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Iterative Contraction at a Point via Wardowski Function

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Abstract. In this paper, we investigate the existence of a fixed point for iterative contraction mappings via Wardowski function in the setting of complete b-metric spaces. Our results improve and extend several results in this direction in the literature. We consider illustrative examples to support our main results.

1. Introduction and Preliminaries

It is an indispensable fact that fixed point theory becomes the fundament of both (nonlinear) functional analysis and topology, in this century. The first formal fixed point theorem was announced by Banach in 1922. His result has naturally researchers attention, as the mentioned theorem not only proves the existence and uniqueness of a fixed point, but also indicates how to reach this point. More precisely, he proved that under the contraction condition, each Picard iteration, for an arbitrary initial point, converges and this limit is the unique fixed point of the given mapping.

The first observation concerned the fact that the contraction condition necessarily implies the continuity of the given mapping. Naturally, researches was wondered whether such a fixed point theorem would be proved for discontinuous mapping. One of the interesting responses to this question was given by Bryant [18]. Roughly speaking, suppose T is discontinuous but T^2 provides the contraction condition, that is, T^2 is continuous and has a unique fixed point. Bryant [18] proved that the fixed point of T^2 forms a fixed point for T, too. Later, Sehgal [37] proved that if for a given initial value x, there is a natural number $n(x)$ such that mapping $T^{n(x)}$ forms a contraction, then it has a fixed point. This trend was followed by Guseman [22], Matkowski [30], Iseki [23], Ray and Rhoades,[36] Jachymski [24].

In this paper, we shall follow the trend of Sehgal [37] by taking the Wardowski function into account. For the sake of completeness, we recollect some basic results and definitions.

First we state the results of Sehgal [37].

Theorem 1.1. *([37]) Let $(\mathcal{X}, d)$ be a complete metric space and $T : \mathcal{X} \leftrightarrow \mathcal{X}$ a continuous mapping. If there exists a real number k, $0 < k < 1$ such that, for each $v \in \mathcal{M}$ there is a positive integer $m(\varsigma)$ such that, for each $z \in \mathcal{X}$,*

$$d(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) \leq k \cdot d(\varsigma, z) \tag{1}$$

then T has a unique fixed point in $\mathcal{X}$.

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Sehgal [37] was improved by Guseman [22] by excluding the continuity condition. After that, it was generalized and extended by many authors, among them being, Iseki [23], J. Matkowski [30], Singh [38], Ray and Rhoades [36]. On the other hand, the result of Kincses and Totik [28] is one of the most interesting results in this direction. Therefore, we collect it here:

Theorem 1.2. *([28]) Let a metric space $(\mathcal{X}, d)$ and T be a self mapping on $\mathcal{X}$ such that for some $k \in [0,1)$ and for all $\varsigma, z \in \mathcal{X}$ we can find a positive integer $m(\varsigma)$ such that*

$$d(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) \leq k \max\left\{d(\varsigma, z), d(z, T^{m(\varsigma)}\varsigma), d(\varsigma, T^{m(\varsigma)}z)\right\}. \tag{2}$$

Then T has a unique fixed point $\varsigma*$.

In what follows, we recall some auxiliary functions than are used in our consideration, namely the Wardowski type function [40] and the simulation function [27]. Let $\mathfrak{F}$ denote the set of all strictly increasing functions $F : (0,\infty) \to \mathbb{R}$ which satisfy the following conditions

(f_a) for every sequence of positive real numbers $\{\eta_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} \eta_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\eta_n) = -\infty;$$

(f_b) there exists $p \in (0,1)$ such that $\lim_{\alpha\to 0^+} (\alpha^p F(t)) = 0$.

Theorem 1.3. *[40] A mapping $T : \mathcal{X} \to \mathcal{X}$ on a complete metric space $(\mathcal{X}, d)$ has a unique fixed point $\varsigma*$ provided that there exist $\tau > 0$ and $F \in \mathfrak{F}$ such that*

$$d(T\varsigma, Tz) > 0 \Rightarrow \tau + F(d(T\varsigma, Tz)) \leq F(d(\varsigma, z)), \tag{3}$$

for all $\varsigma, z \in \mathcal{X}$. Moreover, for all $\varsigma \in \mathcal{X}$ the sequence $\{T^n\varsigma\}$ is convergent to $\varsigma*$.

This result has been continuously improved, either by weakening the conditions imposed on function F, or by changing the argument or by considering more general metric spaces (we refer here to b-metric [2, 4, 5, 10, 14–16, 21, 29, 31], etc.).

Theorem 1.4. *[41] Let $T : \mathcal{X} \to \mathcal{X}$ be a mapping on a complete metric space $(\mathcal{X}, d)$. Suppose that there exist $\tau > 0$ and $F \in \mathfrak{F}$ such that*

$$\begin{array}{c} d(T\varsigma, Tz) > 0 \Rightarrow \\ \tau + F(d(T\varsigma, Tz)) \leq F(\max\left\{d(\varsigma, z), d(\varsigma, T\varsigma), d(z, Tz), \frac{d(\varsigma, Tz)+d(z, T\varsigma)}{2}\right\}, \end{array} \tag{4}$$

for all $\varsigma, z \in \mathcal{X}$. Then, T has a unique fixed point $\varsigma*$ provided that T or F is continuous.

Inspired from the Wardowski function, we shall use the following set of auxiliary function:

$$\mathcal{G} = \{G : (0,\infty) \to \mathbb{R} \quad | \quad G \text{ is strictly increasing and satisfies } (f_a)\}.$$

Let $\mathcal{Z}$ be the family of all functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following axioms:

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(a,b) < b - a$ for all $a, b > 0$;

(ζ_3) if $\{a_n\}, \{b_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n > 0$, then

$$\limsup_{n\to\infty} \zeta(a_n, b_n) < 0. \tag{5}$$

Such functions are called simulation functions [27]. For more on simulation function, see e.g. [7],[8], [11], [3], [12], [9], [35], [13]. In our future study we will consider the set

$$\mathfrak{S}' = \{\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R} : \zeta \text{ satisfies } (\zeta_2)\}.$$

Definition 1.5. *[19] Let $\mathcal{X}$ be a nonempty set. A mapping $\mathfrak{b} : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ is a b-metric if there exists $s \geq 1$, such that*

($\mathfrak{b}_1$) $\mathfrak{b}(\varsigma, z) = 0$ *if and only if* $\varsigma = z$;

($\mathfrak{b}_2$) $\mathfrak{b}(\varsigma, z) = \mathfrak{b}(z, \varsigma)$;

($\mathfrak{b}_3$) $\mathfrak{b}(\varsigma, z) \leq s[\mathfrak{b}(\varsigma, y) + \mathfrak{b}(y, z)]$,

for all $\varsigma, y, z \in \mathcal{X}$. The triplet $(\mathcal{X}, \mathfrak{b}, s)$ is called a b-metric space.

In what follows we look over some of the notions and results in b-metric space.

Definition 1.6. *[17] Let $(\mathcal{X}, \mathfrak{b}, s)$ be a b-metric space. The sequence $\{\varsigma_n\}$ is:*

(c) *convergent if and only if there exists $\varsigma \in \mathcal{X}$ such that $\mathfrak{b}(\varsigma_n, \varsigma) \to 0$ as $n \to \infty$.*

(C) *Cauchy if and only if $\mathfrak{b}(\varsigma_n, \varsigma_p) \to 0$ as $p, n \to \infty$.*

(cC) *The b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ is complete provided that every Cauchy sequence from $(\mathcal{X}, \mathfrak{b}, s)$ is convergent. To indicate that the considered metric space is complete we will use the notation $(\mathcal{X}^*, \mathfrak{b}, s)$.*

Let $(\mathcal{X}, \mathfrak{b}, s)$ be a b-metric space and a mapping $T : \mathcal{X} \to \mathcal{X}$.

1. The mapping T is continuous at a point $\varsigma \in \mathcal{X}$, if $T(\varsigma_n) \to T(\varsigma)$ for every sequence $\{\varsigma_n\}$ in $\mathcal{X}$ such that $\varsigma_n \to \varsigma$. Moreover, if T is continuous at each point $\varsigma \in \mathcal{X}$, then T is said to be continuous on $\mathcal{X}$.
2. Let $\chi_0 \in \mathcal{X}$. The set $O(\chi_0) = \{T^n \chi_0 : n = 0, 1, 2, ...\}$ is called *an orbit* of $\chi_0 \in \mathcal{X}$ and denote by $\rho(\chi_0) = \sup\{\mathfrak{b}(\varsigma, z) : \varsigma, z \in O(\chi_0)\}$ the diameter of the set $O(\chi_0)$.
   If for some $\chi \in \mathcal{X}$, every Cauchy sequence from $O(\chi)$ converges in $\mathcal{X}$ we say that the space is orbitally complete and we will use the notation $(\mathcal{X}^{O^*}, \mathfrak{b}, s)$ for such spaces.

2. Main results

Theorem 2.1. *Let T be a self-mapping on the space $(\mathcal{X}^{O^*}, \mathfrak{b}, s)$, a function $\zeta \in \mathfrak{S}'$ and $a_1, a_2, a_3 \in [0,1)$ such that $a_1 + a_2 + 2sa_3 < \frac{1}{s}$. If for every $\varsigma \in \mathcal{X}$, there is a positive integer $m = m(\varsigma)$ such that*

$$\zeta(\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z), \mathcal{L}(\varsigma, z)) \geq 0, \tag{6}$$

where

$$\mathcal{L}(\varsigma, z) = a_1 \mathfrak{b}(\varsigma, z) + a_2 \mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + a_3 \mathfrak{b}(z, T^{m(\varsigma)}z)$$

holds for every $\varsigma, z \in \mathcal{X}$, then T has a fixed point $\varsigma^* \in \mathcal{X}$.

Proof. First of all, we remark that since $\zeta \in \mathfrak{S}'$, we have

$$0 \leq \zeta(\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z), \mathcal{L}(\varsigma, z)) < \mathcal{L}(\varsigma, z) - \mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z)$$

and the inequality (13) yields

$$\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) < \mathcal{L}(\varsigma, z), \tag{7}$$

for any $\varsigma, z \in \mathcal{X}$.
For an easier reading, we will structure the demonstration in a few steps.

Step 1. First of all, we shall show that the orbit is bounded for every $\varsigma \in \mathcal{X}$.
Let ς be an arbitrary point in $\mathcal{X}$ and $r(\varsigma) = \sup\left\{\mathfrak{b}(\varsigma, T^l\varsigma) : l = 1, 2, ..., m(\varsigma)\right\}$. For a positive integer m there exists $l \geq 1$ such that $lm(\varsigma) < m \leq (l+1)m(\varsigma)$. Therefore, for $1 \leq p \leq m(\varsigma)$ we can write $m = lm(\varsigma) + p$. By $(\mathfrak{b}_3)$, we have

$$\begin{aligned}\mathfrak{b}(\varsigma, T^m\varsigma) &\leq s \cdot [\mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + \mathfrak{b}(T^{m(\varsigma)}\varsigma, T^m\varsigma)] \\ &= s \cdot [\mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + s \cdot \mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}(T^{m-m(\varsigma)}\varsigma))].\end{aligned} \tag{8}$$

Replacing z in (7) with $T^{m-m(\varsigma)}\varsigma$ we get

$$\begin{aligned}\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}(T^{m-m(\varsigma)}\varsigma)) &= \mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}(T^{(l-1)m(\varsigma)+p}\varsigma) < \mathcal{L}(\varsigma, T^{(l-1)m(\varsigma)+p}\varsigma) \\ &= a_1\mathfrak{b}(\varsigma, T^{(l-1)m(\varsigma)+p}\varsigma) + a_2\mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + a_3\mathfrak{b}(T^{(l-1)m(\varsigma)+p}\varsigma, T^{lm(\varsigma)+p}\varsigma) \\ &< a_1\mathfrak{b}(\varsigma, T^{(l-1)m(\varsigma)+p}\varsigma) + a_2\mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + \\ &\quad + sa_3\left[\mathfrak{b}(T^{(l-1)m(\varsigma)+p}\varsigma, \varsigma) + \mathfrak{b}(\varsigma, T^{lm(\varsigma)+p}\varsigma)\right] \\ &= (a_1 + sa_3)\mathfrak{b}(\varsigma, T^{(l-1)m(\varsigma)+p}\varsigma) + a_2\mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + sa_3\mathfrak{b}(\varsigma, T^m\varsigma)\end{aligned}$$

Returning in (8) we get,

$$\mathfrak{b}(\varsigma, T^m\varsigma) = s \cdot \mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + s(a_1 + sa_3)\mathfrak{b}(\varsigma, T^{(l-1)m(\varsigma)+p}\varsigma) + sa_2\mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + s^2a_3\mathfrak{b}(\varsigma, T^m\varsigma)$$

and hence

$$\begin{aligned}\mathfrak{b}(\varsigma, T^m\varsigma) &< \tfrac{s(1+a_2)}{1-s^2a_3}\mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma) + \tfrac{s(a_1+sa_3)}{1-s^2a_3}\mathfrak{b}(\varsigma, T^{(l-1)m(\varsigma)+p}\varsigma) \\ &\leq \tfrac{s(1+a_2)}{1-s^2a_3}r(\varsigma) + \tfrac{s(a_1+sa_3)}{1-s^2a_3}\mathfrak{b}(\varsigma, T^{(l-1)m(\varsigma)+p}\varsigma).\end{aligned}$$

Let denote by $c = a_1 + a_2 + 2a_3s$ and $\alpha = sc < 1$ (since by hypothesis's $c < \frac{1}{s}$). Thus,

$$\frac{s(1+a_2)}{1-s^2a_3} < \frac{s+1}{1-\alpha} \text{ and } \frac{s(a_1+sa_3)}{1-s^2a_3} \leq \alpha$$

and

$$\begin{aligned}\mathfrak{b}(\varsigma, T^m\varsigma) &< \tfrac{s+1}{1-\alpha}r(\varsigma) + \alpha\mathfrak{b}(\varsigma, T^{(l-1)m(\varsigma)+p}\varsigma) \\ &< \tfrac{s+1}{1-\alpha}r(\varsigma) + \alpha[\tfrac{s+1}{1-\alpha}r(\varsigma) + \alpha\mathfrak{b}(\varsigma, T^{(l-2)m(\varsigma)+p}\varsigma)] \\ &= \tfrac{s+1}{1-\alpha}r(\varsigma) + \alpha\tfrac{2}{1-\alpha}r(\varsigma) + \alpha^2\mathfrak{b}(\varsigma, T^{(l-2)m(\varsigma)+p}\varsigma) \\ &\quad ... \\ &\leq \tfrac{s+1}{1-\alpha}(1 + \alpha + \alpha^2 + ... + \alpha^{l-1})r(\varsigma) + \alpha^l\mathfrak{b}(\varsigma, T^p\varsigma) \\ &< \tfrac{s+1}{(1-\alpha)^2}r(\varsigma) + \alpha^l\mathfrak{b}(\varsigma, T^p\varsigma),\end{aligned}$$

whence, the orbit $O(\varsigma) = \{T^n\varsigma : n = 0, 1, 2, ...\}$ is a bounded subset of $\mathcal{X}$.

Step 2. We claim that the sequence $\{\varsigma_n\}$ is Cauchy.
Let ς_0 be a fixed point in $\mathcal{X}$ and $\{\varsigma_n\}_n$ be the sequence defined as follows

$$\varsigma_1 = T^{m_0}\varsigma_0, \varsigma_2 = T^{m_1}\varsigma_1 = T^{m_1+m_0}\varsigma_0, ..., \varsigma_n = T^{m_{n-1}}\varsigma_{n-1} = T^{m_{n-1}+m_{n-2}+...+m_0}\varsigma_0.$$

where $m_i = m(\varsigma_i)$. Of course, the sequence $\{\varsigma_n\}_n \subset \mathcal{O}(\varsigma_0)$.
Let now, ς_k be a fixed term of the sequence $\{\varsigma_n\}_n$. Then, there exists $v = v(\varsigma_0)$ such that $\varsigma_q = T^v\varsigma_k$ is a successor terms of ς_k. By Step 1 and by (7) we have

$$\begin{aligned}\mathfrak{b}(\varsigma_k,\varsigma_q) &= \mathfrak{b}(T^{m_{k-1}}\varsigma_{k-1},T^{v}\varsigma_k) = \mathfrak{b}(T^{m_{k-1}}\varsigma_{k-1},T^{m_{k-1}}(T^{v-m_{k-1}}\varsigma_{k-1})) < \mathcal{L}(\varsigma_{k-1},T^{v-m_{k-1}}\varsigma_{k-1})\\ &< a_1\cdot\mathfrak{b}(\varsigma_{k-1},T^{v-m_{k-1}}\varsigma_{k-1}) + a_2\cdot\mathfrak{b}(\varsigma_{k-1},T^{m_{k-1}}\varsigma_{k-1}) + a_3\cdot\mathfrak{b}(T^{v-m_{k-1}}\varsigma_{k-1},T^{v}\varsigma_{k-1})\\ &\le a_1\cdot\mathfrak{b}(\varsigma_{k-1},T^{v-m_{k-1}}\varsigma_{k-1}) + a_2\cdot\mathfrak{b}(\varsigma_{k-1},T^{m_{k-1}}\varsigma_{k-1})+\\ &\quad +a_3 s\cdot[\mathfrak{b}(T^{v-m_{k-1}}\varsigma_{k-1},\varsigma_{k-1}) + \mathfrak{b}(\varsigma_{k-1},T^{v}\varsigma_{k-1})].\end{aligned}$$

Denoting by $v_0 \in \{m_{k-1}, v, v-m_{k-1}\}$ such that

$$\mathfrak{b}(\varsigma_{k-1},T^{v_0}\varsigma_{k-1}) = \max\{\mathfrak{b}(\varsigma_{k-1},T^{m_{k-1}}\varsigma_{k-1}),\mathfrak{b}(\varsigma_{k-1},T^{v}\varsigma_{k-1}),\mathfrak{b}(\varsigma_{k-1},T^{v-m_{k-1}}\varsigma_{k-1})\},$$

from the above inequality we have

$$\mathfrak{b}(\varsigma_k,\varsigma_q) < (a_1+a_2+2a_3 s)\cdot\mathfrak{b}(\varsigma_{k-1},T^{v_0}\varsigma_{k-1}) = c\cdot\mathfrak{b}(\varsigma_{k-1},T^{v_0}\varsigma_{k-1}). \tag{9}$$

In the same way, if we consider $v_1 \in \{m_{k-2}, v_0, v_0-m_{k-2}\}$ such that

$$\mathfrak{b}(\varsigma_{k-2},T^{v_1}\varsigma_{k-2}) = \max\left\{\mathfrak{b}(\varsigma_{k-2},T^{m_{k-2}}\varsigma_{k-2}),\mathfrak{b}(\varsigma_{k-2},T^{v}_0\varsigma_{k-2}),\mathfrak{b}(\varsigma_{k-2},T^{v_0-m_{k-2}}\varsigma_{k-2})\right\},$$

we have

$$\begin{aligned}\mathfrak{b}(\varsigma_{k-1},T^{v_0}\varsigma_{k-1}) &= \mathfrak{b}(T^{m_{k-2}}\varsigma_{k-2},T^{m_{k-2}}(T^{v_0-m_{k-2}}\varsigma_{k-2})) < \mathcal{L}(\varsigma_{k-2},T^{v_0-m_{k-2}}\varsigma_{k-2})\\ &< a_1\cdot\mathfrak{b}(\varsigma_{k-2},T^{v_0-m_{k-2}}\varsigma_{k-2}) + a_2\cdot\mathfrak{b}(\varsigma_{k-2},T^{m_{k-2}}\varsigma_{k-2}) + a_3\cdot\mathfrak{b}(T^{v_0-m_{k-2}}\varsigma_{k-2},T^{v_0}\varsigma_{k-2})\\ &\le a_1\cdot\mathfrak{b}(\varsigma_{k-2},T^{v_0-m_{k-2}}\varsigma_{k-2}) + a_2\cdot\mathfrak{b}(\varsigma_{k-2},T^{m_{k-2}}\varsigma_{k-2})+\\ &\quad +a_3 s\cdot[\mathfrak{b}(T^{v_0-m_{k-2}}\varsigma_{k-2},\varsigma_{k-2}) + \mathfrak{b}(\varsigma_{k-2},T^{v_0}\varsigma_{k-2})]\\ &\le (a_1+a_2+2a_3 s)\cdot\mathfrak{b}(\varsigma_{k-1},T^{v_1}\varsigma_{k-1})\\ &= c\cdot\mathfrak{b}(\varsigma_{k-2},T^{v_1}\varsigma_{k-2}).\end{aligned}$$

Continuing in the same way, we can find v_{k+1} such that

$$\mathfrak{b}(\varsigma_{k-1},T^{v_0}\varsigma_{k-1}) < c\cdot\mathfrak{b}(\varsigma_{k-2},T^{v_1}\varsigma_{k-2}) < c^2\cdot\mathfrak{b}(\varsigma_{k-3},T^{v_2}\varsigma_{k-3}) < \ldots < c^{k-1}\cdot\mathfrak{b}(\varsigma_0,T^{v_{k-1}}\varsigma_0)$$

and replacing in (9),

$$\mathfrak{b}(\varsigma_k,\varsigma_q) < c^k\cdot\mathfrak{b}(\varsigma_0,T^{v_{k-1}}\varsigma_0) < c^k\rho(\varsigma_0).$$

Let now, $w = w(\varsigma_0)$ such that $\varsigma_p = T^{w(\varsigma_0)}$ is an another successor term of ς_k. We have,

$$\mathfrak{b}(\varsigma_p,\varsigma_q) \le s[\mathfrak{b}(\varsigma_p,\varsigma_k) + \mathfrak{b}(\varsigma_k,\varsigma_q)] \le 2sc^k\rho(\varsigma_0) \to 0 \text{ as } k\to\infty,$$

so that the sequence $\{\varsigma_n\}_n \subset \mathcal{O}(\varsigma_0)$ is Cauchy on the space $(\mathcal{X}^{\mathcal{O}^*},\mathfrak{b},s)$. Thus, there exists $\varsigma^* \in \mathcal{X}$ such that $\lim_{n\to\infty}\varsigma_n = \varsigma^*$.

Step 3. We now claim that ς^* is the unique fixed point of T.
Firstly, we prove that $\lim_{n\to\infty}\mathfrak{b}(T^{m(\varsigma^*)}\varsigma_n,\varsigma_n) = 0$.
By letting $\varsigma = T^{m(\varsigma^*)}\varsigma_n$ and $z = \varsigma_n$ in (7), we have

$$\begin{aligned}\mathfrak{b}(\varsigma_n,T^{m(\varsigma^*)}\varsigma_n) &= \mathfrak{b}(T^{m_{n-1}}\varsigma_{n-1},T^{m_{n-1}}(T^{m(\varsigma^*)-m_{n-1}}\varsigma_{n-1})) < \mathcal{L}(\varsigma_{n-1},T^{m(\varsigma^*)-m_{n-1}}\varsigma_{n-1})\\ &= a_1\mathfrak{b}(\varsigma_{n-1},T^{m(\varsigma^*)-m_{n-1}}\varsigma_{n-1}) + a_2\mathfrak{b}(\varsigma_{n-1},T^{m_{n-1}}\varsigma_{n-1}) + a_3\mathfrak{b}(T^{m(\varsigma^*)-m_{n-1}}\varsigma_{n-1},T^{m(\varsigma^*)}\varsigma_{n-1})\\ &\le a_1\mathfrak{b}(\varsigma_{n-1},T^{m(\varsigma^*)-m_{n-1}}\varsigma_{n-1}) + a_2\mathfrak{b}(\varsigma_{n-1},T^{m_{n-1}}\varsigma_{n-1})+\\ &\quad +a_3 s\cdot[\mathfrak{b}(T^{m(\varsigma^*)-m_{n-1}}\varsigma_{n-1},\varsigma_{n-1}) + \mathfrak{b}(\varsigma_{n-1},T^{m(\varsigma^*)-m_{n-1}}\varsigma_{n-1})].\end{aligned}$$

Letting $s_0 \in \{m(\varsigma^*), m_{n-1}, m(\varsigma^*) - m_{n-1}\}$ such that

$$\mathfrak{b}(\varsigma_{n-1}, T^{s_0}\varsigma_{n-1}) = \max\left\{\mathfrak{b}(\varsigma_{n-1}, T^{m(\varsigma^*)-m_{n-1}}\varsigma_{n-1}), \mathfrak{b}(\varsigma_{n-1}, T^{m(\varsigma^*)}\varsigma_{n-1}), \mathfrak{b}(\varsigma_{n-1}, T^{m_{n-1}}\varsigma_{n-1})\right\}$$

in the above inequality we have

$$\mathfrak{b}(\varsigma_n, T^{m(\varsigma^*)}\varsigma_n) < (a_1 + a_2 + 2a_3 s)\mathfrak{b}(\varsigma_{n-1}, T^{s_0}\varsigma_{n-1}) = c\mathfrak{b}(\varsigma_{n-1}, T^{s_0}\varsigma_{n-1}).$$

Repeating this process, we find that

$$\mathfrak{b}(\varsigma_n, T^{m(\varsigma^*)}\varsigma_n) < c\mathfrak{b}(\varsigma_{n-1}, T^{s_0}\varsigma_{n-1}) < ... < c^n\mathfrak{b}(\varsigma_0, T^{s_n}\varsigma_0) < c^n\rho(\varsigma_0) \to 0.$$

Accordingly,

$$\lim_{n\infty} \mathfrak{b}(\varsigma_n, T^{m(\varsigma^*)}\varsigma_n) = 0. \tag{10}$$

Secondly: $T^{m(\varsigma^*)}\varsigma^* = \varsigma^*$
Supposing that $T^{m(\varsigma^*)}\varsigma^* \neq \varsigma^*$, by $(\mathfrak{b}_3)$ and (7) we have

$$\begin{aligned}0 &< \mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*) \le s\mathfrak{b}(\varsigma^*, \varsigma_n) + s^2\mathfrak{b}(\varsigma_n, T^{m(\varsigma^*)}\varsigma_n) + s^2\mathfrak{b}(T^{m(\varsigma^*)}\varsigma_n, T^{m(\varsigma^*)}\varsigma^*)\\ &< s\mathfrak{b}(\varsigma^*, \varsigma_n) + s^2\mathfrak{b}(\varsigma_n, T^{m(\varsigma^*)}\varsigma_n) + s^2[a_1\mathfrak{b}(\varsigma_n, \varsigma^*) + a_2\mathfrak{b}(\varsigma_n, T^{m(\varsigma^*)}\varsigma_n) + a_3\mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*)]\end{aligned}$$

and letting $n \to \infty$ we get

$$0 < \mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)\varsigma^*}) < sa_3\mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*) < \mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*),$$

which is a contradiction. Therefore, $T^{m(\varsigma^*)}\varsigma^* = \varsigma^*$.
If we suppose that there exists another point z^* in $\mathcal{X}$ such that $T^{m(\varsigma^*)}z^* = z^* \neq \varsigma^*$, then we have

$$\begin{aligned}0 &< \mathfrak{b}(\varsigma^*, z^*) = \mathfrak{b}(T^{m(\varsigma^*)}\varsigma^*, T^{m(\varsigma^*)}z^*) < \mathcal{L}(\varsigma^*, z^*)\\ &= a_1\mathfrak{b}(\varsigma^*, z^*) + a_2\mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*) + a_3\mathfrak{b}(z^*, T^{m(\varsigma^*)}z^*) = a_1\mathfrak{b}(\varsigma^*, z^*) < \mathfrak{b}(\varsigma^*, z^*).\end{aligned}$$

This is a contradiction. Hence the fixed point of $T^{m(\varsigma^*)}$ is unique and from here we can conclude that ς^* is a fixed point of T.
Indeed, we have

$$T\varsigma^* = T(T^{m(\varsigma^*)}\varsigma^*) = T^{m(\varsigma^*)}(T\varsigma^*)$$

which shows that $T\varsigma^*$ is a fixed point of $T^{m(\varsigma^*)}$. Due to uniqueness, $T\varsigma^* = \varsigma^*$.
□

Example 2.2. *Let the set* $\mathcal{X} = [0, 2]$ *endowed with the* 2*-metric* $\mathfrak{b}(\varsigma, z) = |\varsigma - z|^2$. *Let the mapping* $T : \mathcal{X} \to \mathcal{X}$ *be defined as*

$$T\varsigma = \begin{cases} \frac{\varsigma}{2}, & \text{for } \varsigma \in [0, \frac{1}{2}]\\ 0, & \text{for } \varsigma \in (\frac{1}{2}, 1]\\ 1, & \text{for } \varsigma \in (1, 2]\end{cases}$$

For $\varsigma = \frac{5}{6}$ *and* $z = \frac{7}{6}$, *we have*

$$\mathfrak{b}(T\frac{5}{6}, T\frac{7}{6}) = |0 - 1|^2 = 1 > k \cdot \frac{4}{36}, \text{ for any } k \in [0, 1)$$

and

$$\mathfrak{b}(T\frac{5}{6},T\frac{7}{6}) = |0-1|^2 = 1 > a_1\frac{4}{36} + a_2\frac{25}{36} + a_3\frac{1}{36} = a_1\mathfrak{b}(\frac{5}{6},\frac{7}{6}) + a_2\mathfrak{b}(\frac{5}{6},T\frac{5}{6}) + a_3\mathfrak{b}(\frac{7}{6},T\frac{7}{6})$$

for any $a_1, a_2, a_3 \in [0,1)$ *such that* $a_1 + a_2 + 2sa_3 < \frac{1}{s}$.

On the other hand, $T^n(\varsigma) = \begin{cases} \frac{\varsigma}{2^n}, & \text{for } \varsigma \in [0,\frac{1}{2}] \\ 0, & \text{for } \varsigma \in (\frac{1}{2},2] \end{cases}$ *and for example, taking* $a_1 = a_2 = a_3 = \frac{1}{20}$ *and* $m = m(\varsigma) = 3$, *we have:*

1. *for* $\varsigma, z \in [0,\frac{1}{2}]$,

$$\mathfrak{b}(T\varsigma,Tz) = \mathfrak{b}(\frac{\varsigma}{2^3},\frac{z}{2^3}) = \frac{|\varsigma-z|^2}{64} < \frac{|\varsigma-z|^2}{20} = \frac{1}{20}\mathfrak{b}(\varsigma,z) < \mathcal{L}(\varsigma,z);$$

2. *for* $\varsigma, z \in (\frac{1}{2},2]$,

$$\mathfrak{b}(T\varsigma,Tz) = 0 < \mathcal{L}(\varsigma,z);$$

3. *for* $\varsigma \in [0,\frac{1}{2}]$ *and* $z \in (\frac{1}{2},2]$,

$$\begin{aligned} \mathfrak{b}(T\varsigma,Tz) &= \mathfrak{b}(\tfrac{\varsigma}{2^3},0) = \tfrac{\varsigma^2}{64} < \tfrac{1}{20}\left(|\varsigma-z|^2 + \tfrac{49}{64}\varsigma^2 + z^2\right) \\ &< \tfrac{1}{20}\left(2\mathfrak{b}(\varsigma,z) + \mathfrak{b}(\varsigma,\tfrac{\varsigma}{8}) + \mathfrak{b}(z,0)\right) \end{aligned}$$

In conclusion, for any function $\zeta \in \mathfrak{S}'$ *all the hypothesis of Theorem 2.1 hold, thus* T *has a fixed point.*

Corollary 2.3. *Let* T *be a self-mapping on the space* $(\mathcal{X}^{\mathcal{O}^*}, \mathfrak{b}, s)$ *and* $\ell_1, \ell_2, \ell_3 \in [0,1)$ *such that* $\ell_1 + \ell_2 + 2s\ell_3 < \frac{1}{s}$. *If for each* $\varsigma \in \mathcal{X}$, *there is a positive integer* $m = m(\varsigma)$ *such that*

$$\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) \le \ell_1\mathfrak{b}(\varsigma,z) + \ell_2\mathfrak{b}(\varsigma,T^{m(\varsigma)}\varsigma) + \ell_3\mathfrak{b}(z,T^{m(\varsigma)}z) \tag{11}$$

for every $z \in \mathcal{X}$ *then* T *has a fixed point* $\varsigma^* \in \mathcal{X}$.

Proof. Letting $\zeta(a,b) = kb - a$, with $k \in (0,1)$ and $\ell_i = k \cdot a_i$, the proof follows by Theorem 2.1. □

Of course, by choosing s = 1 we have similar results in metric spaces.

Corollary 2.4. *Let* T *be a self-mapping on an orbitally complete metric space* $(\mathcal{X}, d)$, *a function* $\zeta \in \mathfrak{S}'$ *and* $a_1, a_2, a_3 \in [0,1)$ *such that* $a_1 + a_2 + 2a_3 < 1$. *If for every* $\varsigma \in \mathcal{X}$, *there is a positive integer* $m = m(\varsigma)$ *such that*

$$\zeta(d(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z), \mathcal{L}(\varsigma,z)) \ge 0, \tag{12}$$

where

$$\mathcal{L}(\varsigma,z) = a_1 d(\varsigma,z) + a_2 d(\varsigma,T^{m(\varsigma)}\varsigma) + a_3 d(z,T^{m(\varsigma)}z)$$

holds for every $z \in \mathcal{X}$, *then* T *has a fixed point* $\varsigma^* \in \mathcal{X}$.

Corollary 2.5. *Let* $(\mathcal{X}, d)$ *be an orbitally complete metric space and* $\ell_1, \ell_2, \ell_3 \in [0,1)$ *such that* $\ell_1 + \ell_2 + 2\ell_3 < 1$. *If for every* $\varsigma \in \mathcal{X}$, *there is a positive integer* $m = m(\varsigma)$ *such that*

$$d(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) \le \ell_1 d(\varsigma,z) + \ell_2 d(\varsigma,T^{m(\varsigma)}\varsigma) + \ell_3 d(z,T^{m(\varsigma)}z) \tag{13}$$

then T *has a fixed point* $\varsigma^* \in \mathcal{X}$.

In the next Theorem, we investigate the existence and the uniqueness of a fixed point for iterative contraction mappings via Wardowski function.

Theorem 2.6. *Let T be a self-mapping on the space $(\mathcal{X}^{O^*}, \mathfrak{b}, s)$ and assume that there exists $\varsigma_0 \in \mathcal{X}$ such that the orbit $O(\varsigma_0) = \{T^n\varsigma_0 : n = 0, 1, 2, ...\}$ is a bounded subset of $\mathcal{X}$. Let the functions $\zeta \in \mathfrak{S}'$, $G \in \mathcal{G}$ and suppose that for every $\varsigma \in \mathcal{X}$, there is a positive integer $m = m(\varsigma)$ such that*

$$\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) > 0 \Rightarrow \zeta(\tau + G(\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z)), G(\mathcal{R}(\varsigma, z))) \geq 0, \tag{14}$$

where

$$\mathcal{R}(\varsigma, z) = \max\left\{\mathfrak{b}(\varsigma, z), \mathfrak{b}(\varsigma, T^{m(\varsigma)}\varsigma), \mathfrak{b}(\varsigma, T^{m(\varsigma)}z)\right\}$$

holds for every $z \in \mathcal{X}$. Then there exists $\varsigma^* \in \mathcal{X}$ such that $\lim_{n\to\infty} T^n\varsigma_0 = \varsigma^*$.
If ONE of the following holds

- (A) *T is continuous,*
- (B) *the orbit $O(\varsigma^*)$ is bounded,*
- (C) *G and $\mathfrak{b}$ are continuous*

then $\varsigma^* \in \mathcal{X}$ is the unique fixed point of T.

Proof. As in a previous demonstration, since $\zeta \in \mathfrak{S}'$, we have

$$\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) > 0 \Rightarrow \zeta(\tau + G(\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z)), G(\mathcal{R}(\varsigma, z))) < G(\mathcal{R}(\varsigma, z)) - [\tau + G(\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z))]$$

and the inequality (14) yields

$$\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) > 0 \Rightarrow \tau + G(\mathfrak{b}(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z)) < G(\mathcal{R}(\varsigma, z)), \tag{15}$$

for any $\varsigma, z \in \mathcal{X}$.

Let $\sigma \in \mathcal{X}$. We prove that if there is $\beta(\sigma)$ such that $T^{\alpha(\sigma)+\beta(\sigma)}\sigma = T^{\alpha(\sigma)}\sigma$ for every $\alpha(\sigma)$ then $T^{\beta(\sigma)}$ is a fixed point of $T^{\alpha(\sigma)}$.
Indeed, because

$$T^{\alpha(\sigma)}(T^{\beta(\sigma)}\sigma) = T^{\alpha(\sigma)+\beta(\sigma)}\sigma = T^{\beta(\sigma)}\sigma,$$

we get that $\sigma^* = T^{\beta(\sigma)}$ is a fixed point of $T^{\alpha(\sigma)}$. If there is another point, $\sigma' \in \mathcal{X}$, such that $T^{\alpha(\sigma)}\sigma^* = \sigma^* \neq \sigma' = T^{\alpha(\sigma)}\sigma'$, because $\mathfrak{b}(T^{\alpha(\sigma)}\sigma^*, T^{\alpha(\sigma)}\sigma') > 0$ we have from (7)

$$\begin{aligned}
G(\mathfrak{b}(\sigma^*, \sigma')) &= G(\mathfrak{b}(T^{\alpha(\sigma)}\sigma^*, T^{\alpha(\sigma)}\sigma')) < G(\mathcal{R}(\sigma^*, \sigma') - \tau \\
&= G(\max\left\{\mathfrak{b}(\sigma^*, \sigma'), \mathfrak{b}(\sigma^*, T^{\alpha(\sigma)}\sigma^*), \mathfrak{b}(\sigma^*, T^{\alpha(\sigma)}\sigma')\right)\right\} - \tau \\
&= G(\mathfrak{b}(\sigma^*, \sigma'), \mathfrak{b}(\sigma, \sigma), \mathfrak{b}(\sigma^*, \sigma'))) - \tau \\
&= G(\mathfrak{b}(\sigma^*, \sigma')) - \tau < G(\mathfrak{b}(\sigma^*, \sigma')),
\end{aligned}$$

which is a contradiction. Thus, σ^* is the unique fixed point of $T^{\alpha(\sigma)}$. On the other hand, we have

$$T\sigma^* = T(T^{\alpha(\sigma)}\sigma^*) = T^{\alpha(\sigma)}(T\sigma^*)$$

which shows that $T\sigma^*$ is also a fixed point of $T^{\alpha(\sigma)}$. Therefore, taking into account the uniqueness of the fixed point, we have $T\sigma^* = \sigma^*$.

Let ς_0 be a point in $\mathcal{X}$ such that the orbit $O(\varsigma_0) = \{T^n\varsigma_0 : n = 0, 1, 2, ...\}$ is a bounded subset of $\mathcal{X}$; we have that $\rho(\varsigma_0) < \infty$. On the other hand, from the above considerations, we have that $\mathfrak{b}(T^q\varsigma_0, T^p\varsigma_0) > 0$ for every $p, q \in \mathbb{N} \cup \{0\}$ or, equivalent $\rho(\varsigma_0) > 0$.
We build the sequence $\{\varsigma_n\}_n$ defined as follows

$$\varsigma_1 = T^{m_0}\varsigma_0, \varsigma_2 = T^{m_1}\varsigma_1 = T^{m_1+m_0}\varsigma_0, ..., \varsigma_n = T^{m_{n-1}}\varsigma_{n-1} = T^{m_{n-1}+m_{n-2}+...+m_0}\varsigma_0.$$

where $m_i = m(\varsigma_i)$. Of course, it is easy to see that the sequence $\{\varsigma_n\}_n \subset O(\varsigma_0)$.
We claim that the sequence $\{\varsigma_n\}$ is Cauchy.
Let ς_k a fixed term of the sequence $\{\varsigma_n\}_n$. Then, there exists $v = v(\varsigma_0)$ such that $\varsigma_q = T^v\varsigma_k$ is a successor terms of ς_k. Again, as proved above, $\mathfrak{b}(T^{m_{k-1}}\varsigma_{k-1}, T^{m_{k-1}}(T^{v-m_{k-1}}\varsigma_{k-1})) > 0$ and by (15) we have

$$\begin{aligned} G(\mathfrak{b}(\varsigma_k, \varsigma_q) &= G(\mathfrak{b}(T^{m_{k-1}}\varsigma_{k-1}, T^u\varsigma_k) = G(\mathfrak{b}(T^{m_{k-1}}\varsigma_{k-1}, T^{m_{k-1}}(T^{v-m_{k-1}}\varsigma_{k-1}))) \\ &< G(\mathcal{R}(\varsigma_{k-1}, T^{v-m_{k-1}}\varsigma_{k-1})) - \tau \\ &< G(\max\{\mathfrak{b}(\varsigma_{k-1}, T^{v-m_{k-1}}\varsigma_{k-1}), \mathfrak{b}(\varsigma_{k-1}, T^{m_{k-1}}\varsigma_{k-1}), \mathfrak{b}(\varsigma_{k-1}, T^v\varsigma_{k-1})\}) - \tau. \end{aligned}$$

Denoting by $v_0 \in \{m_{k-1}, v, v - m_{k-1}\}$ such that

$$\mathfrak{b}(\varsigma_{k-1}, T^{v_0}\varsigma_{k-1}) = \max\{\mathfrak{b}(\varsigma_{k-1}, T^{m_{k-1}}\varsigma_{k-1}), \mathfrak{b}(\varsigma_{k-1}, T^v\varsigma_{k-1}), \mathfrak{b}(\varsigma_{k-1}, T^{v-m_{k-1}}\varsigma_{k-1})\},$$

from the above inequality we have

$$G(\mathfrak{b}(\varsigma_k, \varsigma_q)) < G(\mathfrak{b}(\varsigma_{k-1}, T^{v_0}\varsigma_{k-1})) - \tau. \tag{16}$$

In the same way, if we consider $v_1 \in \{m_{k-2}, v_0, v_0 - m_{k-2}\}$ such that

$$\mathfrak{b}(\varsigma_{k-2}, T^{v_1}\varsigma_{k-2}) = \max\left\{\mathfrak{b}(\varsigma_{k-2}, T^{m_{k-2}}\varsigma_{k-2}), \mathfrak{b}(\varsigma_{k-2}, T_0^v\varsigma_{k-2}), \mathfrak{b}(\varsigma_{k-2}, T^{v_0-m_{k-2}}\varsigma_{k-2})\right\},$$

we have

$$\begin{aligned} G(\mathfrak{b}(\varsigma_{k-1}, T^{v_1}\varsigma_{k-1})) &= G(\mathfrak{b}(T^{m_{k-2}}\varsigma_{k-2}, T^{m_{k-2}}(T^{v_0-m_{k-2}}\varsigma_{k-2}))) \\ &< G(\mathcal{L}(\varsigma_{k-2}, T^{v_0-m_{k-2}}\varsigma_{k-2})) - \tau \\ &< G(\max\{\mathfrak{b}(\varsigma_{k-2}, T^{v_0-m_{k-2}}\varsigma_{k-2}), \mathfrak{b}(\varsigma_{k-2}, T^{m_{k-2}}\varsigma_{k-2}), \mathfrak{b}(\varsigma_{k-2}, T^{v_0}\varsigma_{k-2})\}) \\ &\leq G(\mathfrak{b}(\varsigma_{k-2}, T^{v_1}\varsigma_{k-2})) - \tau \end{aligned}$$

Continuing, we can find v_{k-1} such that

$$G(\mathfrak{b}(\varsigma_{k-1}, T^{v_0}\varsigma_{k-1})) \quad < G(\mathfrak{b}(\varsigma_{k-2}, T^{v_1}\varsigma_{k-2})) - \tau < ... < G(\mathfrak{b}(\varsigma_0, T^{v_{k-1}}\varsigma_0)) - (k-1)\tau < G(\rho(\varsigma_0)) - (k-1)\tau$$

and replacing in (16),

$$\begin{aligned} G(\mathfrak{b}(\varsigma_k, \varsigma_q)) &< G(\mathfrak{b}(\varsigma_{k-1}, T^{v_0}\varsigma_{k-1})) - \tau \\ &< G(\rho(\varsigma_0)) - k\tau. \end{aligned}$$

Taking in the inequality above $k \to \infty$ we get $\lim_{n\to\infty} G(\mathfrak{b}(\varsigma_k, \varsigma_q)) = -\infty$ and then from (f_a), we have

$$\lim_{n\to\infty} \mathfrak{b}(\varsigma_k, \varsigma_q) = 0. \tag{17}$$

Let now, $w = w(\varsigma_0)$ such that $x_p = T^{w(\varsigma_0)}\varsigma_0$ is another successor term of ς_k. We have,

$$\mathfrak{b}(\varsigma_p, \varsigma_q) \leq s[\mathfrak{b}(\varsigma_p, \varsigma_k) + \mathfrak{b}(\varsigma_k, \varsigma_q)] \to 0 \text{ as } k \to \infty,$$

so that the sequence $\{\varsigma_n\}_n \subset O(\varsigma_0)$ is Cauchy on the space $(\mathcal{X}^{O^*}, \mathfrak{b}, s)$. Thus, there exists $\varsigma^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \varsigma_n = \varsigma^*. \tag{18}$$

We shall show that $T^{m(\varsigma^*)}\varsigma^* = \varsigma^*$, under the assumptions (A) or (B) or (C).

(A) If T is continuous, then $T^{m(\varsigma^*)}$ is continuous and

$$T^{m(\varsigma^*)}\varsigma^* = T^{m(\varsigma^*)}(\lim_{n\to\infty} T^q \varsigma_0) = \lim_{n\to\infty} T^{m(\varsigma^*)+q}\varsigma_0 = \varsigma^*.$$

Hence, ς^* is a fixed point of $T^{m(\varsigma^*)}$.

(B) By $(\mathfrak{b}_3)$, we have

$$\mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*) \leq s\left[\mathfrak{b}(\varsigma^*, \varsigma_k) + \mathfrak{b}(\varsigma_k, T^{m(\varsigma^*)}\varsigma^*)\right] \tag{19}$$

Supposing that $\mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*) > 0$, we have

$$\mathfrak{b}(\varsigma_k, T^{m(\varsigma^*)}\varsigma^*) = \mathfrak{b}(T^{m_{k-1}}\varsigma_{k-1}, T^{m_{k-1}}(T^{m(\varsigma^*)-m_{k-1}}\varsigma^*)) > 0,$$

and by (15),

$$\begin{aligned} G(\mathfrak{b}(\varsigma_k, T^{m(\varsigma^*)}\varsigma^*)) &= G(\mathfrak{b}(T^{m_{k-1}}\varsigma_{k-1}, T^{m_{k-1}}(T^{m(\varsigma^*)-m_{k-1}}\varsigma^*)) < G(\mathcal{R}(\varsigma_{k-1}, T^{m(\varsigma^*)-m_{k-1}}\varsigma^*)) - \tau \\ &= G(\max\left\{\mathfrak{b}(\varsigma_{k-1}, T^{m(\varsigma^*)-m_{k-1}}\varsigma^*), \mathfrak{b}(\varsigma_{k-1}, T^{m_{k-1}}\varsigma_{k-1}), \mathfrak{b}(\varsigma_{k-1}, T^{m(\varsigma^*)}\varsigma^*)\right\}) - \tau \\ &\leq G(\mathfrak{b}(\varsigma_{k-1}, T^{\nu_0}\varsigma^*)) - \tau, \end{aligned}$$

where $\nu_0 \in \{m(\varsigma^*) - m_{k-1}, m(\varsigma^*), m_{k-1}\}$ is choosing such that

$$\mathfrak{b}(\varsigma_{k-1}, T^{\nu_0}\varsigma^*) = \max\left\{\mathfrak{b}(\varsigma_{k-1}, T^{m(\varsigma^*)-m_{k-1}}\varsigma^*), \mathfrak{b}(\varsigma_{k-1}, T^{m_{k-1}}\varsigma_{k-1}), \mathfrak{b}(\varsigma_{k-1}, T^{m(\varsigma^*)}\varsigma^*)\right\}).$$

Thus, continuing in this way, we have

$$G(\mathfrak{b}(\varsigma_k, T^{m(\varsigma^*)}\varsigma^*)) = G(\mathfrak{b}(\varsigma_{k-1}, T^{\nu_0}\varsigma^*)) - \tau < ... < G(\mathfrak{b}(\varsigma_0, T^{\nu_{k-1}}\varsigma^*)) - k\tau,$$

for all $k \in \mathbb{N}$.
On the other hand, we remark that, under the assumption (B), there is a constant $\mathcal{k} > 0$ such that $\mathfrak{b}(\varsigma^*, T^{n_{k-1}}\varsigma^*) < \frac{\mathcal{k}}{2s}$ and $\mathfrak{b}(\varsigma_0, \varsigma^*) < \frac{\mathcal{k}}{2s}$. Thus,

$$\mathfrak{b}(\varsigma_0, T^{\nu_{k-1}}\varsigma^*) \leq s[\mathfrak{b}(\varsigma_0, \varsigma^*) + \mathfrak{b}(\varsigma^*, T^{n_{k-1}}\varsigma^*)] < \mathcal{k}. \tag{20}$$

Taking the limit as $k \to \infty$ and keeping in mind (20), we get $\lim_{n\to\infty} G(\mathfrak{b}(\varsigma_k, T^{m(\varsigma^*)}\varsigma^*)) = -\infty$, which together with (f_a), gives

$$\lim_{n\to\infty} \mathfrak{b}(\varsigma_k, T^{m(\varsigma^*)}\varsigma^*) = 0. \tag{21}$$

$$\mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*) \leq s[\mathfrak{b}(\varsigma^*, \varsigma_k) + \mathfrak{b}(\varsigma_k, T^{m(\varsigma^*)}\varsigma^*)]$$

and taking into account (18) and (21) we have $\mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*) = 0$.

(C) If G and $\mathfrak{b}$ are continuous, we have

$$G(\mathfrak{b}(T^{m(\varsigma^*)}\varsigma^*, T^{m(\varsigma^*)}(T^n\varsigma_0))) < G\left(\max\left\{\mathfrak{b}(\varsigma^*, T^n\varsigma_0), \mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*), \mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}(T^n\varsigma_0))\right\}\right) - \tau.$$

Letting $n \to \infty$ and taking into account (18)

$$G(\mathfrak{b}(T^{m(\varsigma^*)}\varsigma^*, \varsigma^*)) < G(\mathfrak{b}(T^{m(\varsigma^*)}\varsigma^*, \varsigma^*)) - \tau < G(\mathfrak{b}(T^{m(\varsigma^*)}\varsigma^*, \varsigma^*)),$$

which is a contradiction. Therefore, $G(\mathfrak{b}(T^{m(\varsigma^*)}\varsigma^*, \varsigma^*)) = 0$, that is $T^{m(\varsigma^*)}\varsigma^* = \varsigma^*$.
Let ς^*, z^* be two distinct fixed points of $T^{m(\varsigma^*)}$. Then, $\mathfrak{b}(T^{m(\varsigma^*)}\varsigma^*, T^{m(\varsigma^*)}z^*) = \mathfrak{b}(\varsigma^*, z^*) > 0$ implies that

$$\begin{aligned} G(\mathfrak{b}(T^{m(\varsigma^*)}\varsigma^*, T^{m(\varsigma^*)}z^*)) &< G\left(\max\left\{\mathfrak{b}(\varsigma^*, z^*), \mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}\varsigma^*), \mathfrak{b}(\varsigma^*, T^{m(\varsigma^*)}z^*)\right\}\right) - \tau \\ &< G(\max\{\mathfrak{b}(\varsigma^*, z^*), \mathfrak{b}(\varsigma^*, \varsigma^*), \mathfrak{b}(\varsigma^*, z^*)\}) - \tau \\ &< G(\mathfrak{b}(\varsigma^*, z^*)) - \tau \\ &< G(\mathfrak{b}(\varsigma^*, z^*)). \end{aligned}$$

This is a contradiction.

Hence, $T^{m(\varsigma^*)}$ has a unique fixed point, and follow the lines from Theorem 2.1 we conclude that $T\varsigma^* = \varsigma^*$. □

Example 2.7. *Let the set $\mathcal{X} = [0, \infty)$ endowed with the b-metric $\mathfrak{b} = |\varsigma - z|^2$, for every $\varsigma, z \in \mathcal{X}$. Let the map $T : \mathcal{X} \to \mathcal{X}$ be given by* $T\varsigma = \begin{cases} \frac{\varsigma}{3}, & \text{for } \varsigma \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ 1, & \text{for } \varsigma = \frac{1}{2} \end{cases}$.
Since

$$\mathfrak{b}(T0, T\tfrac{1}{2}) = \mathfrak{b}(0, 1) = 1, \quad \mathfrak{b}(0, \tfrac{1}{2}) = \tfrac{1}{4},$$
$$\mathfrak{b}(0, T0) = 0, \quad \mathfrak{b}(\tfrac{1}{2}, T\tfrac{1}{2}) = \mathfrak{b}(\tfrac{1}{2}, 1) = \tfrac{1}{4},$$
$$\mathfrak{b}(0, T\tfrac{1}{2}) = \mathfrak{b}(0, 1) = 1, \quad \mathfrak{b}(\tfrac{1}{2}, T0) = \tfrac{1}{4}$$

we get that T satisfies neither the assumption of Theorem 1.3, nor that of Theorem 1.4.
On the other hand, since $T^2\varsigma = \begin{cases} \frac{\varsigma}{9}, & \text{for } \varsigma \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ \frac{1}{3}, & \text{for } \varsigma = \frac{1}{2} \end{cases}$ *we have*

$$\mathfrak{b}(T^2\varsigma, T^2\tfrac{1}{2}) = \mathfrak{b}(\tfrac{\varsigma}{9}, \tfrac{1}{3}) = \left|\tfrac{\varsigma-3}{9}\right|^2, \quad \mathfrak{b}(\varsigma, \tfrac{1}{2}) = \left|\varsigma - \tfrac{1}{2}\right|^2,$$
$$\mathfrak{b}(\varsigma, T^2\varsigma) = (\tfrac{8\varsigma}{9})^2, \quad \mathfrak{b}(\varsigma, T^2\tfrac{1}{2}) = \mathfrak{b}(\varsigma, \tfrac{1}{3}) = \left|\varsigma - \tfrac{1}{3}\right|^2.$$

and choosing $G(t) = lnt$, $\tau = 2ln\frac{21}{20}$ the inequality (15) becomes

$$\left(\frac{21}{20}\right)^2 \mathfrak{b}(T^2\varsigma, T^2\frac{1}{2}) < \mathcal{R}(\varsigma, \frac{1}{2}),$$

where

$$\mathcal{R}(\varsigma, \tfrac{1}{2}) = \max\left\{\mathfrak{b}(\varsigma, \tfrac{1}{2}), \mathfrak{b}(\varsigma, T^2\varsigma), \mathfrak{b}(\varsigma, T^2\tfrac{1}{2})\right\} = \max\left\{\left|\varsigma - \tfrac{1}{2}\right|^2, (\tfrac{8\varsigma}{9})^2, \left|\varsigma - \tfrac{1}{3}\right|^2\right\}.$$

If $\varsigma > \frac{1}{3}$, we have

$$\left(\frac{21}{20} \cdot \frac{3-\varsigma}{9}\right)^2 < \left(\frac{8\varsigma}{9}\right)^2 \leq \mathcal{R}(\varsigma, \frac{1}{2})$$

and if $\varsigma \leq \frac{1}{3}$ we get

$$\left(\frac{21}{20} \cdot \frac{3-\varsigma}{9}\right)^2 < \left(\varsigma - \frac{1}{2}\right)^2 \leq \mathcal{R}(\varsigma, \frac{1}{2}).$$

Therefore, because G and $\mathfrak{b}$ are continuous, by Theorem 2.6, it follows that T has a fixed point.

Corollary 2.8. *Let T be a self-mapping on the complete orbitally metric space $(\mathcal{X}^{O^*}, d)$ and assume that there exists $\varsigma_0 \in \mathcal{X}$ such that the orbit $O(\varsigma_0) = \{T^n\varsigma_0 : n = 0, 1, 2, ...\}$ is a bounded subset of $\mathcal{X}$. Let the functions $\zeta \in \mathfrak{S}'$, $G \in \mathcal{G}$ and suppose that for every $\varsigma \in \mathcal{X}$, there is a positive integer $m = m(\varsigma)$ such that*

$$d(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z) > 0 \Rightarrow \zeta(\tau + G(d(T^{m(\varsigma)}\varsigma, T^{m(\varsigma)}z)), G(\mathcal{R}(\varsigma, z))) \geq 0, \tag{22}$$

where

$$\mathcal{R}(\varsigma, z) = \max\left\{d(\varsigma, z), d(\varsigma, T^{m(\varsigma)}\varsigma), d(\varsigma, T^{m(\varsigma)}z)\right\}$$

holds for every $z \in \mathcal{X}$. If either T or G is continuous, then T has a unique fixed point of $\varsigma^* \in \mathcal{X}$. Moreover, $\lim_{n\to\infty} T^n\varsigma_0 = \varsigma^*$.

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