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A Note on the Potential Function of an Arbitrary Graph H

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Abstract. Given a graph H, a graphic sequence π is *potentially H-graphic* if there is a realization of π containing H as a subgraph. In 1991, Erdős et al. introduced the following problem: determine the minimum even integer $\sigma(H,n)$ such that each n-term graphic sequence with sum at least $\sigma(H,n)$ is potentially H-graphic. This problem can be viewed as a "potential" degree sequence relaxation of the Turán problems. Let H be an arbitrary graph of order k. Ferrara et al. [Combinatorica, 36(2016)687–702] established an upper bound on $\sigma(H,n)$: if $\omega=\omega(n)$ is an increasing function that tends to infinity with n, then there exists an $N=N(\omega,H)$ such that $\sigma(H,n)\le\widetilde{\sigma}(H)n+\omega(n)$ for any $n\ge N$, where $\widetilde{\sigma}(H)$ is a parameter only depending on the graph H. Recently, Yin [European J. Combin., 85(2020)103061] obtained a new upper bound on $\sigma(H,n)$: there exists an $M=M(k,\alpha(H))$ such that $\sigma(H,n)\le\widetilde{\sigma}(H)n+k^2-3k+4$ for any $n\ge M$. In this paper, we investigate the precise behavior of $\sigma(H,n)$ for arbitrary H with $\widetilde{\sigma}_{\alpha(H)+1}(H)<\widetilde{\sigma}(H)$ or $\nabla_{\alpha(H)+1}(H)\ge 2$, where $\nabla_{\alpha(H)+1}(H)=\min\{\Delta(F)|F$ is an induced subgraph of H and $|V(F)|=\alpha(H)+1\}$ and $\widetilde{\sigma}_{\alpha(H)+1}(H)=2(k-\alpha(H)-1)+\nabla_{\alpha(H)+1}(H)-1$. Moreover, we also show that $\sigma(H,n)=(k-\alpha(H)-1)(2n-k+\alpha(H))+2$ for those H so that $\nabla_{\alpha(H)+1}(H)=1$, $\widetilde{\sigma}_{\alpha(H)+1}(H)=\widetilde{\sigma}(H)$, $\widetilde{\sigma}_p(H)<\widetilde{\sigma}(H)$ for $\alpha(H)+2\le p\le k$ and there is an $F<H$ with $|V(F)|=\alpha(H)+1$ and $\pi(F)=(1^2,0^{\alpha(H)-1})$.

1. Introduction

A sequence $\pi=(d_1,\ldots,d_n)$ of non-negative integers is said to be a *graphic sequence* if it is realizable by a simple graph G on n vertices. In this case, G is referred to as a *realization* of π. The set of all sequences $\pi=(d_1,\ldots,d_n)$ of non-negative, non-increasing integers with $d_1\le n-1$ is denoted by NS_n. The set of all graphic sequences in NS_n is denoted by GS_n. For a sequence $\pi=(d_1,\ldots,d_n)$, we denote $\sigma(\pi)=d_1+\cdots+d_n$ and $p(\pi)=\max\{i|d_i\ge 1\}$. Given a (simple) graph H, a graphic sequence π is said to be *potentially* (respectively, *forcibly*) *H-graphic* if there exists a realization of π containing H as a subgraph (respectively, each realization of π contains H as a subgraph).

One of the classical extremal problems is to determine the minimum integer m such that every graph G on n vertices with edge number $e(G)\ge m$ contains H as a subgraph. This m is denoted by $ex(H,n)$, and is called the *Turán number* of H. In terms of graphic sequences, the number $2ex(H,n)$ is the minimum even

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integer such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geq 2ex(H, n)$ is forcibly H-graphic. In 1991, Erdős et al. [2] introduced the following problem: determine the minimum even integer $\sigma(H, n)$ such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic. We will refer to $\sigma(H, n)$ as the *potential number* or *potential function* of H. As $\sigma(\pi)$ is twice the number of edges in any realization of π, this problem can be viewed as a potential degree sequence relaxation of the Turán problems.

In [2], Erdős et al. conjectured that $\sigma(K_r, n) = (r-2)(2n-r+1)+2$, where K_r is the complete graph on r vertices. The cases $r = 3$, 4 and 5 were proved separately (see respectively [2], and [7,11], and [12]), and Li et al. [13] proved the conjecture true for $r \geq 6$ and $n \geq \binom{r-1}{2} + 3$. In addition to these results for complete graphs, the value of $\sigma(H, n)$ has been determined exactly for a number of specific graph classes (c.f. [3,4,6,7,10,15,16]). For an arbitrarily chosen H, Ferrara and Schmitt [6] gave a construction that yields the best known lower bound on $\sigma(H, n)$.

We assume that H is an arbitrary graph of order k and $\alpha(H)$ is the independent number of H. We let $\Delta(F)$ denote the maximum degree of a graph F, and let $F < H$ denote that F is an induced subgraph of H. For each $p \in \{\alpha(H)+1, \ldots, k\}$, let

$$\nabla_p(H) = \min\{\Delta(F) | F < H \text{ and } |V(F)| = p\}.$$

Clearly, $1 \leq \nabla_{\alpha(H)+1}(H) \leq \cdots \leq \nabla_k(H) \leq k-1$. Let

$$\widetilde{\pi}_p(H, n) = ((n-1)^{k-p}, (k-p+\nabla_p(H)-1)^{n-k+p})$$

if $(n-k+p)(\nabla_p(H)-1)$ is even, and

$$\widetilde{\pi}_p(H, n) = ((n-1)^{k-p}, (k-p+\nabla_p(H)-1)^{n-k+p-1}, k-p+\nabla_p(H)-2)$$

if $(n-k+p)(\nabla_p(H)-1)$ is odd, where the symbol x^y in a sequence stands for y consecutive terms, each equal to x. Then,

$$\sigma(\widetilde{\pi}_p(H, n)) = (2(k-p)+\nabla_p(H)-1)n-(k-p)(k-p+\nabla_p(H))$$

if $(n-k+p)(\nabla_p(H)-1)$ is even, and

$$\sigma(\widetilde{\pi}_p(H, n)) = (2(k-p)+\nabla_p(H)-1)n-(k-p)(k-p+\nabla_p(H))-1$$

if $(n-k+p)(\nabla_p(H)-1)$ is odd. Ferrara et al. [5] showed that $\widetilde{\pi}_p(H, n)$ is graphic and is not potentially H-graphic for all $p \in \{\alpha(H)+1, \ldots, k\}$, thus establishing a lower bound on $\sigma(H, n)$.

Proposition 1.1 $\sigma(H, n) \geq \sigma(\widetilde{\pi}_p(H, n)) + 2$ *for* $p \in \{\alpha(H)+1, \ldots, k\}$.

Let

$$\widetilde{\sigma}_p(H) = 2(k-p)+\nabla_p(H)-1,$$

and let

$$\widetilde{\sigma}(H) = \max\{\widetilde{\sigma}_p(H) | p = \alpha(H)+1, \ldots, k\}.$$

Ferrara et al. [5] established an upper bound on $\sigma(H, n)$ and determined $\sigma(H, n)$ asymptotically.

Theorem 1.1 [5] *Let H be a graph, and let $\omega = \omega(n)$ be an increasing function that tends to infinity with n. There exists an $N = N(\omega, H)$ such that for any $n \geq N$,*

$$\sigma(H, n) \leq \widetilde{\sigma}(H)n + \omega(n).$$

Theorem 1.2 [5] *Let H be a graph of order k and let n be a positive integer. Then*

$$\sigma(H, n) = \widetilde{\sigma}(H)n + o(n).$$

Recently, Yin [14] established a new upper bound on $\sigma(H, n)$ as follows.

Theorem 1.3 [14] *Let H be a graph of order k. There exists an $M = M(k, \alpha(H))$ such that for any $n \geq M$,*

$$\sigma(H, n) \leq \widetilde{\sigma}(H)n + k^2 - 3k + 4.$$

The focus of this paper is the precise behavior of the potential number for arbitrary H. As such, for $p \in \{\alpha(H)+1,\dots,k\}$ and a graph F with $|V(F)| = p$, we denote $\pi(F) = (d_1,\dots,d_p)$ to be the degree sequence of F with $d_1 \geq \cdots \geq d_p$. We say that $(d_1,\dots,d_p) \geq (d'_1,\dots,d'_p)$ if $d_i \geq d'_i$ for $1 \leq i \leq p$. We now choose $\rho_p(H) = (d_1,\dots,d_p) \in NS_p$ with $d_1 \leq p-2$ and $d_p \geq \nabla_p(H)-1$ so that $\rho_p(H) \not\geq \pi(F)$ for each $F < H$ with $|V(F)| = p$ and $\sigma(\rho_p(H))$ is maximal, and let

$$\pi^*_p(H,n) = ((n-1)^{k-p}, k-p+d_1,\dots,k-p+d_p,(k-p+\nabla_p(H)-1)^{n-k}) \tag{1}$$

if $\sum_{i=1}^{p} d_i + (\nabla_p(H)-1)(n-k)$ is even, and

$$\pi^*_p(H,n) = ((n-1)^{k-p}, k-p+d_1,\dots,k-p+d_p,(k-p+\nabla_p(H)-1)^{n-k-1}, k-p+\nabla_p(H)-2) \tag{2}$$

if $\sum_{i=1}^{p} d_i + (\nabla_p(H)-1)(n-k)$ is odd.

Clearly, $\widetilde{\sigma}_p(H)$ is also the leading coefficient of $\sigma(\pi^*_p(H,n))$, and $\nabla_p(H)-1 \leq p-2$ and $((\nabla_p(H)-1)^p) \not\geq \pi(F)$ for each $F < H$ with $|V(F)| = p$. Thus $\sigma(\pi^*_p(H,n)) + 2 \geq \sigma(\widetilde{\pi}_p(H,n)) + 2$ for all $p \in \{\alpha(H)+1,\dots,k\}$.

For $\nabla_p(H) \geq 2$, applying Erdős-Gallai characterization, we can see that $\pi^*_p(H,n)$ is graphic for n sufficiently large. Every realization G of $\pi^*_p(H,n)$ is a complete graph on $k-p$ vertices joined to an $(n-k+p)$-vertex graph G_p with degree sequence $(d_1,\dots,d_p,(\nabla_p(H)-1)^{n-k})$ or $(d_1,\dots,d_p,(\nabla_p(H)-1)^{n-k-1},\nabla_p(H)-2)$. Any k-vertex subgraph of G contains at least p vertices in G_p. If G contains H as a subgraph, then G_p contains an $F < H$ with $|V(F)| = p$ as a subgraph. This implies $\rho_p(H) \geq \pi(F)$, a contradiction. Thus H is not a subgraph of G. In other words, $\pi^*_p(H,n)$ is not potentially H-graphic. This also establishes a lower bound on $\sigma(H,n)$ as follows.

Proposition 1.2 *$\sigma(H,n) \geq \sigma(\pi^*_p(H,n)) + 2$ for $\alpha(H)+1 \leq p \leq k$ and $\nabla_p(H) \geq 2$.*

For $\alpha(H)+1 \leq i \leq k$, we can see that $\nabla_i(H) = 1$ implies $\nabla_{\alpha(H)+1}(H) = 1$ and $\widetilde{\sigma}_i(H) \leq \widetilde{\sigma}_{\alpha(H)+1}(H)$. Therefore, if $\widetilde{\sigma}_{\alpha(H)+1}(H) < \widetilde{\sigma}(H)$ or $\nabla_{\alpha(H)+1}(H) \geq 2$, then

$$\max\{\sigma(\pi^*_p(H,n)) + 2 | \alpha(H)+1 \leq p \leq k \text{ and } \nabla_p(H) \geq 2\} \geq \sigma(\pi^*_i(H,n)) + 2$$

for $\alpha(H)+1 \leq i \leq k$ and n sufficiently large. In this paper, we determine the precise value of $\sigma(H,n)$ if $\widetilde{\sigma}_{\alpha(H)+1}(H) < \widetilde{\sigma}(H)$ or $\nabla_{\alpha(H)+1}(H) \geq 2$.

Theorem 1.4 *Let H be a graph of order k, with $\pi^*_p(H,n)$ as given in (1) or (2) for each $p \in \{\alpha(H)+1,\dots,k\}$, and let n be a sufficiently large integer. If $\widetilde{\sigma}_{\alpha(H)+1}(H) < \widetilde{\sigma}(H)$ or $\nabla_{\alpha(H)+1}(H) \geq 2$, then*

$$\sigma(H,n) = \max\{\sigma(\pi^*_p(H,n)) + 2 | \alpha(H)+1 \leq p \leq k \text{ and } \nabla_p(H) \geq 2\}.$$

Moreover, we also prove the following Theorem 1.5.

Theorem 1.5 *Let H be a graph of order k with $\nabla_{\alpha(H)+1}(H) = 1$, $\widetilde{\sigma}_{\alpha(H)+1}(H) = \widetilde{\sigma}(H)$ and $\widetilde{\sigma}_p(H) < \widetilde{\sigma}(H)$ for $\alpha(H)+2 \leq p \leq k$, and let n be a sufficiently large integer. If there is an $F < H$ with $|V(F)| = \alpha(H)+1$ so that $\pi(F) = (1^2, 0^{\alpha(H)-1})$, then*

$$\sigma(H,n) = (k-\alpha(H)-1)(2n-k+\alpha(H)) + 2.$$

We can see that Theorem 1.5 covers a number of specific graph families, including complete graphs, disjoint unions of cliques, matchings, odd cycles, (generalized) friendship graphs, intersecting cliques, etc. We will adopt the method of the reference [14] to prove Theorem 1.4–1.5.

2. Proof of Theorem 1.4

The following known results will be useful. For $\pi = (d_1, \ldots, d_n) \in NS_n$, let $d'_1 \geq \cdots \geq d'_{n-1}$ be the rearrangement in non-increasing order of $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. We say that $\pi' = (d'_1, \ldots, d'_{n-1})$ is the *residual sequence* of π.

Theorem 2.1 [8,9] *Let $\pi = (d_1, \ldots, d_n) \in NS_n$. Then π is graphic if and only if π' is graphic.*

Theorem 2.2 [1] *Let $\pi = (d_1, \ldots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then π is graphic if and only if $\sum\limits_{i=1}^{h} d_i \leq h(h-1) + \sum\limits_{j=h+1}^{n} \min\{h, d_j\}$ for each h with $1 \leq h \leq n-1$.*

Theorem 2.3 [16] *Let $\pi = (d_1, \ldots, d_n) \in NS_n$, $x = d_1$ and $\sigma(\pi)$ be even. If there is an integer n_1 with $1 \leq n_1 \leq n$ such that $d_{n_1} \geq y \geq 1$ and $n_1 \geq \frac{1}{y}\left\lfloor \frac{(x+y+1)^2}{4} \right\rfloor$, then π is graphic.*

Theorem 2.4 [17] *Let $n \geq r$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $d_r \geq r-1$. If $d_i \geq 2r-2-i$ for $i = 1, \ldots, r-2$, then π is potentially K_r-graphic.*

In this section, we always assume that H is a graph of order k with $\widetilde{\sigma}_{\alpha+1}(H) < \widetilde{\sigma}(H)$ or $\nabla_{\alpha+1}(H) \geq 2$, and n is a sufficiently large integer relative to k and $\alpha(H)$. We need some lemmas. For convenience, we denote $\Sigma = \max\{\sigma(\pi^*_p(H,n)) + 2 | \alpha(H) + 1 \leq p \leq k \text{ and } \nabla_p(H) \geq 2\}$ and $\alpha = \alpha(H)$.

Lemma 2.1 *Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. Then*

(a) $d_k \geq k - \alpha$;

(b) If there is an h with $0 \leq h \leq k-\alpha-1$ so that $d_h \geq k-1$ and $d_{h+1} \leq k-2$, then there is an $F < H$ with $|V(F)| = k-h$ so that $(d_{h+1}-h, \ldots, d_k-h) \geq \pi(F)$.

Proof. (a) If $d_k \leq k-\alpha-1$, by Theorem 2.2, then

$$\begin{aligned}
\sigma(\pi) &= \sum_{i=1}^{k-1} d_i + \sum_{i=k}^{n} d_i \\
&\leq (k-1)(k-2) + \sum_{i=k}^{n} \min\{k-1, d_i\} + \sum_{i=k}^{n} d_i \\
&= (k-1)(k-2) + 2\sum_{i=k}^{n} d_i \\
&\leq (k-1)(k-2) + 2(n-k+1)(k-\alpha-1) \\
&= 2(k-\alpha-1)n + (k-1)(2\alpha-k).
\end{aligned}$$

However, if $\widetilde{\sigma}_{\alpha+1}(H) < \widetilde{\sigma}(H)$ or $\nabla_{\alpha+1}(H) \geq 2$, then $\widetilde{\sigma}(H) > 2(k-\alpha-1)$. This implies that $\sigma(\pi) \geq \Sigma > 2(k-\alpha-1)n + (k-1)(2\alpha-k)$, a contradiction.

(b) If $d_k \leq h + \nabla_{k-h}(H) - 2$, then

$$\begin{aligned}
\sigma(\pi) &\leq (n-1)h + (k-2)(k-h-1) + (h + \nabla_{k-h}(H) - 2)(n-k+1) \\
&= (2h + \nabla_{k-h}(H) - 2)n - h + (k-2)(k-h-1) - (h + \nabla_{k-h}(H) - 2)(k-1) \\
&< \sigma(\widetilde{\pi}_{k-h}(H,n)) + 2 \\
&\leq \sigma(\pi^*_{k-h}(H,n)) + 2,
\end{aligned}$$

a contradiction. Hence $d_k \geq h + \nabla_{k-h}(H) - 1$. If $d_k \geq h + \nabla_{k-h}(H)$, then there is an $F < H$ with $|V(F)| = k-h$ and $\Delta(F) = \nabla_{k-h}(H)$ so that $(d_{h+1}-h, \ldots, d_k-h) \geq \pi(F)$. Assume $d_k = h + \nabla_{k-h}(H) - 1$. By $\sigma(\pi) \geq \sigma(\pi^*_{k-h}(H,n)) + 2$, we have

$$\begin{aligned}
\sum_{i=1}^{k-h} (d_{h+i} - h) &= \sigma(\pi) - \sum_{i=1}^{h} d_i - h(k-h) - \sum_{i=k+1}^{n} d_i \\
&\geq \sigma(\pi^*_{k-h}(H,n)) + 2 - (n-1)h - h(k-h) - (h + \nabla_{k-h}(H) - 1)(n-k) \\
&\geq \sigma(\rho_{k-h}(H)) + 1.
\end{aligned}$$

It follows from $d_{h+1} - h \leq (k-h) - 2$ and the definition of $\rho_{k-h}(H)$ that there is an $F < H$ with $|V(F)| = k-h$ so that $(d_{h+1}-h, \ldots, d_k-h) \geq \pi(F)$. □

Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. By Lemma 2.1(a), $d_k \geq k - \alpha$. Denote $\pi_0 = (d_1^{(0)}, \ldots, d_n^{(0)})$, where $d_i^{(0)} = d_i$ for $1 \leq i \leq n$. We construct $\pi_1, \ldots, \pi_k$ depending on two cases.

Case 1. $d_{k-\alpha} \geq k - 1$.

For $i = 1, \ldots, k$ in turn, we construct $\pi_i = (d_{i+1}^{(i)}, \ldots, d_k^{(i)}, d_{k+1}^{(i)}, \ldots, d_n^{(i)})$, by deleting $d_i^{(i-1)}$ from $\pi_{i-1} = (d_i^{(i-1)}, \ldots, d_k^{(i-1)}, d_{k+1}^{(i-1)}, \ldots, d_n^{(i-1)})$, reducing the first $d_i^{(i-1)}$ nonzero remaining terms of π_{i-1} by one, and then reordering the last $n - k$ terms to be non-increasing.

Case 2. There is an h with $0 \leq h \leq k - \alpha - 1$ so that $d_h \geq k - 1$ and $d_{h+1} \leq k - 2$.

By Lemma 2.1(b), there is an $F < H$ with $|V(F)| = k - h$ so that $(d_{h+1} - h, \ldots, d_k - h) \geq \pi(F) = (d_1, \ldots, d_p)$. Let $d_{h+j} - h = f_j + d_j$ for $j = 1, \ldots, k - h$. In this case, we first construct π_i, $1 \leq i \leq h$ as above, and then we construct π_i, $h + 1 \leq i \leq k$ from π_{i-1} by deleting $d_i^{(i-1)}$, reducing the first f_{i-h} nonzero terms, *starting with* $d_{k+1}^{(i-1)}$ by one, and then reordering the last $n - k$ terms to be non-increasing.

Thus by Lemmas 2.2 and 2.3 of [14], the following Lemmas 2.2 and 2.3 are obvious and immediately.

Lemma 2.2 [14] *Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$, and let $d_{k-\alpha} \geq k - 1$. Then*

(i) If $\pi_{k-\alpha-1} = (d_{k-\alpha}^{(k-\alpha-1)}, \ldots, d_k^{(k-\alpha-1)}, d_{k+1}^{(k-\alpha-1)}, \ldots, d_n^{(k-\alpha-1)})$ satisfies $d_k^{(k-\alpha-1)} \geq d_{k+1}^{(k-\alpha-1)}$, then π is potentially H-graphic;

(ii) If $d_k^{(k-\alpha-1)} < d_{k+1}^{(k-\alpha-1)}$ and π_k is graphic, then π is potentially H-graphic.

Lemma 2.3 [14] *Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. Assume that there is an h with $0 \leq h \leq k - \alpha - 1$ so that $d_h \geq k - 1$ and $d_{h+1} \leq k - 2$. If π_k is graphic, then π is potentially H-graphic.*

Lemma 2.4 *Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. If $d_{k-\alpha} \geq k - 1$, then π is potentially H-graphic.*

Proof. By Theorem 2.4, we may assume $d_k \leq 2k - 4$. By Lemma 2.2, we further assume $d_k^{(k-\alpha-1)} < d_{k+1}^{(k-\alpha-1)}$, and only need to check that π_k is graphic. Let $d_k^{(\ell)} < d_{k+1}^{(\ell)}$ so that ℓ is minimal. Then $1 \leq \ell \leq k - \alpha - 1$ and $d_k^{(\ell-1)} \geq d_{k+1}^{(\ell-1)}$. Moreover, $d_k^{(j)} \geq d_{k+1}^{(j)}$ for $1 \leq j \leq \ell - 1$. This implies that π_{j+1} is the residual sequence of π_j for $0 \leq j \leq \ell - 2$, where $\pi_0 = \pi$. By Theorem 2.1, π_j is graphic for $1 \leq j \leq \ell - 1$. Moreover, by $\widetilde{\sigma}_{\alpha+1}(H) < \widetilde{\sigma}(H)$ or $\nabla_{\alpha+1}(H) \geq 2$, $\pi_{\ell-1} = (d_\ell^{(\ell-1)}, \ldots, d_n^{(\ell-1)})$ satisfies that $d_\ell^{(\ell-1)} \geq \cdots \geq d_n^{(\ell-1)}$, $d_j^{(\ell-1)} = d_j - (\ell - 1)$ for $j = \ell, \ldots, k$,

$$\begin{array}{rcl} \sigma(\pi_{\ell-1}) & = & \sigma(\pi) - 2d_1 - 2(d_2 - 1) - \cdots - 2(d_{\ell-1} - \ell + 2) \\ & \geq & \sigma(\pi) - 2(n-1) - 2(n-2) - \cdots - 2(n - \ell + 1) \\ & = & \sigma(\pi) - 2(\ell - 1)n + \ell(\ell - 1) \\ & > & 2(k - \alpha - 1)n - 2(\ell - 1)n \\ & = & 2(k - \ell - \alpha)n \\ & \geq & 2n, \end{array}$$

and

$$n - \ell - 1 \geq d_\ell^{(\ell-1)} \geq \cdots \geq d_k^{(\ell-1)} = \cdots = d_{(\ell-1)+d_\ell^{(\ell-1)}+2}^{(\ell-1)} \geq d_{(\ell-1)+d_\ell^{(\ell-1)}+3}^{(\ell-1)} \geq \cdots \geq d_n^{(\ell-1)}.$$

The rest proof is the same as the proof of Lemma 2.4 of [14], we omit it here. □

Lemma 2.5 *Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. If there is an h with $0 \leq h \leq k - \alpha - 1$ so that $d_h \geq k - 1$ and $d_{h+1} \leq k - 2$, then π is potentially H-graphic.*

Proof. By Lemma 2.3, it is enough to check that π_k is graphic. If $\nabla_{\alpha+1}(H) \geq 2$, then $\nabla_{k-h}(H) \geq 2$, and hence

$$\begin{array}{rcl} \sigma(\pi_h) & = & \sigma(\pi) - 2d_1 - 2(d_2 - 1) - \cdots - 2(d_h - h + 1) \\ & \geq & \sigma(\pi) - 2(n-1) - 2(n-2) - \cdots - 2(n - h) \\ & = & \sigma(\pi) - 2hn + h(h+1) \\ & > & \widetilde{\sigma}(H)n - 2hn - \frac{n}{2} \\ & \geq & \widetilde{\sigma}_{k-h}(H)n - 2hn - \frac{n}{2} \\ & = & (\nabla_{k-h}(H) - 1)n - \frac{n}{2} \\ & \geq & \frac{n}{2}. \end{array}$$

If $\widetilde{\sigma}_{\alpha+1}(H) < \widetilde{\sigma}(H)$, then similarly

$$\begin{array}{rcl}\sigma(\pi_h) & \geq & \sigma(\pi) - 2(n-1) - 2(n-2) - \cdots - 2(n-h) \\ & = & \sigma(\pi) - 2hn + h(h+1) \\ & > & \widetilde{\sigma}(H)n - 2hn - \frac{n}{2} \\ & \geq & \widetilde{\sigma}_{\alpha+1}(H)n - 2hn + \frac{n}{2} \\ & \geq & 2(k-\alpha-1-h)n + \frac{n}{2} \\ & \geq & \frac{n}{2}.\end{array}$$

Thus by $(p(\pi_h) - h)(k-2) \geq \sigma(\pi_h)$, we can see that $p(\pi_h) - h$ is sufficiently large. This implies that $p(\pi_k) - k$ is also sufficiently large as $f_j \leq k-2$ for $1 \leq j \leq k-h$. Therefore, π_k is graphic by $d_{k+1}^{(k)} \leq k-2$ and Theorem 2.3. □

Proof of Theorem 1.4. Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq \Sigma$. It is enough to show that π is potentially H-graphic. It is trivial for $k = 1, 2$. If $\alpha = k$, then $H = \overline{K_k}$, and so π is clearly potentially H-graphic. Assume $k \geq 3$ and $\alpha \leq k-1$. If $d_{k-\alpha} \geq k-1$, by Lemma 2.4, then π is potentially H-graphic. If there is an h with $0 \leq h \leq k-\alpha-1$ so that $d_h \geq k-1$ and $d_{h+1} \leq k-2$, by Lemma 2.5, then π is potentially H-graphic. □

3. Proof of Theorem 1.5

In this section, we always assume that H is a graph of order k with $\nabla_{\alpha(H)+1}(H) = 1$, $\widetilde{\sigma}_{\alpha(H)+1}(H) = \widetilde{\sigma}(H)$ and $\widetilde{\sigma}_p(H) < \widetilde{\sigma}(H)$ for $\alpha(H)+2 \leq p \leq k$, n is a sufficiently large integer relative to k and $\alpha(H)$ and there is an $F < H$ with $|V(F)| = \alpha(H)+1$ so that $\pi(F) = (1^2, 0^{\alpha(H)-1})$. Clearly, $\widetilde{\sigma}_{\alpha(H)+1}(H) = 2(k-\alpha-1)$. We also need some lemmas. For convenience, we denote $\alpha = \alpha(H)$.

Lemma 3.1 *Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq (k-\alpha-1)(2n-k+\alpha)+2$. Then*

(a) $d_k \geq k-\alpha-1$;

(b) $d_{k-\alpha+1} \geq k-\alpha$;

(c) If there is an h with $0 \leq h \leq k-\alpha-2$ so that $d_h \geq k-1$ and $d_{h+1} \leq k-2$, then there is an $F < H$ with $|V(F)| = k-h$ and $\Delta(F) = \nabla_{k-h}(H)$ so that $(d_{h+1}-h, \ldots, d_k-h) \geq \pi(F)$.

Proof. (a) If $d_k \leq k-\alpha-2$, by Theorem 2.2, then

$$\begin{array}{rcl}\sigma(\pi) & = & \sum\limits_{i=1}^{k-1} d_i + \sum\limits_{i=k}^{n} d_i \\ & \leq & (k-1)(k-2) + \sum\limits_{i=k}^{n} \min\{k-1, d_i\} + \sum\limits_{i=k}^{n} d_i \\ & = & (k-1)(k-2) + 2\sum\limits_{i=k}^{n} d_i \\ & \leq & (k-1)(k-2) + 2(n-k+1)(k-\alpha-2) \\ & = & 2(k-\alpha-2)n + (k-1)(2\alpha-k+2) \\ & < & \sigma(\pi),\end{array}$$

a contradiction.

(b) If $d_{k-\alpha+1} \leq k-\alpha-1$, by Theorem 2.2, then

$$\begin{array}{rcl}\sigma(\pi) & = & \sum\limits_{i=1}^{k-\alpha} d_i + \sum\limits_{i=k-\alpha+1}^{n} d_i \\ & \leq & (k-\alpha)(k-\alpha-1) + \sum\limits_{i=k-\alpha+1}^{n} \min\{k-\alpha, d_i\} + \sum\limits_{i=k-\alpha+1}^{n} d_i \\ & = & (k-\alpha)(k-\alpha-1) + 2\sum\limits_{i=k-\alpha+1}^{n} d_i \\ & \leq & (k-\alpha)(k-\alpha-1) + 2(n-k+\alpha)(k-\alpha-1) \\ & = & (k-\alpha-1)(2n-k+\alpha) \\ & < & \sigma(\pi),\end{array}$$

a contradiction.

(c) If $d_k \le h + \nabla_{k-h}(H) - 1$, by $\widetilde{\sigma}_{\alpha+1}(H) = \widetilde{\sigma}(H) > \widetilde{\sigma}_{k-h}(H)$, then

$$\begin{array}{rcl}\sigma(\pi) & \le & (n-1)h + (k-2)(k-h-1) + (h + \nabla_{k-h}(H) - 1)(n-k+1) \\ & = & (2h + \nabla_{k-h}(H) - 1)n - h + (k-2)(k-h-1) - (h + \nabla_{k-h}(H) - 1)(k-1) \\ & = & \widetilde{\sigma}_{k-h}(H)n - h + (k-2)(k-h-1) - (h + \nabla_{k-h}(H) - 1)(k-1) \\ & < & \sigma(\widetilde{\pi}_{\alpha+1}(H, n)) \\ & = & (k-\alpha-1)(2n-k+\alpha),\end{array}$$

a contradiction. Hence $d_k \ge h + \nabla_{k-h}(H)$. This implies that there is an $F < H$ with $|V(F)| = k - h$ and $\Delta(F) = \nabla_{k-h}(H)$ so that $(d_{h+1} - h, \dots, d_k - h) \ge \pi(F)$. □

Let $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) \ge (k-\alpha-1)(2n-k+\alpha)+2$. By Lemma 3.1(a), $d_k \ge k-\alpha-1$. Denote $\pi_0 = (d_1^{(0)}, \dots, d_n^{(0)})$, where $d_i^{(0)} = d_i$ for $1 \le i \le n$. We construct $\pi_1, \dots, \pi_k$ depending on two cases.

Case 1. $d_{k-\alpha-1} \ge k-1$.

For $i = 1, \dots, k$ in turn, we construct $\pi_i = (d_{i+1}^{(i)}, \dots, d_k^{(i)}, d_{k+1}^{(i)}, \dots, d_n^{(i)})$, by deleting $d_i^{(i-1)}$ from $\pi_{i-1} = (d_i^{(i-1)}, \dots, d_k^{(i-1)}, d_{k+1}^{(i-1)}, \dots, d_n^{(i-1)})$, reducing the first $d_i^{(i-1)}$ nonzero remaining terms of π_{i-1} by one, and then reordering the last $n-k$ terms to be non-increasing.

Case 2. There is an h with $0 \le h \le k-\alpha-2$ so that $d_h \ge k-1$ and $d_{h+1} \le k-2$.

By Lemma 3.1(c), there is an $F < H$ with $|V(F)| = k-h$ and $\Delta(F) = \nabla_{k-h}(H)$ so that $(d_{h+1}-h, \dots, d_k - h) \ge \pi(F) = (d_1, \dots, d_p)$. Let $d_{h+j} - h = f_j + d_j$ for $j = 1, \dots, k-h$. In this case, we first construct π_i, $1 \le i \le h$ as above, and then we construct π_i, $h+1 \le i \le k$ from π_{i-1} by deleting $d_i^{(i-1)}$, reducing the first f_{i-h} nonzero terms, *starting with* $d_{k+1}^{(i-1)}$ by one, and then reordering the last $n-k$ terms to be non-increasing.

Thus by Lemmas 2.2 and 2.3 of [14], and Lemma 3.1(b), the following Lemmas 3.2 and 3.3 are also obvious and immediately.

Lemma 3.2 [14] *Let $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) \ge (k-\alpha-1)(2n-k+\alpha)+2$, and let $d_{k-\alpha-1} \ge k-1$. Then*

(i) If $\pi_{k-\alpha-1} = (d_{k-\alpha}^{(k-\alpha-1)}, \dots, d_k^{(k-\alpha-1)}, d_{k+1}^{(k-\alpha-1)}, \dots, d_n^{(k-\alpha-1)})$ satisfies $d_k^{(k-\alpha-1)} \ge d_{k+1}^{(k-\alpha-1)}$, then π is potentially H-graphic;

(ii) If $d_k^{(k-\alpha-1)} < d_{k+1}^{(k-\alpha-1)}$ and π_k is graphic, then π is potentially H-graphic.

Lemma 3.3 [14] *Let $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) \ge (k-\alpha-1)(2n-k+\alpha)+2$. Assume that there is an h with $0 \le h \le k-\alpha-2$ so that $d_h \ge k-1$ and $d_{h+1} \le k-2$. If π_k is graphic, then π is potentially H-graphic.*

Lemma 3.4 *Let $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) \ge (k-\alpha-1)(2n-k+\alpha)+2$. If $d_{k-\alpha-1} \ge k-1$, then π is potentially H-graphic.*

Proof. By Theorem 2.4, we may assume $d_k \le 2k-4$. By Lemma 3.2, we further assume $d_k^{(k-\alpha-1)} < d_{k+1}^{(k-\alpha-1)}$, and only need to check that π_k is graphic. Let $d_k^{(\ell)} < d_{k+1}^{(\ell)}$ so that ℓ is minimal. Then $1 \le \ell \le k-\alpha-1$ and $d_k^{(\ell-1)} \ge d_{k+1}^{(\ell-1)}$. Moreover, $d_k^{(j)} \ge d_{k+1}^{(j)}$ for $1 \le j \le \ell-1$. This implies that π_{j+1} is the residual sequence of π_j for $0 \le j \le \ell-2$, where $\pi_0 = \pi$. By Theorem 2.1, π_j is graphic for $1 \le j \le \ell-1$. Moreover, $\pi_{\ell-1} = (d_\ell^{(\ell-1)}, \dots, d_n^{(\ell-1)})$ satisfies that $d_\ell^{(\ell-1)} \ge \cdots \ge d_n^{(\ell-1)}$, $d_j^{(\ell-1)} = d_j - (\ell-1)$ for $j = \ell, \dots, k$,

$$\begin{array}{rcl}\sigma(\pi_{\ell-1}) & \ge & \sigma(\pi) - 2(n-1) - 2(n-2) - \cdots - 2(n-\ell+1) \\ & = & \sigma(\pi) - 2(\ell-1)n + \ell(\ell-1) \\ & \ge & (k-\alpha-1)(2n-k+\alpha) + 2 - 2(\ell-1)n + \ell(\ell-1) \\ & > & n,\end{array}$$

and

$$n - \ell - 1 \ge d_\ell^{(\ell-1)} \ge \cdots \ge d_k^{(\ell-1)} = \cdots = d_{(\ell-1)+d_\ell^{(\ell-1)}+2}^{(\ell-1)} \ge d_{(\ell-1)+d_\ell^{(\ell-1)}+3}^{(\ell-1)} \ge \cdots \ge d_n^{(\ell-1)}.$$

The rest proof is the same as the proof of Lemma 2.4 of [14], we omit it here. □

Lemma 3.5 *Let $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) \ge (k-\alpha-1)(2n-k+\alpha)+2$. If there is an h with $0 \le h \le k-\alpha-2$ so that $d_h \ge k-1$ and $d_{h+1} \le k-2$, then π is potentially H-graphic.*

Proof. By Lemma 3.3, it is enough to check that π_k is graphic. Clearly,

$$\begin{array}{rcl}\sigma(\pi_h) & \geq & \sigma(\pi) - 2(n-1) - 2(n-2) - \cdots - 2(n-h) \\ & = & \sigma(\pi) - 2hn + h(h+1) \\ & \geq & (k-\alpha-1)(2n-k+\alpha) + 2 - 2hn + h(h+1) \\ & \geq & n.\end{array}$$

Thus by $(p(\pi_h) - h)(k-2) \geq \sigma(\pi_h)$, we can see that $p(\pi_h) - h$ is sufficiently large. This implies that $p(\pi_k) - k$ is also sufficiently large as $f_j \leq k-2$ for $1 \leq j \leq k-h$. Therefore, π_k is graphic by $d_{k+1}^{(k)} \leq k-2$ and Theorem 2.3. □

Proof of Theorem 1.5. Let $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq (k-\alpha-1)(2n-k+\alpha)+2$. It is enough to show that π is potentially H-graphic. It is trivial for $k = 1, 2$. If $\alpha = k$, then $H = \overline{K_k}$, and so π is clearly potentially H-graphic. Assume $k \geq 3$ and $\alpha \leq k-1$. If $d_{k-\alpha-1} \geq k-1$, by Lemma 3.4, then π is potentially H-graphic. If there is an h with $0 \leq h \leq k-\alpha-2$ so that $d_h \geq k-1$ and $d_{h+1} \leq k-2$, by Lemma 3.5, then π is potentially H-graphic. □

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