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Existence Criteria of Positive Solutions for Fractional p-Laplacian Boundary Value Problems

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Abstract. By means of the Bai-Ge's fixed point theorem, this paper shows the existence of positive solutions for nonlinear fractional p-Laplacian differential equations. Here, the fractional derivative is the standard Riemann-Liouville one. Finally, an example is given to illustrate the importance of results obtained.

1. Introduction

We are concerned with the multiple positive solutions of the following fractional differential equation with integral boundary conditions:

$$\begin{cases} D_{0^+}^{\vartheta_1}(\phi_p(D_{0^+}^{\vartheta_2}u(t))) = f(t,u(t),D_{0^+}^{\vartheta_2}u(t)), \quad t \in (0,1), \\ D_{0^+}^{\vartheta_2}u(0) = 0, \quad D_{0^+}^{\vartheta_1-2}(\phi_p(D_{0^+}^{\vartheta_2}u(0))) = D_{0^+}^{\vartheta_1-1}(\phi_p(D_{0^+}^{\vartheta_2}u(1))) = \int_0^1 g(s)\phi_p(D_{0^+}^{\vartheta_2}u(s))ds, \\ u(0) = 0, \quad D_{0^+}^{\vartheta_2-2}u(0) = D_{0^+}^{\vartheta_2-1}u(1) = \int_0^1 h(s)u(s)ds, \end{cases} \tag{1}$$

where $2 < \vartheta_1, \vartheta_2 \leq 3$, $5 < \vartheta_1 + \vartheta_2 \leq 6$, $D^{(.)}$ denotes the Riemann-Liouville fractional derivative of order (.), $f \in C([0,1] \times [0,\infty) \times (-\infty,+\infty), [0,\infty))$ and $g, h \in C([0,1],[0,\infty))$ with $\int_0^1 \frac{s^{\vartheta_1-1}+s^{\vartheta_1-2}(\vartheta_1-1)}{\Gamma(\vartheta_1)} g(s)ds < 1$, $\int_0^1 \frac{s^{\vartheta_2-1}+s^{\vartheta_2-2}(\vartheta_2-1)}{\Gamma(\vartheta_2)} h(s)ds < 1$, $\phi_p(s)$ is the p-Laplacian operator, *i.e.*, $\phi_p(s) = |s|^{p-2}s,\ p > 1, (\phi_p)^{-1} = \phi_q,\ \frac{1}{p} + \frac{1}{q} = 1$.

In recent years, the subject of fractional calculus and fractional differential equations has obtained a considerable popularity and importance, mostly by virtue of their demonstrated applications in widespread fields of science and engineering. For the related applications and details about fractional calculus and fractional differential equation, see [1, 2, 4–16] and the references therein. In [7], the authors studied the following fractional boundary value problem

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$$\begin{cases} D_{0^+}^{\alpha}(\phi_p({}^cD_{0^+}^{\beta}u(t))) + f(t,u(t),{}^cD_{0^+}^{\beta}u(t)), \quad t \in (0,1), \\ {}^cD_{0^+}^{\beta}u(0) = u'(0) = 0, \quad u(0) = \int_0^1 g_0(s)u(s)ds, \\ D_{0^+}^{\alpha-1}(\phi_p({}^cD_{0^+}^{\beta}u(1))) = \int_0^1 g_1(s)\phi_p({}^cD_{0^+}^{\beta}u(s))ds, \end{cases}$$

where $\alpha, \beta \in (1,2]$, $D_{0^+}^{\alpha}$ is the Riemann Liouville fractional derivative operator, ${}^cD_{0^+}^{\beta}$ is the Caputo fractional derivative operator, ϕ_p is the p-Laplacian operator, $f \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, $g_0, g_1 \in C([0,1]\times[0,+\infty))$. They obtained the existence of at least three positive solutions by using the generalization of the Leggett-Williams fixed point theorem.

In [1], Ahmad, Ntouyas and Alsaedi considered the following nonlinear fractional differential equations of order $q \in (1,2]$ with three-point integral boundary condition given by

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(0) = 0, \quad x(1) = a\int_0^{\eta} x(s)ds, \quad 0 < \eta < 1, \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1]\times X \to X$ is continuous, and $a \in \mathbb{R}$ is such that $a \neq 2/\eta^2$. Some new existence and uniqueness results are obtained by using the standard fixed point theorems and Leray-Schauder degree theory.

Inspired by the works mentioned above, we establish the multiplicity results of positive solutions for the boundary value problem (1). Here, our nonlinear function f is independent of fractional derivative of unknown function $u(t)$. This makes the problem more difficult and complicated. Only a few papers cover fractional differential equations with fractional orders $2 < \vartheta_1, \vartheta_2 \leq 3$. Also, due to the singularity of the Riemann-Liouville fractional derivative, it is very difficult to determine the initial value.

2. Preliminaries

We first wish to collect the background knowledge of fractional calculus (see [5, 10]).

Definition 2.1. The Riemann Liouville fractional integral of order $\vartheta \in \mathbb{R}^+$ of a function $g : (0,\infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\vartheta}g(t) = \frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1}g(s)ds,$$

whenever the right-hand side is defined. Similarly, with $\vartheta > 0$ and $\vartheta \in \mathbb{R}$, we define the Riemann-Liouville fractional derivative to be

$$D_{0^+}^{\vartheta}g(t) = \Big(\frac{d}{dt}\Big)^n I_{0^+}^{n-\vartheta}g(t) = \frac{1}{\Gamma(n-\vartheta)}\Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\vartheta-1}g(s)ds,$$

where n is the smallest integer greater than or equal to ϑ, whenever the right-hand side is defined. In particular, for $\vartheta = n$, $D_{0^+}^n g(t) = g^{(n)}(t)$.

Lemma 2.2. *The general solution to $D_{0^+}^{\vartheta}g = 0$ is the function $g(t) = c_1t^{\vartheta-1} + c_2t^{\vartheta-2} + \cdots + c_nt^{\vartheta-n}$, $c_i \in \mathbb{R}$, where $n-1 < \vartheta \leq n$ and $\vartheta > 0$, $i = 1,\ldots,n$.*

It is easy to see that the boundary value problem (1) can be decomposed into the following coupled boundary value problems:

$$\begin{cases} D_{0^+}^{\vartheta_1} v(t) = f(t, u(t), D_{0^+}^{\vartheta_2} u(t)), \quad t \in (0,1), \\ v(0) = 0, \\ D_{0^+}^{\vartheta_1 - 2} v(0) = D_{0^+}^{\vartheta_1 - 1} v(1) = \int_0^1 g(s) v(s) ds \end{cases} \tag{2}$$

and

$$\begin{cases} D_{0^+}^{\vartheta_2} u(t) = \phi_q(v(t)), \quad t \in (0,1), \\ u(0) = 0, \\ D_{0^+}^{\vartheta_2 - 2} u(0) = D_{0^+}^{\vartheta_2 - 1} u(1) = \int_0^1 h(s) u(s) ds. \end{cases} \tag{3}$$

Lemma 2.3. *If $k \in C([0,1])$, then*

$$\begin{cases} D_{0^+}^{\vartheta_1} v(t) = k(t), \quad t \in (0,1), \\ v(0) = 0, \\ D_{0^+}^{\vartheta_1 - 2} v(0) = D_{0^+}^{\vartheta_1 - 1} v(1) = \int_0^1 g(s) v(s) ds \end{cases} \tag{4}$$

has a unique solution v and v can be expressed in the form

$$v(t) = -\int_0^1 H_1(t,s) k(s) ds,$$

where

$$H_1(t,s) = G_1(t,s) + \frac{t^{\vartheta_1 - 1} + t^{\vartheta_1 - 2} (\vartheta_1 - 1)}{\Delta_1 \Gamma(\vartheta_1)} \int_0^1 G_1(t,s) g(t) dt, \tag{5}$$

$$G_1(t,s) = \frac{1}{\Gamma(\vartheta_1)} \begin{cases} t^{\vartheta_1 - 1}, & 0 \le t \le s \le 1, \\ t^{\vartheta_1 - 1} - (t-s)^{\vartheta_1 - 1}, & 0 \le s \le t \le 1, \end{cases} \tag{6}$$

and $\Delta_1 = 1 - \int_0^1 \frac{s^{\vartheta_1 - 1} + s^{\vartheta_1 - 2} (\vartheta_1 - 1)}{\Gamma(\vartheta_1)} g(s) ds.$

Proof. Let v verify (4). Then the general solution of (4) is given by

$$v(t) = \frac{1}{\Gamma(\vartheta_1)} \int_0^t (t-s)^{\vartheta_1 - 1} k(s) ds + c_1 t^{\vartheta_1 - 1} + c_2 t^{\vartheta_1 - 2} + c_3 t^{\vartheta_1 - 3}. \tag{7}$$

The boundary condition $v(0) = 0$ implies that $c_3 = 0$. By the boundary conditions $D_{0^+}^{\vartheta_1 - 2} v(0) = \int_0^1 g(s) v(s) ds$ and $D_{0^+}^{\vartheta_1 - 1} v(1) = \int_0^1 g(s) v(s) ds$, we have

$$c_2 = \frac{1}{\Gamma(\vartheta_1 - 1)} \int_0^1 g(s) v(s) ds$$

and

$$c_1 = -\frac{1}{\Gamma(\vartheta_1)}\int_0^1 k(s)ds + \frac{1}{\Gamma(\vartheta_1)}\int_0^1 g(s)v(s)ds.$$

Substituting c_1 and c_2 into (7), we obtain

$$\begin{aligned} v(t) &= \frac{1}{\Gamma(\vartheta_1)}\int_0^t (t-s)^{\vartheta_1-1}k(s)ds - \frac{t^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\left[\int_0^1 k(s)ds - \int_0^1 g(s)v(s)ds\right] + \frac{t^{\vartheta_1-2}}{\Gamma(\vartheta_1-1)}\int_0^1 g(s)v(s)ds \\ &= -\int_0^1 G_1(t,s)k(s)ds + \frac{t^{\vartheta_1-1}+t^{\vartheta_1-2}(\vartheta_1-1)}{\Gamma(\vartheta_1)}\int_0^1 g(s)v(s)ds \end{aligned} \tag{8}$$

where

$$\int_0^1 g(s)v(s)ds = \int_0^1 g(s)\Big[-\int_0^1 G_1(s,\tau)k(\tau)d\tau + \frac{s^{\vartheta_1-1}+s^{\vartheta_1-2}(\vartheta_1-1)}{\Gamma(\vartheta_1)}\int_0^1 g(\tau)v(\tau)d\tau\Big]ds.$$

So,

$$\int_0^1 g(s)v(s)ds = \frac{1}{1-\int_0^1 \frac{s^{\vartheta_1-1}+s^{\vartheta_1-2}(\vartheta_1-1)}{\Gamma(\vartheta_1)}g(s)ds}\int_0^1 g(s)\Big[-\int_0^1 G_1(s,\tau)k(\tau)d\tau\Big]ds. \tag{9}$$

Putting (9) into (8), we have

$$\begin{aligned} v(t) &= -\int_0^1 G_1(t,s)k(s)ds + \frac{t^{\vartheta_1-1}+t^{\vartheta_1-2}(\vartheta_1-1)}{\Delta_1\Gamma(\vartheta_1)}\int_0^1\Big[-\int_0^1 G_1(t,s)g(t)dt\Big]k(s)ds \\ &= -\int_0^1 H_1(t,s)k(s)ds. \end{aligned}$$

□

Lemma 2.4. *Let $v \in C([0,1])$, then*

$$\begin{cases} D_{0^+}^{\vartheta_2}u(t) = \phi_q(v(t)), \quad t \in (0,1), \\ u(0) = 0, \\ D_{0^+}^{\vartheta_2-2}u(0) = D_{0^+}^{\vartheta_2-1}u(1) = \int_0^1 h(s)u(s)ds \end{cases} \tag{10}$$

has a unique solution u and u can be expressed in the form

$$u(t) = -\int_0^1 H_2(t,s)\phi_q(v(s))ds,$$

where

$$H_2(t,s) = G_2(t,s) + \frac{t^{\vartheta_2-1}+t^{\vartheta_2-2}(\vartheta_2-1)}{\Delta_2\Gamma(\vartheta_2)}\int_0^1 G_2(t,s)h(t)dt, \tag{11}$$

$$G_2(t,s)=\frac{1}{\Gamma(\vartheta_2)}\begin{cases}t^{\vartheta_2-1}, & 0\le t\le s\le 1,\\ t^{\vartheta_2-1}-(t-s)^{\vartheta_2-1}, & 0\le s\le t\le 1,\end{cases} \tag{12}$$

and $\Delta_2=1-\int_0^1\frac{s^{\vartheta_2-1}+s^{\vartheta_2-2}(\vartheta_2-1)}{\Gamma(\vartheta_2)}h(s)ds.$

Lemma 2.5. *Assume $\eta\in(0,1)$, then the functions $G_i(t,s)$, (i=1,2) given by* (6) *and* (12) *satisfy*

(i) $G_i(t,s)$ are continuous functions and $G_i(t,s)\ge 0$ for any $t,s\in[0,1]$, $i=1,2$.

(ii)$G_i(t,s)\le G_i(1,s)$ for any $t,s\in[0,1]$, $i=1,2$.

(iii)$G_1(t,s)\ge t^{\vartheta_1-1}G_1(t,s)$ and $G_2(t,s)\ge t^{\vartheta_2-1}G_2(t,s)$ for any $t,s\in[0,1]$.

(iv)$G_1(t,s)\ge \eta^{\vartheta_1-1}G_1(t,s)$ and $G_2(t,s)\ge \eta^{\vartheta_2-1}G_2(t,s)$ for any $t\in[\eta,1]$ and $s\in[0,1]$.

Proof. We can easily see that (*i*) and (*iii*)-(*iv*) hold. Next, we show that (*ii*) holds. If $t\ge s$, then

$$\begin{aligned}G_{1t}(t,s)&=\frac{(\vartheta_1-1)t^{\vartheta_1-2}-(\vartheta_1-1)(t-s)^{\vartheta_1-2}}{\Gamma(\vartheta_1)}\\&=\frac{(\vartheta_1-1)\left[t^{\vartheta_1-2}-t^{\vartheta_1-2}(1-\frac{s}{t})^{\vartheta_1-2})\right]}{\Gamma(\vartheta_1)}\\&\ge\frac{(\vartheta_1-1)t^{\vartheta_1-2}\left[1-(1-s)^{\vartheta_1-2})\right]}{\Gamma(\vartheta_1)}\\&\ge 0.\end{aligned}$$

Since $G_{1t}(t,s)\ge 0$, $G_1(t,s)$ is increasing on $[s,1]$ with respect to t. Hence, we have $G_1(t,s)\le G_1(1,s)$.
Now, we will show that, $G_1(t,s)\le G_1(1,s)$ when $s\ge t$. Let $f(s)=1-s^{\vartheta_1-1}-(1-s)^{\vartheta_1-1}$. Then $f''(s)=-(\vartheta_1-1)(\vartheta_1-2)s^{\vartheta_1-3}-(\vartheta_1-1)(\vartheta_1-2)(1-s)^{\vartheta_1-3}\le 0$. From the fact that $f''(s)\le 0$, the graph of f is concave down on $[0,1]$. Using $f(1)=f(0)=0$, we get $f(s)\ge 0$. Thus,

$$G_1(t,s)=\frac{t^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\le\frac{s^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\le\frac{1-(1-s)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}=G_1(1,s)$$

Thus, $G_1(t,s)\le G_1(1,s)$ for any $t,s\in[0,1]$. Similar to the proof of $G_1(t,s)\le G_1(1,s)$ for any $t,s\in[0,1]$, we obtain $G_2(t,s)\le G_2(1,s)$.
The proof is completed. □

Lemma 2.6. *The functions $H_i(t,s)$, (i=1,2) given by* (5) *and* (11) *verify*

(i) $H_1(t,s)\le\Theta_1(s)$ and $H_2(t,s)\le\Theta_2(s)$ for any $t,s\in[0,1]$.

(ii)$H_1(t,s)\ge\eta^{\vartheta_1-1}\Theta_1(s)$ and $H_2(t,s)\ge\eta^{\vartheta_2-1}\Theta_2(s)$ for any $t\in[\eta,1]$ and $s\in[0,1]$,

where

$$\Theta_1(s)=G_1(1,s)+\frac{\vartheta_1}{\Delta_1\Gamma(\vartheta_1)}\int_0^1 G_1(t,s)g(t)dt, \tag{13}$$

$$\Theta_2(s) = G_2(1,s) + \frac{\vartheta_2}{\Delta_2\Gamma(\vartheta_2)}\int_0^1 G_2(t,s)h(t)dt. \tag{14}$$

Proof. It follows from Lemma 2.5 and the definition of $G_1(t,s)$ and $G_2(t,s)$ that (i) holds. At the moment, we prove that (ii) also holds. For any $t \in [\eta, 1]$ and $s \in [0,1]$,

$$\begin{aligned}
H_1(t,s) &= G_1(t,s) + \frac{t^{\vartheta_1-1} + t^{\vartheta_1-2}(\vartheta_1-1)}{\Delta_1\Gamma(\vartheta_1)}\int_0^1 G_1(t,s)g(t)dt \\
&\geq \eta^{\vartheta_1-1}G_1(1,s) + \frac{\eta^{\vartheta_1-1} + \eta^{\vartheta_1-2}(\vartheta_1-1)}{\Delta_1\Gamma(\vartheta_1)}\int_0^1 G_1(t,s)g(t)dt \\
&\geq \eta^{\vartheta_1-1}G_1(1,s) + \frac{\eta^{\vartheta_1-1} + \eta^{\vartheta_1-1}(\vartheta_1-1)}{\Delta_1\Gamma(\vartheta_1)}\int_0^1 G_1(t,s)g(t)dt \\
&= \eta^{\vartheta_1-1}\left[G_1(1,s) + \frac{\vartheta_1}{\Delta_1\Gamma(\vartheta_1)}\int_0^1 G_1(t,s)g(t)dt\right] \\
&= \eta^{\vartheta_1-1}\Theta_1(s).
\end{aligned} \tag{15}$$

Similarly, we can prove that $H_2(t,s) \geq \eta^{\vartheta_2-1}\Theta_2(s)$ for any $t \in [\eta,1]$ and $s \in [0,1]$.
The proof is completed. □

Let $E = C^{\vartheta_2}[0,1] := \left\{u : u \in C[0,1], D_{0^+}^{\vartheta_2}u \in C[0,1]\right\}$ be a Banach space with the norm

$$\|u\|_{\vartheta_2} = \max\{\|u\|_\infty, \|D_{0^+}^{\vartheta_2}u\|_\infty\},$$

where $\|u\|_\infty = \max_{t\in[0,1]}|u(t)|$, $\|D_{0^+}^{\vartheta_2}u\|_\infty = \max_{t\in[0,1]}|D_{0^+}^{\vartheta_2}u(t)|$. Define the cone $P \subset E$ by

$$P = \left\{u \in E : u(t) \geq 0, \ \min_{t\in[\eta,1]} u(t) \geq \eta^{\vartheta_2-1}\max_{t\in[0,1]} u(t)\right\}.$$

Let $A : P \to E$, by

$$Au(t) = \int_0^1 H_2(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),D_{0^+}^{\vartheta_2}u(\tau))d\tau\Big)ds. \tag{16}$$

Then, the fixed point of the operator A is the solution of the problem (1).

Lemma 2.7. *$A : P \to P$ is a completely continuous operator.*

Proof. We first show that $A : P \to P$ is well defined. Let $u \in P$. Considering the definition of A, one gets $Au(t) \geq 0$ for $t \in [0,1]$. Moreover, by Lemma 2.5 and Lemma 2.6, we have,

$$\begin{aligned}
\max_{t\in[0,1]} Au(t) &= \int_0^1 H_2(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),D_{0^+}^{\vartheta_2}u(\tau))d\tau\Big)ds \\
&\leq \int_0^1 \Theta_2(s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),D_{0^+}^{\vartheta_2}u(\tau))d\tau\Big)ds.
\end{aligned}$$

And next,

$$\begin{aligned}\min_{t\in[\eta,1]} Au(t) &= \int_0^1 H_2(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),D_{0^+}^{\vartheta_2}u(\tau))d\tau\Big)ds\\ &\geq \int_0^1 \eta^{\vartheta_2-1}\Theta_2(s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),D_{0^+}^{\vartheta_2}u(\tau))d\tau\Big)ds\\ &\geq \eta^{\vartheta_2-1}\max_{t\in[0,1]} Au(t).\end{aligned}$$

So, $Au \in P$ and then $A(P) \subset P$. Also, in view of the Arzela Ascoli theorem and the standard arguments, one can see easily that $A : P \to P$ is completely continuous.
□

We are now ready to apply the fixed point theorem due to Bai and Ge to the operator A in order to get sufficient conditions for the existence of multiple positive solutions to the BVP (1).

Let $r > a > 0,\ L > 0$ be given constants and ζ be a nonnegative continuous concave functional and α, β be a nonnegative continuous convex functional on the cone P. Define bounded convex sets by

$$P(\alpha^r,\beta^L) = \{u \in P :\ \alpha(u) < r,\ \beta(u) < L\},$$

$$\overline{P}(\alpha^r,\beta^L) = \{u \in P :\ \alpha(u) \leq r,\ \beta(u) \leq L\},$$

$$P(\alpha^r,\beta^L,\zeta_a) = \{u \in P :\ \alpha(u) < r,\ \beta(u) < L,\ \zeta(u) > a\},$$

$$\overline{P}(\alpha^r,\beta^L,\zeta_a) = \{u \in P :\ \alpha(u) \leq r,\ \beta(u) \leq L,\ \zeta(u) \geq a\}.$$

The nonnegative continuous convex functionals α, β on cone P satisfy

(E_1):There exists $M > 0$ such that $\|u\| \leq M max\{\alpha(u), \beta(u)\}\ for\ all\ u \in P;$

(E_2): $\Omega = \{u \in P : \alpha(u) < r,\ \beta(u) < L\} \neq \emptyset,$ for any $r > 0,\ L > 0.$

The following fixed point theorem is fundamental and important for the proof of our main result.

Theorem 2.8. *[3] Let $\mathbb{B}$ be a Banach space, $P \subset \mathbb{B}$ be a cone and $r_2 \geq d > b > r_1 > 0,\ L_2 \geq L_1 > 0$ be given. Assume that α, β are nonnegative continuous convex functionals on P, such that (E_1) and (E_2) are satisfied, ζ is a nonnegative continuous concave functional on P, such that $\zeta(u) \leq \alpha(u)$ for all $u \in \overline{P}(\alpha^{r_2},\beta^{L_2})$ and let $A : \overline{P}(\alpha^{r_2},\beta^{L_2}) \to \overline{P}(\alpha^{r_2},\beta^{L_2})$ be a completely continuous operator.*

(B1) $\{u \in \overline{P}(\alpha^d,\beta^{L_2},\zeta_b) : \zeta(u) > b\} \neq \emptyset,$ *and* $\zeta(Au) > b$ *for* $u \in \overline{P}(\alpha^d,\beta^{L_2},\zeta_b),$

(B2) $\alpha(Au) < r_1,\ \beta(Au) < L_1,$ *for all* $u \in \overline{P}(\alpha^{r_1},\beta^{L_1}),$

(B3) $\zeta(Au) > b,$ *for all* $u \in \overline{P}(\alpha^{r_2},\beta^{L_2},\zeta_b)$ *with* $\alpha(Au) > d.$

Then A has at least three fixed points u_1, u_2, u_3 in $\overline{P}(\alpha^{r_2},\beta^{L_2})$ with

$$u_1 \in P(\alpha^{r_1},\beta^{L_1}),\ u_2 \in \{\overline{P}(\alpha^{r_2},\beta^{L_2},\zeta_b) :\ \zeta(u) > b\},$$

$$u_3 \in \overline{P}(\alpha^{r_2},\beta^{L_2}) \setminus \left(\overline{P}(\alpha^{r_2},\beta^{L_2},\zeta_b) \cup \overline{P}(\alpha^{r_1},\beta^{L_1})\right).$$

For the readers convenience, let us denote

$$
\begin{aligned}
c &= \eta^{\vartheta_2-1}\int_\eta^1 \Theta_2(s)\phi_q\Big(\eta^{\vartheta_1-1}\int_\eta^1 \Theta_1(\tau)d\tau\Big)ds,\\
L &= \int_0^1 \Theta_2(s)\phi_q\Big(\int_0^1 \Theta_1(\tau)d\tau\Big)ds,\\
B &= \phi_q\Big(\int_0^1 \Theta_1(s)ds\Big).
\end{aligned}
$$

3. Existence theorem

Theorem 3.1. *Assume that there exist constants $\eta^{\vartheta_1+\vartheta_2-2}r_2 > b > r_1 > 0$, $L_2 \geq L_1 > 0$ such that $\frac{b}{c} \leq \min\left\{\frac{r_2}{L}, \frac{L_2}{B}\right\}$. Assume*

(*a*) $f(t,u,v) \leq \min\left\{\phi_p(\frac{r_2}{L}), \phi_p(\frac{L_2}{B})\right\}$ *for* $t \in [0,1]$, $u \in [0,r_2]$, $v \in [-L_2, 0]$.

(*b*) $f(t,u,v) > \phi_p(\frac{b}{c})$ *for* $t \in [\eta,1]$, $u \in [b, \frac{b}{\eta^{\vartheta_1+\vartheta_2-2}}]$, $v \in [-L_2, 0]$.

(*c*) $f(t,u,v) < \min\left\{\phi_p(\frac{r_1}{L}), \phi_p(\frac{L_1}{B})\right\}$ *for* $t \in [0,1]$, $u \in [0,r_1]$, $v \in [-L_1, 0]$.

Then the problem (1) *has at least three positive solutions* u_i $(i = 1,2,3)$ *with*

$$
\begin{aligned}
0 \leq \max_{t\in[0,1]} u_1(t) \leq r_1,\ \max_{t\in[0,1]}|D_{0^+}^{\vartheta_2}u_1(t)| \leq L_1;\\
b < \min_{t\in[\eta,1]} u_2(t) \leq \max_{t\in[0,1]} u_2(t) \leq r_2,\ \max_{t\in[0,1]}|D_{0^+}^{\vartheta_2}u_2(t)| \leq L_2;\\
\min_{t\in[\eta,1]} u_3(t) \leq b,\ r_1 \leq \max_{t\in[0,1]} u_3(t) \leq r_2,\ \max_{t\in[0,1]}|D_{0^+}^{\vartheta_2}u_3(t)| \leq L_2.
\end{aligned}
$$

Proof. Let P and A be defined as above. Define α, β and ζ by

$$
\begin{aligned}
\alpha(u) = \max_{t\in[0,1]}|u(t)|,\ \beta(u) = \max_{t\in[0,1]}\left|D_{0^+}^{\vartheta_2}u(t)\right|,\\
\zeta(u) = \min_{t\in[\eta,1]}|u(t)|,
\end{aligned}
$$

for $u \in P$. Obviously, α, $\beta : P \to [0,+\infty)$ are nonnegative continuous convex functionals, ζ is nonnegative continuous concave functional with $\zeta(u) \leq \alpha(u)$ for all $u \in \mathbb{B}$. For any $r > 0$ and $L > 0$, let $u_1(t) = at^{\vartheta_2-1}$, where $0 < a < r$. Then $u_1(t) = at^{\vartheta_2-1} \in P(\alpha^r, \beta^L) \neq \emptyset$. Thus, (E_1), (E_2) are satisfied. Then $\alpha(u)$, $\beta(u)$, $\zeta(u)$ satisfy the conditions in Theorem 2.8. Now, we set out to verify that the operator A satisfies all conditions in Theorem 2.8 which will prove the existence of three fixed points of A. It follows from Lemma 2.7 that A is completely continuous. First of all, we show that $A : \overline{P}(\alpha^{r_2}, \beta^{L_2}) \to \overline{P}(\alpha^{r_2}, \beta^{L_2})$. Let $u \in \overline{P}(\alpha^{r_2}, \beta^{L_2})$. Then $\alpha(u) \leq r_2$, $\beta(u) \leq L_2$. By condition (*a*), we can get

$$
\begin{aligned}
\alpha(Au) = \max_{t\in[0,1]}|Au(t)| &= \max_{t\in[0,1]}\left|\int_0^1 H_2(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,u(\tau),D_{0^+}^{\vartheta_2}u(\tau))d\tau\Big)ds\right|\\
&\leq \int_0^1 \Theta_2(s)\phi_q\Big(\int_0^1 \Theta_1(\tau)f(\tau,u(\tau),D_{0^+}^{\vartheta_2}u(\tau))d\tau\Big)ds \qquad (17)\\
&\leq \frac{r_2}{L}\int_0^1 \Theta_2(s)\phi_q\Big(\int_0^1 \Theta_1(\tau)d\tau\Big)ds = r_2
\end{aligned}
$$

and

$$\begin{aligned}\beta(Au) = \max_{t\in[0,1]} |D_{0^+}^{\vartheta_2} u(t)| &= \max_{t\in[0,1]} \Big| -\phi_q\Big(\int_0^1 H_1(t,s) f(s,u(s),D_{0^+}^{\vartheta_2}u(s))ds\Big)\Big| \\ &\leq \phi_q\Big(\int_0^1 \Theta_1(s) f(s,u(s),D_{0^+}^{\vartheta_2}u(s))ds\Big) \\ &\leq \frac{L_2}{B}\phi_q\Big(\int_0^1 \Theta_1(s)ds\Big) = L_2.\end{aligned} \tag{18}$$

So, we have $A : \overline{P}(\alpha^{r_2}, \beta^{L_2}) \to \overline{P}(\alpha^{r_2}, \beta^{L_2})$. With assumption (c) by the similar argument, we can get that $A : \overline{P}(\alpha^{r_1}, \beta^{L_1}) \to P(\alpha^{r_1}, \beta^{L_1})$. Hence, condition $(B2)$ in Theorem 2.8 is satisfied. Next, to check the condition $(B1)$ of Theorem 2.8, we choose $u(t) = \frac{b}{\eta^{\vartheta_1+\vartheta_2-2}} t^{\vartheta_2-1}$ for any $t \in [0,1]$. We can easily get

$$\alpha(u) = \max_{t\in[0,1]} |\frac{b}{\eta^{\vartheta_1+\vartheta_2-2}} t^{\vartheta_2-1}| \leq \frac{b}{\eta^{\vartheta_1+\vartheta_2-2}},$$

$$\beta(u) = \max_{t\in[0,1]} |D_{0^+}^{\vartheta_2} \frac{b}{\eta^{\vartheta_1+\vartheta_2-2}} t^{\vartheta_2-1}| = 0,$$

and

$$\zeta(u) = \min_{t\in[\eta,1]} |\frac{b}{\eta^{\vartheta_1+\vartheta_2-2}} t^{\vartheta_2-1}| = \frac{b}{\eta^{\vartheta_1-1}} > b.$$

Then $u(t) = \frac{b}{\eta^{\vartheta_1+\vartheta_2-2}} t^{\vartheta_2-1} \in \overline{P}(\alpha^{\frac{b}{\eta^{\vartheta_1+\vartheta_2-2}}}, \beta^{L_2}, \zeta_b)$ and $\zeta(u) > b$. So $\Big\{u \in \overline{P}(\alpha^{\frac{b}{\eta^{\vartheta_1+\vartheta_2-2}}}, \beta^{L_2}, \zeta_b) : \zeta(u) > b\Big\} \neq \emptyset$. If we choose $u \in \overline{P}(\alpha^{\frac{b}{\eta^{\vartheta_1+\vartheta_2-2}}}, \beta^{L_2}, \zeta_b)$, then $u(t) \in [b, \frac{b}{\eta^{\vartheta_1+\vartheta_2-2}}]$ for any $t \in [\eta, 1]$. Thus, from assumption (b) we get

$$\begin{aligned}\zeta(Au) = \min_{t\in[\eta,1]} |Au(t)| &\geq \int_\eta^1 \eta^{\vartheta_2-1}\Theta_2(s)\phi_q\Big(\int_\eta^1 \eta^{\vartheta_1-1}\Theta_1(\tau) f(\tau,u(\tau),D_{0^+}^{\vartheta_2}u(\tau))d\tau\Big)ds \\ &> \frac{b}{c}\int_\eta^1 \eta^{\vartheta_2-1}\Theta_2(s)\phi_q\Big(\int_\eta^1 \eta^{\vartheta_1-1}\Theta_1(\tau)d\tau\Big)ds \\ &= b.\end{aligned}$$

So, we obtain that $\zeta(Au) > b$. Thus the condition $(B1)$ of Theorem 2.8 is verified. Finally, we show that the last condition of Theorem 2.8 is satisfied. Assume that $u \in \overline{P}(\alpha^{r_2}, \beta^{L_2}, \zeta_b)$ with $\alpha(Au) > \frac{b}{\eta^{\vartheta_1+\vartheta_2-2}}$. Then, in view of the definition of ζ and $Au \in P$, we obtain that

$$\begin{aligned}\min_{t\in[\eta,1]} Au(t) &\geq \eta^{\vartheta_2-1} \max_{t\in[0,1]} |Au(t)| \\ &> \eta^{\vartheta_2-1}\frac{b}{\eta^{\vartheta_1+\vartheta_2-2}} \\ &= \frac{b}{\eta^{\vartheta_1-1}} > b.\end{aligned}$$

Hence, $\zeta(Au) > b$, for all $u \in \overline{P}(\alpha^{r_2}, \beta^{L_2}, \zeta_b)$. That is condition $(B3)$ of Theorem 2.8 hold.

Consequently, Theorem 2.8 yields that the operator A has at least three positive solutions u_i, $(i = 1, 2, 3)$ with

$$u_1 \in P(\alpha^{r_1}, \beta^{L_1}),\ u_2 \in \{\overline{P}(\alpha^{r_2}, \beta^{L_2}, \zeta_b) :\ \zeta(u) > b\},$$
$$u_3 \in \overline{P}(\alpha^{r_2}, \beta^{L_2}) \setminus \left(\overline{P}(\alpha^{r_2}, \beta^{L_2}, \zeta_b) \cup \overline{P}(\alpha^{r_1}, \beta^{L_1})\right).$$

The proof is completed. □

Example 3.2. *We consider the following boundary value problem*

$$\begin{cases} D_{0^+}^{11/4}(\phi_2(D_{0^+}^{11/4}u(t))) = f(t, u(t), D_{0^+}^{11/4}u(t)), \quad t \in (0, 1), \\ D_{0^+}^{11/4}u(0) = 0, \quad D_{0^+}^{3/4}(\phi_2(D_{0^+}^{11/4}u(0))) = D_{0^+}^{7/4}(\phi_2(D_{0^+}^{11/4}u(1))) = \dfrac{1}{2}\displaystyle\int_0^1 \phi_2(D_{0^+}^{11/4}u(s))ds, \\ u(0) = 0, \quad D_{0^+}^{3/4}u(0) = D_{0^+}^{7/4}u(1) = \dfrac{1}{2}\displaystyle\int_0^1 u(s)ds, \end{cases} \tag{19}$$

where $\vartheta_1 = \vartheta_2 = \frac{11}{4}$ *and* $5 < \vartheta_1 + \vartheta_2 \le 6$, $p = 2$, $g(s) = h(s) = \frac{1}{2}$,

$$f(t, u, v) = \begin{cases} \left(\dfrac{1}{10}\right)^t + \dfrac{4u}{3} + \dfrac{|v|}{10^5}, & u \in [0, 15], \\ \left(\dfrac{1}{10}\right)^t + 156u - 2320 + \dfrac{|v|}{10^5}, & u \in [15, 20], \\ \left(\dfrac{1}{10}\right)^t + \dfrac{u - 20}{25} + 800 + \dfrac{|v|}{10^5}, & u \in [20, \infty). \end{cases}$$

By easy calculation, we obtain $\Delta_1 = \Delta_2 \approx 0.576078$, $L \approx 0.626309$, $B \approx 0.791397$. *Let* $\eta = \frac{1}{2}$, *then* $c \approx 0.025431$. *Choosing* $r_1 = 15$, $b = 20$, $r_2 = 520$, $L_1 = 50$, $L_2 = 700$ *and* $d = \dfrac{b}{\eta^{\vartheta_1+\vartheta_2-2}} = 160\sqrt{2}$, *one gets*

$$f(t, u, v) \le \min\left\{\phi_p(\tfrac{r_2}{L}), \phi_p(\tfrac{L_2}{B})\right\} \approx 830.261, \textit{for } t \in [0, 1],\ u \in [0, 520],\ v \in [-700, 0],$$

$$f(t, u, v) > \phi_p(\frac{b}{c}) \approx 786.442, \textit{for } t \in [\tfrac{1}{2}, 1],\ u \in [20, 160\sqrt{2}],\ v \in [-700, 0],$$

$$f(t, u, v) < \min\left\{\phi_p(\tfrac{r_1}{L}), \phi_p(\tfrac{L_1}{B})\right\} \approx 23.95 \textit{ for } t \in [0, 1],\ u \in [0, 15],\ v \in [-50, 0],$$

i.e., f holds the conditions of Theorem 3.1. Therefore, Theorem 3.1 implies that the problem (19) *has at least three positive solutions u_i for $i \in \{1, 2, 3\}$ with*

$$0 \le \max_{t\in[0,1]} u_1(t) \le 15,\ \max_{t\in[0,1]} |D_{0^+}^{\vartheta_2}u_1(t)| \le 50;$$

$$20 < \min_{t\in[\eta,1]} u_2(t) \le \max_{t\in[0,1]} u_2(t) \le 520,\ \max_{t\in[0,1]} |D_{0^+}^{\vartheta_2}u_2(t)| \le 700;$$

$$\min_{t\in[1/2,1]} u_3(t) \le 20,\ 15 \le \max_{t\in[0,1]} u_3(t) \le 520,\ \max_{t\in[0,1]} |D_{0^+}^{\vartheta_2}u_3(t)| \le 700.$$

References

[1] B. Ahmad, S. K. Ntouyas and A. Alsaedi, New Existence Results for Nonlinear Fractional Differential Equations with Three-Point Integral Boundary Conditions, Adv. Differ. Equ., Volume 2011, Article ID 107384, 11 p.

[2] C. Bai, Triple positive solutions for a boundary value problem of nonlinear fractional differential equation, Electron. J. Qual. Theory Differ. Equ., no. 24, pp. 1-10, 2008.

[3] Z. Bai and W. Ge, Existence of three positive solutions for some second-order boundary value problems. Comput. Math. Appl., 48, 699-707 (2004).

[4] G. Chai, Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator, Bound. Value Probl., 2012 (2012), 20 pages.

[5] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.

[6] Y. Li and G. Li, Positive solutions of p-Laplacian fractional differential equations with integral boundary value conditions, J. Nonlinear Sci. Appl., 9 (2016), 717–726.

[7] X. Liu and M. Jia, The Positive Solutions for Integral Boundary Value Problem of Fractional p-Laplacian Equation with Mixed Derivatives, Mediterr. J. Math., (2017) 14:94.

[8] S. K. Ntouyas, S. Etemad, On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions, Appl. Math. Comput., 266, 235-243 (2015).

[9] N. Nyamoradi, Multiple positive solutions for fractional differential systems, Ann. Univ. Ferrara, (2012) 58:359-369.

[10] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[11] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives: theory and applications, Gordon and Breach, Switzerland, (1993).2.

[12] X. Li, X. Liu, M. Jia, Y. Li and S. Zhang, Existence of positive solutions for integral boundary value problems of fractional differential equations on infinite interval, Math. Methods Appl. Sci., DOI: 10.1002/mma.4106.

[13] G. Wang, S. Liu, R. P. Agarwal and L. Zhang, Positive Solutions of Integral Boundary Value Problem Involving Riemann-Liouville Fractional Derivative, J. Fract. Calc. Appl., Vol. 4(2) July 2013, pp. 312-321.

[14] S. Vong, Positive solutions of singular fractional differential equations with integral boundary conditions, Math. Comput. Model., Volume 57, Issues 5-6, March 2013, Pages 1053-1059.

[15] A. Yang and H. Wang, Positive solutions for higher-order nonlinear fractional differential equation with integral boundary condition, Electron. J. Qual. Theory Differ. Equ., 2011, No. 1, 1-15.

[16] X. Zhang, L. Liu, B. Wiwatanapataphee and Y. Wu, The eigenvalue for a class of singular p-Laplacian fractional differential equations involving the Riemann-Stieltjes integral boundary condition, Appl. Math. Comput., 235 (2014), 412-422.