



A Note on the Schwarz Lemma for Harmonic Functions

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Abstract. In this note we consider some generalizations of the Schwarz lemma for harmonic functions on the unit disk, whereby values of such functions and the norms of their differentials at the point $z = 0$ are given.

1. Introduction

1.1. A summary of some results

In this paper we consider some generalizations of the Schwarz lemma for harmonic functions from the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ to the interval $(-1, 1)$ (or to itself).

First, we cite a theorem which is known as the Schwarz lemma for harmonic functions and is considered a classical result.

Theorem 1 ([10],[9, p.77]). *Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a harmonic function such that $f(0) = 0$. Then*

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|, \quad \text{for all } z \in \mathbb{U},$$

and this inequality is sharp for each point $z \in \mathbb{U}$.

In 1977, H. W. Hethcote [11] improved this result by removing the assumption $f(0) = 0$ and proved the following theorem.

Theorem 2 ([11, Theorem 1] and [29, Theorem 3.6.1]). *Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a harmonic function. Then*

$$\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0) \right| \leq \frac{4}{\pi} \arctan |z|, \quad \text{for all } z \in \mathbb{U}.$$

As was written in [25], it seems that researchers had some difficulties handling the case $f(0) \neq 0$, where f is a harmonic mapping from \mathbb{U} to itself. Before we can explain the essence of these difficulties, it is necessary to recall a particular mapping and some of its properties. We emphasize that this mapping and its properties have an important role in our results.

Let $\alpha \in \mathbb{U}$ be arbitrary. Then for $z \in \mathbb{U}$ we define $\varphi_\alpha(z) = \frac{\alpha + z}{1 + \bar{\alpha}z}$. It is well known that φ_α is a conformal automorphism of \mathbb{U} . Also, for $\alpha \in (-1, 1)$ we have

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1° φ_α is increasing on $(-1, 1)$ and maps $(-1, 1)$ onto itself;

2° $\varphi_\alpha([-r, r]) = [\varphi_\alpha(-r), \varphi_\alpha(r)] = \left[\frac{\alpha - r}{1 - \alpha r}, \frac{\alpha + r}{1 + \alpha r} \right]$, where $r \in [0, 1)$.

Now we can explain the previously mentioned difficulties. If f is a holomorphic mapping from \mathbb{U} to \mathbb{U} , such that $f(0) = b$, then using the mapping $g = \varphi_{-b} \circ f$ we can reduce the problem to the case $f(0) = 0$. But, if f is a harmonic mapping from \mathbb{U} to \mathbb{U} such that $f(0) = b$, then the mapping $g = \varphi_{-b} \circ f$ does not have to be a harmonic mapping.

In a previous joint work [25] of the author with M. Mateljević, the Theorem 1 was proved in a different way than previously found in the literature (for example, see [10] and [9]). Modifying that proof, the following theorem (which can be considered an improvement of the H. W. Hethcote result) has also been proved in [25].

Theorem 3 ([25, Theorem 6]). Let $u : \mathbb{U} \rightarrow (-1, 1)$ be a harmonic function such that $u(0) = b$. Then

$$\frac{4}{\pi} \arctan \varphi_a(-|z|) \leq u(z) \leq \frac{4}{\pi} \arctan \varphi_a(|z|), \quad \text{for all } z \in \mathbb{U}.$$

Here $a = \tan \frac{b\pi}{4}$. Also, these inequalities are both sharp at each point $z \in \mathbb{U}$.

As one corollary of Theorem 3 it is possible to prove the following theorem.

Theorem 4 ([26, Theorem 1]). Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a harmonic function such that $f(0) = b$. Then

$$|f(z)| \leq \frac{4}{\pi} \arctan \varphi_A(|z|), \quad \text{for all } z \in \mathbb{U}.$$

Here $A = \tan \frac{|b|\pi}{4}$.

This paper expands on that previous research. We give further generalizations of Theorems 3 and 4. These generalizations (see Theorems 11 and 12) consist of considering harmonic functions on the unit disk \mathbb{U} with following additional conditions:

- 1) the value at the point $z = 0$ is given;
- 2) the norm of its differential at the point $z = 0$ is given.

In the literature one can find the following two generalizations of the Schwarz lemma for holomorphic functions.

Theorem 5 ([17, Proposition 2.2.2 (p. 32)]. Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a holomorphic function. Then

$$|f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}, \quad \text{for all } z \in \mathbb{U}.$$

The following theorem is in fact a corollary of Theorem 5, by considering the holomorphic function

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \in \mathbb{U} \setminus \{0\}, \\ 0, & z = 0. \end{cases}$$

Theorem 6 ([17, Proposition 2.6.3 (p. 60)], [28, Lemma 2]). Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a holomorphic function such that $f(0) = 0$. Then

$$|f(z)| \leq |z| \frac{|f'(0)| + |z|}{1 + |f'(0)||z|}, \quad \text{for all } z \in \mathbb{U}.$$

S. G. Krantz in his book [17] attributes Theorem 5 to Lindelöf. Note that Theorem 4 could be considered a harmonic version of Theorem 5. Similarly, one of the main results of this paper (Theorem 12) could be considered a harmonic version of Theorem 6.

1.2. Hyperbolic metric and the Schwarz-Pick type estimates

By Ω we denote a simply connected plane domain different from \mathbb{C} (we call these domains hyperbolic). By Riemann's Mapping Theorem, it follows that any such domain is conformally equivalent to the unit disk \mathbb{U} . The domain Ω is also equipped with the hyperbolic metric $\rho_{\Omega}(z)|dz|$. More precisely, by definition we have

$$\rho_{\mathbb{U}}(z) = \frac{2}{1 - |z|^2}$$

and if $f : \Omega \rightarrow \mathbb{U}$ a conformal isomorphism, then also by definition, we have

$$\rho_{\Omega}(w) = \rho_{\mathbb{U}}(f(w))|f'(w)|.$$

The hyperbolic metric induces a hyperbolic distance on Ω in the following way

$$d_{\Omega}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \rho_{\Omega}(z)|dz|,$$

where the infimum is taken over all C^1 curves γ joining z_1 to z_2 in Ω . For example, one can show that

$$d_{\mathbb{U}}(z_1, z_2) = 2 \operatorname{artanh} \left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right|,$$

where $z_1, z_2 \in \mathbb{U}$.

Hyperbolic metric and hyperbolic distance do not increase under a holomorphic function. More precisely, the following well-known theorem holds.

Theorem 7 (The Schwarz-Pick lemma for simply connected domains, [3, Theorem 6.4.]). *Let Ω_1 and Ω_2 be hyperbolic domains and $f : \Omega_1 \rightarrow \Omega_2$ be a holomorphic function. Then*

$$\rho_{\Omega_2}(f(z))|f'(z)| \leq \rho_{\Omega_1}(z), \quad \text{for all } z \in \Omega_1, \quad (1)$$

and

$$d_{\Omega_2}(f(z_1), f(z_2)) \leq d_{\Omega_1}(z_1, z_2), \quad \text{for all } z_1, z_2 \in \Omega_1. \quad (2)$$

If f is a conformal isomorphism from Ω_1 onto Ω_2 then in (1) and (2) equalities hold. On the other hand if either equality holds in (1) at one point z or for a pair of distinct points in (2) then f is a conformal isomorphism from Ω_1 onto Ω_2 .

For a holomorphic function $f : \Omega_1 \rightarrow \Omega_2$ (where Ω_1 and Ω_2 are hyperbolic domains) the *hyperbolic derivative* of f at $z \in \Omega_1$ (for motivation and details see Section 5 in [3], cf. [2]) is defined as follows:

$$f^h(z) = \frac{\rho_{\Omega_2}(f(z))}{\rho_{\Omega_1}(z)} f'(z).$$

Note that by Theorem 7 we also have $|f^h(z)| \leq 1$ for all $z \in \Omega_1$.

Using this notion, in 1992, A. F. Beardon and T. K. Carne proved the following theorem, which is stronger than Theorem 7.

Theorem 8 ([2]). *Let Ω_1 and Ω_2 be hyperbolic domains and $f : \Omega_1 \rightarrow \Omega_2$ be a holomorphic function. Then for all $z, w \in \Omega_1$,*

$$d_{\Omega_2}(f(z), f(w)) \leq \log(\cosh d_{\Omega_1}(z, w) + |f^h(w)| \sinh d_{\Omega_1}(z, w)). \quad (3)$$

Let us note that Theorem 8 is of crucial importance for our research (see proof of Theorem 11).

Note that in [2], Theorem 8 is formulated and proved for $\Omega_1 = \Omega_2 = \mathbb{U}$. Using the fact that Ω_1 and Ω_2 are conformally equivalent to \mathbb{U} , one can easily prove that this result remains valid for hyperbolic domains.

Next, since for all $t \in \mathbb{R}$ we have $\log(\cosh t + \sinh t) = \log(e^t) = t$, it follows that for $|f^h(w)| = 1$ inequality (3) becomes (1). On the other hand, for all $t \in [0, +\infty)$ the function $h : [0, 1] \rightarrow [0, +\infty)$ defined by $h(a) = \log(\cosh t + a \sinh t)$ is monotonically increasing. Hence, if $|f^h(w)| < 1$ then (3) is stronger inequality than (1).

There are many papers where authors have considered various versions of Schwarz-Pick type estimates for harmonic functions and related problems (see [13], [4], [16], [8], [12], [6], [19], [27] and [18]). In this regard, we note that M. Mateljević [24] cf. [23] recently explained one method (we will refer to it as the strip method) which enabled that some of these results to be proven in an elegant way.

For completeness we will shortly reproduce the strip method. In order to do so, we will first introduce the appropriate notation and specify some simple facts.

By \mathbb{S} we denote the strip $\{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1\}$. The mapping φ defined by $\varphi(z) = \tan\left(\frac{\pi}{4}z\right)$ is a conformal isomorphism from \mathbb{S} onto \mathbb{U} and by ϕ we denote the inverse mapping of φ (see also Example 1 in [25]). Throughout this paper by φ and ϕ we always denote these mappings.

Using the mapping φ one can derive the following equality

$$\rho_{\mathbb{S}}(z) = \rho_{\mathbb{U}}(\varphi(z))|\varphi'(z)| = \frac{\pi}{2} \frac{1}{\cos\left(\frac{\pi}{2}\operatorname{Re} z\right)}, \quad \text{for all } z \in \mathbb{S}.$$

By ∇u we denote the gradient of real-valued C^1 function u , i.e. $\nabla u = (u_x, u_y) = u_x + iu_y$. If $f = u + iv$ is complex-valued C^1 function, where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, then we use notation

$$f_x = u_x + iv_x \quad \text{and} \quad f_y = u_y + iv_y,$$

as well as

$$f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

Finally, by $df(z)$ we denote differential of the function f at point z , i.e. the Jacobian matrix

$$\begin{pmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{pmatrix}.$$

The matrix $df(z)$ is an \mathbb{R} -linear operator from the tangent space $T_z\mathbb{R}^2$ to the tangent space $T_{f(z)}\mathbb{R}^2$. By $\|df(z)\|$ we denote norm of this operator. It is not difficult to prove that $\|df(z)\| = |f_z(z)| + |f_{\bar{z}}(z)|$.

Briefly, the strip method consists of the following elementary considerations (see [24, 25]):

(I) Suppose that $f : \mathbb{U} \rightarrow \mathbb{S}$ be a holomorphic function. Then by Theorem 7 we have

$$\rho_{\mathbb{S}}(f(z))|f'(z)| \leq \rho_{\mathbb{U}}(z),$$

for all $z \in \mathbb{U}$.

(II) If $f = u + iv$ is a harmonic function and $F = U + iV$ is a holomorphic function on a domain D such that $\operatorname{Re} f = \operatorname{Re} F$ on D (in this setting we say that F is associated to f or to u), then

$$F' = U_x + iV_x = U_x - iU_y = u_x - iu_y.$$

Hence $F' = \overline{\nabla u}$ and $|F'| = |\overline{\nabla u}| = |\nabla u|$.

(III) Suppose that D is a simply connected plane domain and $f : D \rightarrow \mathbb{S}$ is a harmonic function. Then it is known from the standard course of complex analysis that there is a holomorphic function F on D such that $\operatorname{Re} f = \operatorname{Re} F$ on D , and it is clear that $F : D \rightarrow \mathbb{S}$.

(IV) The hyperbolic density $\rho_{\mathbb{S}}$ at point z depends only on $\operatorname{Re} z$.

From (I)-(IV) we immediately obtain:

Theorem 9 ([24, Proposition 2.4], [12],[6]). *Let $u : \mathbb{U} \rightarrow (-1, 1)$ be a harmonic function and let F be a holomorphic function which is associated to u . Then*

$$\rho_{\mathbb{S}}(u(z))|\nabla u(z)| = \rho_{\mathbb{S}}(F(z))|F'(z)| \leq \rho_{\mathbb{U}}(z), \quad \text{for all } z \in \mathbb{U}.$$

In other words

$$|\nabla u(z)| \leq \frac{4}{\pi} \frac{\cos\left(\frac{\pi}{2}u(z)\right)}{1 - |z|^2}, \quad \text{for all } z \in \mathbb{U}. \quad (4)$$

If u is the real part of a conformal isomorphism from \mathbb{U} onto \mathbb{S} then in (4) equality holds for all $z \in \mathbb{U}$ and vice versa.

In 1989, F. Colonna [8] proved the following version of the Schwarz-Pick lemma for harmonic functions.

Theorem 10 ([8, Theorem 3] and [24, Proposition 2.8], cf. [1, Theorem 6.26]). *Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a harmonic function. Then*

$$\|df(z)\| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}, \quad \text{for all } z \in \mathbb{U}. \quad (5)$$

In particular,

$$\|df(0)\| \leq \frac{4}{\pi}. \quad (6)$$

Remark 1. *The inequality (5) is sharp in the following sense: for all $z \in \mathbb{U}$ there exists a harmonic function $f_{[z]} : \mathbb{U} \rightarrow \mathbb{U}$ (which depends on z) such that*

$$\|df_{[z]}(z)\| = \frac{4}{\pi} \frac{1}{1 - |z|^2}.$$

One such function is defined by $f_{[z]}(\zeta) = \operatorname{Re}(\phi(\varphi_{-z}(\zeta)))$. For more details see Theorem 4 in [8].

Remark 2. *The inequality (6) could not be improved even if we add the assumption that $f(0) = 0$. More precisely, if $f(\zeta) = \operatorname{Re} \phi(\zeta)$ then f satisfies all assumptions of Theorem 10, $f(0) = 0$ and $\|df(0)\| = \frac{4}{\pi}$ (see also [24, Proposition 2.8] and [1, Theorem 6.26]).*

Remark 3. *It seems that the question: “Is it possible to improve the inequality (5) if we add the assumption $f(0) = b$, where $b \neq 0$?” is an open problem (see [24, Problem 2]).*

Note that the inequalities (4) and (6) naturally impose assumptions in Theorems 11 and 12 below.

2. Main results

Theorem 11. *Let $u : \mathbb{U} \rightarrow (-1, 1)$ be a harmonic function such that:*

$$(R1) \quad u(0) = b \text{ and}$$

$$(R2) \quad |\nabla u(0)| = d, \text{ where } d \leq \frac{4}{\pi} \cos\left(\frac{\pi}{2}b\right).$$

Then, for all $z \in \mathbb{U}$,

$$\frac{4}{\pi} \arctan \varphi_a(-|z|\varphi_c(|z|)) \leq u(z) \leq \frac{4}{\pi} \arctan \varphi_a(|z|\varphi_c(|z|)). \tag{7}$$

Here $a = \tan \frac{b\pi}{4}$ and $c = \frac{\pi}{4} \frac{1}{\cos \frac{\pi}{2}b} d$. These inequalities are sharp for each point $z \in \mathbb{U}$ in the following sense: for arbitrary $z \in \mathbb{U}$ there exist harmonic functions $\hat{u}_{[z]}, \check{u}_{[z]} : \mathbb{U} \rightarrow (-1, 1)$, which depend on z , such that they satisfy (R1) and (R2) and also

$$\hat{u}_{[z]}(z) = \frac{4}{\pi} \arctan \varphi_a(-|z|\varphi_c(|z|)) \quad \text{and} \quad \check{u}_{[z]}(z) = \frac{4}{\pi} \arctan \varphi_a(|z|\varphi_c(|z|)).$$

Remark 4. Formally, if $c = 1$ then function φ_c is not defined. In this case we mean that $\varphi_c(|z|) = 1$ for all $z \in \mathbb{U}$.

Corollary 1. Let $u : \mathbb{U} \rightarrow (-1, 1)$ be a harmonic function such that $u(0) = 0$ and $\nabla u(0) = (0, 0)$. Then, for all $z \in \mathbb{U}$,

$$|u(z)| \leq \frac{4}{\pi} \arctan |z|^2.$$

Theorem 12. Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a harmonic function such that:

(C1) $f(0) = 0$ and

(C2) $\|df(0)\| = d$, where $d \leq \frac{4}{\pi}$.

Then, for all $z \in \mathbb{U}$

$$|f(z)| \leq \frac{4}{\pi} \arctan (|z|\varphi_C(|z|)), \tag{8}$$

where $C = \frac{\pi}{4}d$.

Corollary 2. Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a harmonic function such that $f(0) = 0$ and $\|df(0)\| = 0$. Then, for all $z \in \mathbb{U}$,

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|^2.$$

Remark 5. Formally, if $C = 1$ then function φ_C is not defined. In this case we mean that $\varphi_C(|z|) = 1$ for all $z \in \mathbb{U}$.

3. Proofs of main results

3.1. Proof of Theorem 11

In order to prove Theorem 11, we recall the following definitions and one lemma from [25].

Let $\lambda > 0$ be arbitrary. By $\overline{D}_\lambda(\zeta) = \{z \in \mathbb{U} : d_{\mathbb{U}}(z, \zeta) \leq \lambda\}$ (respectively $\overline{S}_\lambda(\zeta) = \{z \in \mathbb{S} : d_{\mathbb{S}}(z, \zeta) \leq \lambda\}$) we denote the hyperbolic closed disc in \mathbb{U} (respectively in \mathbb{S}) with hyperbolic center $\zeta \in \mathbb{U}$ (respectively $\zeta \in \mathbb{S}$) and hyperbolic radius λ . Specifically, if $\zeta = 0$ we omit ζ from the notation.

Let $r \in (0, 1)$ be arbitrary. By \overline{U}_r we denote the Euclidean closed disc

$$\{z \in \mathbb{C} : |z| \leq r\}.$$

Also, let

$$\lambda(r) = d_{\mathbb{U}}(r, 0) = \log \frac{1+r}{1-r} = 2 \operatorname{artanh} r.$$

Since $d_{\mathbb{U}}(z, 0) = \log \frac{1 + |z|}{1 - |z|} = 2 \operatorname{artanh} |z|$, for all $z \in \mathbb{U}$, we have

$$\overline{D}_{\lambda(r)} = \{z \in \mathbb{C} : 2 \operatorname{artanh} |z| \leq 2 \operatorname{artanh} r\} = \{z \in \mathbb{C} : |z| \leq r\} = \overline{U}_r.$$

Let $b \in (-1, 1)$ be arbitrary and $a = \tan \frac{b\pi}{4}$. By Theorem 7 we have

$$\overline{S}_{\lambda(r)}(b) = \overline{S}_{\lambda(r)}(\phi(\varphi_a(0))) = \phi(\varphi_a(\overline{D}_{\lambda(r)})) = \phi(\varphi_a(\overline{U}_r)),$$

where ϕ is the conformal isomorphism from \mathbb{U} onto \mathbb{S} defined in subsection 1.2. Further, one can show that (see Figure 1):

- i) $\overline{S}_{\lambda(r)}(b)$ is symmetric with respect to the x -axis;
- ii) $\overline{S}_{\lambda(r)}(b)$ is Euclidean convex (see [3, Theorem 7.11]).

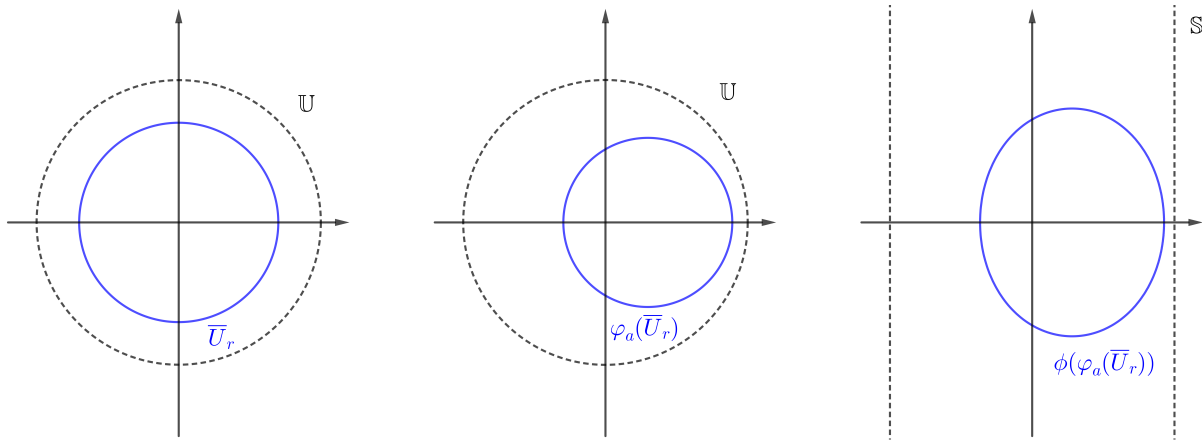


Figure 1: Disks \overline{U}_r , $\varphi_a(\overline{U}_r)$ and $\phi(\varphi_a(\overline{U}_r))$

From i)-ii) we immediately obtain:

Lemma 1 ([25, Lemma 3]). *Let $r \in (0, 1)$ and $b \in (-1, 1)$ be arbitrary. Then*

$$R_e(\overline{S}_{\lambda(r)}(b)) = \left[\frac{4}{\pi} \arctan \varphi_a(-r), \frac{4}{\pi} \arctan \varphi_a(r) \right].$$

Here $a = \tan \frac{b\pi}{4}$ and $R_e : \mathbb{C} \rightarrow \mathbb{R}$ is defined by $R_e(z) = \operatorname{Re} z$.

Proof. [Proof of Theorem 11] Applying the strip method we obtain that there exists holomorphic function $f : \mathbb{U} \rightarrow \mathbb{S}$ such that $\operatorname{Re} f = u$, $f(0) = b$ and $|f'(0)| = d$. Also, we have

$$|f^h(0)| = \frac{\rho_{\mathbb{S}}(f(0))}{\rho_{\mathbb{U}}(0)} |f'(0)| = \frac{\pi}{4} \frac{1}{\cos \frac{\pi}{2} b} d = c.$$

Let $z \in \mathbb{U}$ be arbitrary. By Theorem 8, taking $\Omega_1 = \mathbb{U}$ and $\Omega_2 = \mathbb{S}$, we have

$$d_{\mathbb{S}}(f(z), b) \leq \log(\cosh d_{\mathbb{U}}(z, 0) + |f^h(0)| \sinh d_{\mathbb{U}}(z, 0)) = \log \left(\frac{1 + |z|^2 + 2c|z|}{1 - |z|^2} \right).$$

Now, we chose a point $R(z) \in [0, 1)$ such that

$$d_{\mathbb{U}}(R(z), 0) = \log \left(\frac{1 + |z|^2 + 2c|z|}{1 - |z|^2} \right). \tag{9}$$

Note that the equality (9) is equivalent to the equality

$$\frac{1 + R(z)}{1 - R(z)} = \frac{1 + |z|^2 + 2c|z|}{1 - |z|^2}$$

and hence we obtain $R(z) = |z| \frac{c + |z|}{1 + c|z|} = |z|\varphi_c(|z|)$. Therefore

$$d_{\mathbb{S}}(f(z), b) \leq d_{\mathbb{U}}(|z|\varphi_c(|z|), 0),$$

i.e. $f(z) \in \bar{S}_{\lambda(|z|\varphi_c(|z|))}(b)$. Finally, by Lemma 1

$$u(z) = \operatorname{Re} f(z) \in \left[\frac{4}{\pi} \arctan \varphi_a(-|z|\varphi_c(|z|)), \frac{4}{\pi} \arctan \varphi_a(|z|\varphi_c(|z|)) \right].$$

If $z = 0$ then it is clear that the inequality (7) is sharp.

In order to prove that inequality (7) is sharp in the case $z \in \mathbb{U} \setminus \{0\}$, we first define the functions $\hat{\Phi}, \check{\Phi} : \mathbb{U} \rightarrow \mathbb{S}$ as follows

$$\hat{\Phi}(\zeta) = \phi \left(\varphi_a(-\zeta \cdot \varphi_c(\zeta)) \right)$$

and

$$\check{\Phi}(\zeta) = \phi \left(\varphi_a(\zeta \cdot \varphi_c(\zeta)) \right).$$

Let $z \in \mathbb{U} \setminus \{0\}$. Define the functions $\hat{u}_{[z]}, \check{u}_{[z]} : \mathbb{U} \rightarrow (-1, 1)$ (which depend on z) in the following way:

$$\hat{u}_{[z]}(\zeta) = \operatorname{Re} \hat{\Phi}(e^{-i \arg z} \zeta)$$

and

$$\check{u}_{[z]}(\zeta) = \operatorname{Re} \check{\Phi}(e^{-i \arg z} \zeta).$$

It is easy to check that the functions $\hat{u}_{[z]}$ and $\check{u}_{[z]}$ are harmonic and that they satisfy assumptions (R1) and (R2). Also

$$\hat{u}_{[z]}(z) = \frac{4}{\pi} \arctan \varphi_a(-|z|\varphi_c(|z|))$$

and

$$\check{u}_{[z]}(z) = \frac{4}{\pi} \arctan \varphi_a(|z|\varphi_c(|z|)).$$

□

3.2. Proof of Theorem 12

In order to prove Theorem 12, we need two lemmas.

Lemma 2 ([8, Lemma 1]). *Let $z, w \in \mathbb{C}$. Then*

$$\max_{\theta \in \mathbb{R}} |w \cos \theta + z \sin \theta| = \frac{1}{2} (|w + iz| + |w - iz|).$$

Lemma 3. *Fix $z \in \mathbb{U}$. Function $h : (-1, 1) \rightarrow \mathbb{R}$ defined by $h(t) = \frac{t + |z|}{1 + t|z|}$ is monotonically increasing.*

Proof. The proof follows directly from the fact $h'(t) = \frac{1 - |z|^2}{(1 + t|z|)^2} > 0$ for all $t \in (-1, 1)$. \square

Proof. [Proof of Theorem 12] Denote by u and v real and imaginary part of f , respectively. Let $\theta \in \mathbb{R}$ be arbitrary. It is clear that the function U defined by

$$U(z) = \cos \theta u(z) + \sin \theta v(z)$$

is harmonic on the unit disk \mathbb{U} , $U(0) = 0$ and $|U(z)| \leq |f(z)| < 1$ for all $z \in \mathbb{U}$. By Theorem 11 we have

$$U(z) \leq \frac{4}{\pi} \arctan (|z|\varphi_c(|z|)), \quad \text{for all } z \in \mathbb{U}, \tag{10}$$

where $c = \frac{\pi}{4} |\nabla U(0)|$.

Since

$$\begin{aligned} \nabla U(z) &= \cos \theta \nabla u(z) + \sin \theta \nabla v(z) \\ &= \cos \theta (u_x(z) + iu_y(z)) + \sin \theta (v_x(z) + iv_y(z)), \end{aligned}$$

by Lemma 2 we get

$$\begin{aligned} \max_{\theta \in \mathbb{R}} |\nabla U(z)| &= \max_{\theta \in \mathbb{R}} |\cos \theta (u_x(z) + iu_y(z)) + \sin \theta (v_x(z) + iv_y(z))| \\ &= \frac{1}{2} (|u_x(z) + iu_y(z) + i(v_x(z) + iv_y(z))| + |u_x(z) + iu_y(z) - i(v_x(z) + iv_y(z))|) \\ &= \frac{1}{2} \left(\sqrt{(u_x(z) - v_y(z))^2 + (u_y(z) + v_x(z))^2} + \sqrt{(u_x(z) + v_y(z))^2 + (u_y(z) - v_x(z))^2} \right) \\ &= |f_z(z)| + |f_{\bar{z}}(z)| = \|df(z)\|. \end{aligned}$$

Hence

$$|\nabla U(0)| \leq \|df(0)\|$$

and

$$c = \frac{\pi}{4} |\nabla U(0)| \leq \frac{\pi}{4} \|df(0)\| = \frac{\pi}{4} d = C.$$

By Lemma 3, from (10) we obtain

$$U(z) \leq \frac{4}{\pi} \arctan (|z|\varphi_C(|z|)), \quad \text{for all } z \in \mathbb{U}. \tag{11}$$

Finally, let $z \in \mathbb{U}$ be such that $f(z) \neq 0$ and let θ such that

$$\cos \theta = \frac{u(z)}{|f(z)|} \quad \text{and} \quad \sin \theta = \frac{v(z)}{|f(z)|}.$$

Then $U(z) = |f(z)|$ and hence from (11) we get the inequality (8).

If $z \in \mathbb{U}$ is such that $f(z) = 0$ then the inequality (8) is trivial. \square

4. Appendix

4.1. Harmonic quasiregular mappings and the Schwarz-Pick type estimates

Taking into account Remark 3 we mention some results related to harmonic quasiregular mappings.

Let D and G be domains in \mathbb{C} and $K \geq 1$. A C^1 mapping $f : D \rightarrow G$ we call K -quasiregular mapping if

$$\|df(z)\|^2 \leq K |J_f(z)|, \quad \text{for all } z \in D.$$

Here J_f is the Jacobian determinant of f . In particular, a K -quasiconformal mapping is a K -quasiregular mapping that is also a homeomorphism.

In [16], M. Knežević and M. Mateljević proved the following result (which can be considered as generalization of Theorem 10):

Theorem 13. Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a harmonic K -quasiconformal mapping. Then

$$\|df(z)\| \leq K \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad \text{for all } z \in \mathbb{U}.$$

One result of this type was also obtained by H. H. Chen [5]:

Theorem 14. Let $f : \mathbb{U} \rightarrow \mathbb{U}$ be a harmonic K -quasiconformal mapping. Then

$$\|df(z)\| \leq \frac{4}{\pi} K \frac{\cos(|f(z)|\pi/2)}{1 - |z|^2}, \quad \text{for all } z \in \mathbb{U}.$$

For further results related to harmonic quasiconformal and hyperbolic harmonic quasiconformal mappings we refer the interested reader to [21], [30], [20], [7], [22], [14], [15] and literature cited there.

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