$$

Since $BD = \lambda A^\pi AB$ and $BC = 0$, we get $BDC = (\lambda A^\pi AB)C = 0$. As $DC = \lambda^{-1}D^\pi CAA^\pi$, we have

$$PQ = \begin{pmatrix} 0 & A^\pi AB \\ DC & 0 \end{pmatrix} = \lambda^{-1}\begin{pmatrix} 0 & BD \\ D^\pi CAA^\pi & 0 \end{pmatrix} = \lambda^{-1}P^\pi QPQ^\pi.$$

In view of Theorem 2.4, we obtain

$$\begin{aligned} M^d &= Q^\pi P^d + Q^dP^\pi + \sum_{n=0}^{\infty}(Q^d)^{n+2}PM^nP^\pi \\ &+ Q^\pi \sum_{n=0}^{\infty} M^nQ(P^d)^{n+2} \\ &- \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(Q^d)^{k+1}PM^{n+k}Q(P^d)^{n+2} \\ &- \sum_{n=0}^{\infty}(Q^d)^{n+2}PM^nQP^d. \end{aligned}$$

Since $BD = \lambda A^\pi AB$, we infer that $A^dBD = A^d(\lambda A^\pi AB) = 0$, and then

$$Q^dP = \begin{pmatrix} A^d & (A^d)^2B \\ C(A^d)^2 & C(A^d)^3B \end{pmatrix}\begin{pmatrix} AA^\pi & 0 \\ 0 & D \end{pmatrix} = 0.$$

Therefore

$$M^d = P^d + Q^d + Q^\pi \sum_{n=0}^{\infty} M^nQ(P^d)^{n+2}.$$

Moreover, we have $BDC = (\lambda A^\pi AB)C = 0$. By induction, $BD^nC = 0$ for any $n \in \mathbb{N}$. Accordingly,

$$\begin{aligned} & Q^\pi \sum_{n=0}^{\infty} M^nQ(P^d)^{n+2} \\ =& \sum_{n=0}^{\infty}\begin{pmatrix} A^\pi & -A^dB \\ -CA^d & I - C(A^d)^2B \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n\begin{pmatrix} 0 & B(D^d)^{n+2} \\ 0 & 0 \end{pmatrix} \\ =& \sum_{n=0}^{\infty}\begin{pmatrix} A^\pi & -A^dB \\ -CA^d & I - C(A^d)^2B \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ & \begin{pmatrix} 0 & B(D^d)^{n+2} \\ 0 & 0 \end{pmatrix} \\ =& \sum_{n=0}^{\infty}\begin{pmatrix} A^\pi & -A^dB \\ -CA^d & I - C(A^d)^2B \end{pmatrix}\begin{pmatrix} A^n & 0 \\ \sum_{k=1}^{n} D^{k-1}CA^{n-k} & 0 \end{pmatrix} \\ & \begin{pmatrix} 0 & B(D^d)^{n+2} \\ 0 & 0 \end{pmatrix} \\ =& \begin{pmatrix} 0 & \sum_{n=1}^{\infty} A^nB(D^d)^{n+2} \\ 0 & \sum_{n=0}^{\infty}\sum_{k=1}^{n} D^{k-1}CA^{n-k}B(D^d)^{n+2} \end{pmatrix}, \end{aligned}$$

as desired. □

Corollary 4.2. *Let A and D have g-Drazin inverses. If $CA = \lambda D^\pi DC, AB = \lambda A^\pi BDD^\pi$ and $CB = 0$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} A^d + B(D^d)^3C + \sum_{n=0}^{\infty}\sum_{k=1}^{n} A^{k-1}BD^{n-k}C(A^d)^{n+2} & B(D^d)^2 \\ (D^d)^2C + \sum_{n=0}^{\infty} D^nC(A^d)^{n+2} & D^d \end{pmatrix}.$$

Proof. Obviously,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Applying Theorem 4.1 to $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$, we see that it has g-Drazin inverse and

$$\begin{aligned} & \begin{pmatrix} D & C \\ B & A \end{pmatrix}^d \\ = & \begin{pmatrix} D^d & (D^d)^2C + \sum\limits_{n=0}^{\infty} D^nC(A^d)^{n+2} \\ B(D^d)^2 & A^d + B(D^d)^3C + \sum\limits_{n=0}^{\infty}\sum\limits_{k=1}^{n} A^{k-1}BD^{n-k}C(A^d)^{n+2} \end{pmatrix}. \end{aligned}$$

Since $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, we derive

$$M^d = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix}^d \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

as desired. □

Now we are ready to prove the other main theorem in this section.

Theorem 4.3. *Let A and D have g-Drazin inverses. If $AB = \lambda A^{\pi}BD, DC = \lambda D^{\pi}CA$ and $BC = 0$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} A^d & 0 \\ 0 & D^d \end{pmatrix} + \sum_{n=0}^{\infty} M^n \begin{pmatrix} 0 & B(D^d)^{n+2} \\ C(A^d)^{n+2} & 0 \end{pmatrix}.$$

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Then

$$P^d = \begin{pmatrix} A^d & 0 \\ 0 & D^d \end{pmatrix}, P^{\pi} = \begin{pmatrix} A^{\pi} & 0 \\ 0 & D^{\pi} \end{pmatrix}.$$

As $BC = 0$, we see that $Q^3 = 0$, and so $Q^d = 0, Q^{\pi} = I$. One easily checks that

$$PQ = \begin{pmatrix} 0 & AB \\ DC & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & A^{\pi}BD \\ D^{\pi}CA & 0 \end{pmatrix} = \lambda P^{\pi}QPQ^{\pi}.$$

Since $Q^d = 0$, it follows by Theorem 2.4 that

$$M^d = P^d + \sum_{n=0}^{\infty} M^nQ(P^d)^{n+2}.$$

Moreover, we have

$$\sum_{n=0}^{\infty} M^nQ(P^d)^{n+2} = \sum_{n=1}^{\infty} M^n \begin{pmatrix} 0 & B(D^d)^{n+2} \\ C(A^d)^{n+2} & 0 \end{pmatrix},$$

as required. □

The following example illustrates that Theorem 4.3 is a nontrivial generalization of [11, Theorem 10].

Example 4.4. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_8(\mathbb{C})$, *where*

$$A = D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_4(\mathbb{C}).$$

Then

$$AB = 3A^{\pi}BD, DC = 3D^{\pi}CA \textit{ and } BC = 0.$$

Proof. Since A, B, C, D are nilpotent, we have $A^{\pi} = B^{\pi} = C^{\pi} = D^{\pi} = I_4$. It is clear by computing that

$$AB = 3A^{\pi}BD, DC = 3D^{\pi}CA \text{ and } BC = 0.$$

In this case, $AB \neq A^{\pi}BD$. □

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