Filomat 34:11 (2020), 3675–3687
https://doi.org/10.2298/FIL2011675L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Nonsingularity and Group Invertibility of Combinations of Two Group Invertible Matrices

Yu Li[a], Kezheng Zuo[a]

[a]*Department of Mathematics, Hubei Normal University, Hubei, Huangshi, China*

Abstract. Let A and B be two group invertible matrices, we study the rank, the nonsingularity and the group invertibility of $A-B$, $AA^\sharp - BB^\sharp$, $c_1A + c_2B$, $c_1A + c_2B + c_3AA^\sharp B$ where c_1, c_2 are nonzero complex numbers. Under some special conditions, the necessary and sufficient conditions of $c_1A + c_2B + c_3AB$ and $c_1A + c_2B + c_3AB + c_4BA$ to be nonsingular and group invertible are presented, which generalized some related results of Benítez, Liu, Koliha and Zuo [4, 17, 19, 25].

1. Introduction

The symbol $\mathbb{C}^{m\times n}$ will denote the set of all $m \times n$ matrices over complex numbers $\mathbb{C}$ and $\mathbb{C}^n$ will denote the linear space of all column vectors of dimension n over $\mathbb{C}$. We use the symbol $\mathbb{C} \setminus \{0\}$ to denote the set of all nonzero complex numbers. For $c \in \mathbb{C}$, we denote by $\bar{c}$ the conjugate of c. For $A \in \mathbb{C}^{m\times n}$, denote by $r(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$, A^*, A^T the rank, the null space, the column space, the conjugate transpose and transpose of A respectively. For $A \in \mathbb{C}^{n\times n}$, we use $\sigma(A)$ to represent the spectrum of A and $\sigma(A) \setminus \{0\}$ to represent the spectrum of A exclusive of zero. I_n will stand for the identity matrix of order n.

Next we review some well known facts about the group inverse of a matrix(for a deeper insight, the interested reader can consult [3, Section 4.4] and [8, Section 3-5]). Let $A \in \mathbb{C}^{n\times n}$. If there exists $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A, XAX = X, AX = XA, \tag{1.1}$$

then it is said that A is group invertible. It can be proved that for a given square matrix A, there is at most one matrix X satisfying (1.1) and is denoted by $A^\sharp$. The group inverse does not exist for all square matrices: a square matrix A has a group inverse if and only if $r(A) = r(A^2)$. Let $\mathbb{C}_n^{GI}$ be the set of all group invertible matrices of order n over $\mathbb{C}$, i.e.,

$$\mathbb{C}_n^{GI} = \{A \in \mathbb{C}^{n\times n} | r(A) = r(A^2)\}.$$

It is easy to see that nonsingular and diagonal matrices are all group invertible matrices. Specially, idempotent matrix ($A^2 = A$) and k-potent matrix ($A^k = A, k \geq 2$) are all group invertible matrices.

2010 *Mathematics Subject Classification*. Primary 15A09; Secondary 15A24
Keywords. rank; group invertibility; invertibility; combination
Received: 22 November 2019; Revised: 05 January 2020; Accepted: 20 January 2020
Communicated by Dragana Cvetković-Ilić
Corresponding author: Kezheng Zuo
Research supported by NSFC of China (Grant No. 11961076)
Email addresses: 335972971@qq.com (Yu Li), xiangzuo28@163.com (Kezheng Zuo)

It is straightforward to prove that A is group invertible if and only if A^* is group invertible, and in this case, one has $(A^\sharp)^* = (A^*)^\sharp$. Also, it should be evident that if $A \in \mathbb{C}^{n\times n}$ and $S \in \mathbb{C}^{n\times n}$ is nonsingular, then A is group invertible if and only if $S^{-1}AS$ is group invertible, and in this situation, one has $(S^{-1}AS)^\sharp = S^{-1}A^\sharp S$.

Let $\mathbb{C}_n^P$ be the set of all idempotent matrices of order n over $\mathbb{C}$, i.e.,

$$\mathbb{C}_n^P = \{A \in \mathbb{C}^{n\times n} | A^2 = A\},$$

and $\mathbb{C}_n^{k-P}$ the set of all k-potent matrices of order n over $\mathbb{C}$, i.e.,

$$\mathbb{C}_n^{k-P} = \{A \in \mathbb{C}^{n\times n} | A^k = A\}.$$

Recently, many researchers have studied the properties of the sum, the difference, the linear combinations and the combinations of two matrices satisfying some conditions. For example, in [1,18,23-24,26], the authors studied some properties of the rank, the null space and the column space of the sum, the difference, the linear combinations and the combinations of two idempotent matrices(orthogonal idempotent matrices). In [2,5-6,9-11,13-14,16-17,20,22,25], the authors described the nonsingularity, Drazin invertibility, group invertibility of the linear combinations of two idempotent matrices. In particular, the authors studied the nonsingularity and group invertibility of linear combinations of two k-potent matrices and two group invertible matrices in [4, 19]. The invertibility of combinations of k-potent operators and commuting generalized and hypergeneralized projectors was considered in [12] and [21]. Under the above work, in this paper, we study the rank, the null space, the column space, the nonsingularity, the group invertibility of $A - B$, $AA^\sharp - BB^\sharp$, $c_1A + c_2B$, $c_1A + c_2B + c_3AA^\sharp B$ of two group invertible matrices. When A and B are satisfying some special conditions, the necessary and sufficient conditions of $c_1A + c_2B + c_3AB$ and $c_1A + c_2B + c_3AB + c_4BA$ to be nonsingular and group invertible are presented, which generalized some related results of Benítez, Liu, Koliha and Zuo [4, 17, 19, 25].

For the proof of main theorems, we need some following lemmas.

Lemma 1.1. *Let $A, B \in \mathbb{C}^{n\times n}$ be given. Then*
(a) $\mathcal{N}(A)^\perp = \mathcal{R}(A^*)$.
(b) $\left(\mathcal{N}(A) + \mathcal{N}(B)\right)^\perp = \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$.
(c) $\left(\mathcal{N}(A) \cap \mathcal{N}(B)\right)^\perp = \mathcal{R}(A^*) + \mathcal{R}(B^*)$.

The following Lemma 1.2 is a well-known result of group invertible matrix, we only lay it out without proof for simplicity.

Lemma 1.2. *Let $A \in \mathbb{C}^{n\times n}$ be given. The following are equivalent:*
(a) $A \in \mathbb{C}_n^{GI}$.
(b) $\mathbb{C}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$.
(c) there exists a nonsingular matrix $X \in \mathbb{C}^n$, such that

$$A = X^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}X,$$

where D is a nonsingular matrix of order r, $r = r(A)$.
(d) the orders of blocks whose diagonal elements are zeros in the Jordan canonical form of A are all ones or $0 \notin \sigma(A)$.

Next, a result of contraction of linear transformation will be given by the notation used in reference [15]. Let $A \in \mathbb{C}^{n\times n}$. Then a linear transformation $\mathcal{A}$ on $\mathbb{C}^n$ can be derived from A as following:

$$\mathcal{A} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$\alpha \mapsto A\alpha, \quad \alpha \in \mathbb{C}^n. \tag{1.2}$$

It is easy to prove that the matrix of $\mathcal{A}$ under the basis $\varepsilon_1 = (1, 0, \cdots, 0)^T, \varepsilon_2 = (0, 1, \cdots, 0)^T, \cdots, \varepsilon_n = (0, 0, \cdots, 1)^T$ of $\mathbb{C}^n$ is A. We call that $\mathcal{A}$ is the linear transformation determined by the matrix A.

Lemma 1.3. *Let $A, B \in \mathbb{C}_n^{GI}$ be such that $AB = BA$. Then there exists a nonsingular matrix X such that*

$$A = X^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}X,\ \ B = X^{-1}\begin{pmatrix} M & 0 \\ 0 & L \end{pmatrix}X,$$

where D is a nonsingular matrix of order r, $r = r(A)$, $DM = MD$ and $\sigma(\mathcal{B}|_{\mathcal{R}(A)}) = \sigma(M)$, $\sigma(\mathcal{B}|_{\mathcal{N}(A)}) = \sigma(L)$. Specially, $\mathcal{B}|_{\mathcal{R}(A)}$ is nonsingular iff M is nonsingular, $\mathcal{B}|_{\mathcal{N}(A)}$ is nonsingular iff L is nonsingular, where $\mathcal{B}$ is the linear transformation determined by the matrix B.

Proof. Since $A \in \mathbb{C}_n^{GI}$ and from Lemma 1.2, there exists a nonsingular matrix X such that

$$A = X^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}X,$$

where D is a nonsingular matrix of order r, $r = r(A)$. Let

$$B = X^{-1}\begin{pmatrix} M & N \\ K & L \end{pmatrix}X,$$

where $M \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{(n-r)\times(n-r)}$. The condition $AB = BA$ yields that $DM = MD$ and $N = 0, K = 0$. Therefore

$$B = X^{-1}\begin{pmatrix} M & 0 \\ 0 & L \end{pmatrix}X,\ \ DM = MD.$$

In order to prove that $\sigma(\mathcal{B}|_{\mathcal{R}(A)}) = \sigma(M)$ and $\sigma(\mathcal{B}_{\mathcal{N}(A)}) = \sigma(L)$, without loss of generality, we suppose that

$$A = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} M & 0 \\ 0 & L \end{pmatrix},\ \ DM = MD,$$

where D is a nonsingular matrix of order r. The condition $AB = BA$ implies that $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are both invariant subspaces of $\mathcal{B}$, hence the symbols $\mathcal{B}|_{\mathcal{R}(A)}$ and $\mathcal{B}|_{\mathcal{N}(A)}$ are defined. Let $\varepsilon_1 = (1, 0, \cdots, 0)^T, \varepsilon_2 = (0, 1, \cdots, 0)^T, \cdots, \varepsilon_n = (0, 0, \cdots, 1)^T$, the nonsingularity of D follows that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ is a basis of $\mathcal{R}(A)$ and the matrix of $\mathcal{B}|_{\mathcal{R}(A)}$ under the basis is M. Therefore $\sigma(\mathcal{B}|_{\mathcal{R}(A)}) = \sigma(M)$. Similar discussion follows that $\varepsilon_{r+1}, \varepsilon_{r+2}, \cdots, \varepsilon_n$ is a basis of $\mathcal{N}(A)$, and the matrix of $\mathcal{B}|_{\mathcal{N}(A)}$ under the basis is L. Therefore $\sigma(\mathcal{B}_{\mathcal{N}(A)}) = \sigma(L)$. □

2. The rank, the null space, the column space and nonsingularity of combinations of two group invertible matrices

The nonsingularity of the difference of two idempotent matrices was considered by Koliha[17]:

$$P - Q \text{ is nonsingular} \Leftrightarrow \mathcal{R}(P) \cap \mathcal{R}(Q) = \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$$

$$\Leftrightarrow I - PQ \text{ and } P + Q \text{ are both nonsingular.}$$

Recently, invertibility of the sum (the difference) of two given operators was considered in [7] with a special emphasis on some cases such as when A and B are idempotents, when $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, when A and B are injective, etc. In this section the main results concern the nonsingularity of the difference of two group invertible matrices as well as the nonsingularity of some forms under some assumptions.

Theorem 2.1. *Let $A, B \in \mathbb{C}_n^{GI}$. Then*
(a) $\mathcal{N}(AA^\sharp - BB^\sharp) = \big(\mathcal{R}(A) \cap \mathcal{R}(B)\big) \oplus \big(\mathcal{N}(A) \cap \mathcal{N}(B)\big)$.
(b) $\mathcal{R}(AA^\sharp - BB^\sharp) = \big(\mathcal{R}(A) + \mathcal{R}(B)\big) \cap \big(\mathcal{N}(A) + \mathcal{N}(B)\big)$.
(c) $r(AA^\sharp - BB^\sharp) = r\begin{pmatrix} A \\ B \end{pmatrix} + r(A, B) - r(A) - r(B)$.

Proof. (a) It is clear that $\mathcal{N}(A) \cap \mathcal{N}(B) \subseteq \mathcal{N}(AA^\sharp - BB^\sharp)$. For any $\alpha \in \mathcal{R}(A) \cap \mathcal{R}(B)$, there exist $\beta, \gamma \in \mathbb{C}^n$ such that $\alpha = A\beta = B\gamma$. Then $\alpha = AA^\sharp\alpha = BB^\sharp\alpha$, which then leads to that $\alpha \in \mathcal{N}(AA^\sharp - BB^\sharp)$. Therefore $\mathcal{R}(A) \cap \mathcal{R}(B) \subseteq \mathcal{N}(AA^\sharp - BB^\sharp)$. From the above two subspace inclusions we have $\big(\mathcal{R}(A) \cap \mathcal{R}(B)\big) + \big(\mathcal{N}(A) \cap \mathcal{N}(B)\big) \subseteq \mathcal{N}(AA^\sharp - BB^\sharp)$. Since $A \in \mathbb{C}_n^{GI}$, then $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$, thus the sum $\big(\mathcal{R}(A) \cap \mathcal{R}(B)\big) + \big(\mathcal{N}(A) \cap \mathcal{N}(B)\big)$ is direct sum. For any $\alpha \in \mathcal{N}(AA^\sharp - BB^\sharp)$, we have $AA^\sharp\alpha = BB^\sharp\alpha$. Multiplying by A left to both sides of the equality we have $A\alpha = ABB^\sharp\alpha$, which implies that $\alpha - BB^\sharp\alpha \in \mathcal{N}(A)$. The decomposition of $\alpha = BB^\sharp\alpha + (\alpha - BB^\sharp\alpha) \in \mathcal{R}(A) \cap \mathcal{R}(B) + \mathcal{N}(A) \cap \mathcal{N}(B)$ then establishes the reverse inclusion of the spaces. Therefore, we have $\mathcal{N}(AA^\sharp - BB^\sharp) = \big(\mathcal{R}(A) \cap \mathcal{R}(B)\big) \oplus \big(\mathcal{N}(A) \cap \mathcal{N}(B)\big)$.

(b) Since $A, B \in \mathbb{C}_n^{GI}$, then $A^*, B^* \in \mathbb{C}_n^{GI}$. Applying the result of (a) to the matrices A^* and B^*, we have

$$\mathcal{N}\Big(A^*(A^*)^\sharp - B^*(B^*)^\sharp\Big) = \Big(\mathcal{R}(A^*) \cap \mathcal{R}(B^*)\Big) \oplus \Big(\mathcal{N}(A^*) \cap \mathcal{N}(B^*)\Big). \tag{2.1}$$

Taking orthogonal complement to both sides of the equality and properties of Lemma 1.1 will lead to (b).

(c) By calculating the dimensions of both sides of the equality $\mathcal{N}(AA^\sharp - BB^\sharp) = \big(\mathcal{R}(A) \cap \mathcal{R}(B)\big) \oplus \big(\mathcal{N}(A) \cap \mathcal{N}(B)\big)$, we have $n - r(AA^\sharp - BB^\sharp) = \dim\big(\mathcal{R}(A) \cap \mathcal{R}(B)\big) + \dim\big(\mathcal{N}(A) \cap \mathcal{N}(B)\big) = \dim\big(\mathcal{R}(A)\big) + \dim\big(\mathcal{R}(B)\big) - \dim\big(\mathcal{R}(A) + \mathcal{R}(B)\big) + \dim\mathcal{N}\begin{pmatrix} A \\ B \end{pmatrix} = r(A) + r(B) - r(A, B) + n - r\begin{pmatrix} A \\ B \end{pmatrix}$. Hence $r(AA^\sharp - BB^\sharp) = r\begin{pmatrix} A \\ B \end{pmatrix} + r(A, B) - r(A) - r(B)$.
□

Having in mind that for group invertible matrix X we have that $XX^\sharp$ is an idempotent and that $\mathcal{R}(XX^\sharp) = \mathcal{R}(X)$, $\mathcal{N}(XX^\sharp) = \mathcal{N}(X)$, using Koliha's result we can get some necessary and sufficient conditions for the nonsingularity of $AA^\sharp - BB^\sharp$, in the case when $A, B \in \mathbb{C}_n^{GI}$. Also, some other equivalents for the nonsingularity of $AA^\sharp - BB^\sharp$ are given in the next theorem:

Theorem 2.2. *Let $A, B \in \mathbb{C}_n^{GI}$. The following are equivalent:*
(1) $AA^\sharp - BB^\sharp$ is nonsingular.
(2)$\mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$.
(3)$I_n - AA^\sharp BB^\sharp$ and $AA^\sharp + BB^\sharp$ are nonsingular.
(4)$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{N}(A) + \mathcal{N}(B) = \mathbb{C}^n$.
(5)$r(A, B) = r(A) + r(B)$ and $r\begin{pmatrix} A \\ B \end{pmatrix} = n$.
(6) $r\begin{pmatrix} A \\ B \end{pmatrix} + r(A, B) = r(A) + r(B) + n$.

Proof. It is easy to prove by using the results of Koliha's result and Theorem 2.1. □

Remark 2.3. *In Theorem 2.2, if $A, B \in \mathbb{C}_n^P$ (then $A, B \in \mathbb{C}_n^{GI}$) and $A^\sharp = A$, $B^\sharp = B$, we get that*

$$A - B \text{ is nonsingular } \Leftrightarrow \mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\},$$

which is a generalization of Theorem 1.2 from [17].

From Theorem 2.2 we can see that nonsingularity of $I_n - AA^\sharp BB^\sharp$ is a necessary condition for the nonsingularity of $AA^\sharp - BB^\sharp$. In the next theorem we consider when this condition implies nonsingularity of $A - B$.

Theorem 2.4. *Let $A, B \in \mathbb{C}_n^{GI}$ and let $I_n - AA^\sharp BB^\sharp$ be nonsingular. If there exist $c_1, c_2, c_3, c_4 \in \mathbb{C}$ such that $c_1A + c_2B + c_3AA^\sharp B + c_4BB^\sharp A$ is nonsingular, then $A - B$ is also nonsingular.*

Proof. Let $\alpha \in \mathcal{N}(A - B)$, then $A\alpha = B\alpha$, so $(I_n - AA^\sharp BB^\sharp)(c_1A + c_2B + c_3AA^\sharp B + c_4BB^\sharp A)\alpha = (I_n - AA^\sharp BB^\sharp)\big((c_1 + c_2 + c_3 + c_4)A\alpha\big) = (c_1 + c_2 + c_3 + c_4)(A\alpha - AA^\sharp BB^\sharp A\alpha) = (c_1 + c_2 + c_3 + c_4)(A\alpha - AA^\sharp B\alpha) = (c_1 + c_2 + c_3 + c_4)(A\alpha - A\alpha) = 0$. Under the assumption that $I_n - AA^\sharp BB^\sharp$ and $c_1A + c_2B + c_3AA^\sharp B + c_4BB^\sharp A$ are nonsingular, we have $\alpha = 0$, which means that $\mathcal{N}(A - B) = \{0\}$. This completes the proof. □

Remark 2.5. *From the proof of Theorem 2.4, we have the following two results:*

(a) Let $A, B \in \mathbb{C}_n^{GI}$*. If there exist* $c_1, c_2, c_3, c_4 \in \mathbb{C}$*,* $c_1 + c_2 + c_3 + c_4 = 0$ *such that* $c_1A + c_2B + c_3AA^\sharp B + c_4BB^\sharp A$ *is nonsingular, then* $A - B$ *is also nonsingular.*

(b) Let $A, B \in \mathbb{C}_n^{GI}$*, satisfying that* $I_n - AA^\sharp BB^\sharp$ *is nonsingular. If there exist* $c_i \in \mathbb{C}(i = 1, 2, \cdots, 4k)$ *such that* $c_1A + c_2B + c_3BB^\sharp A + c_4AA^\sharp B + c_5AA^\sharp BB^\sharp A + c_6BB^\sharp AA^\sharp B + c_7BB^\sharp AA^\sharp BB^\sharp A + c_8AA^\sharp BB^\sharp AA^\sharp B \cdots + c_{4k-3}(AA^\sharp BB^\sharp)^{k-1}A + c_{4k-2}(BB^\sharp AA^\sharp)^{k-1}B + c_{4k-1}BB^\sharp(AA^\sharp BB^\sharp)^{k-1}A + c_{4k}AA^\sharp(BB^\sharp AA^\sharp)^{k-1}B$ *is nonsingular, then* $A - B$ *is also nonsingular.*

Theorem 2.6. *Let* $A, B \in \mathbb{C}_n^{GI}$*,* $c_1, c_2, c_2 + c_3, c_1 + c_2 + c_3 \in \mathbb{C} \setminus \{0\}$*. If*

$$\mathcal{R}(A - AA^\sharp B) \cap \mathcal{R}(B - BB^\sharp A) = \{0\} = \mathcal{N}(A) \cap \mathcal{N}(B),$$

then $c_1A + c_2B + c_3AA^\sharp B$ *is nonsingular.*

Proof. For any $\alpha \in \mathcal{N}(c_1A + c_2B + c_3AA^\sharp B)$, then

$$(c_1A + c_2B + c_3AA^\sharp B)\alpha = 0. \tag{2.2}$$

Premultiplying (2.2) by $AA^\sharp$, we have

$$(c_1A + c_2AA^\sharp B + c_3AA^\sharp B)\alpha = 0. \tag{2.3}$$

(2.2) and (2.3) yields that $AA^\sharp B\alpha = B\alpha$. Substituting this equality to (2.2) we have

$$[c_1A + (c_2 + c_3)B]\alpha = 0. \tag{2.4}$$

Premultiplying (2.4) by $BB^\sharp$, we have

$$[c_1BB^\sharp A + (c_2 + c_3)B]\alpha = 0. \tag{2.5}$$

From (2.4) and (2.5), we obtain $BB^\sharp A\alpha = A\alpha$. Now, the equality (2.4) yields that $(c_1 + c_2 + c_3)A\alpha = c_1A\alpha + (c_2 + c_3)A\alpha = -(c_2 + c_3)B\alpha + (c_2 + c_3)BB^\sharp A\alpha = -(c_2 + c_3)(B - BB^\sharp A)\alpha \in \mathcal{R}(B - BB^\sharp A)$ and $(c_1 + c_2 + c_3)A\alpha = c_1A\alpha + (c_2 + c_3)A\alpha = -(c_2 + c_3)B\alpha + (c_2 + c_3)A\alpha = -(c_2 + c_3)AA^\sharp B\alpha + (c_2 + c_3)A\alpha = (c_2 + c_3)(A - AA^\sharp B)\alpha \in \mathcal{R}(A - AA^\sharp B)$, which implies that $A\alpha \in \mathcal{R}(A - AA^\sharp B) \cap \mathcal{R}(B - BB^\sharp A) = \{0\}$. Therefore $A\alpha = 0$. The equality (2.4) and condition $c_2 + c_3 \neq 0$ will lead to that $B\alpha = 0$. So $\alpha \in \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$, hence $\alpha = 0$. This proves that $c_1A + c_2B + c_3AA^\sharp B$ is nonsingular. □

Theorem 2.7. *Let* $A, B \in \mathbb{C}_n^{GI}$*,* $a, b, c, d \in \mathbb{C} \setminus \{0\}$*. The following hold:*

(a) $\mathcal{N}\Big(aA + bB - (a + b)AA^\sharp B\Big) = \mathcal{N}(A - B)$.

(b) $\mathcal{R}\Big(cA + dB - (c + d)ABB^\sharp\Big) = \mathcal{R}(A - B)$.

(c) $r\Big(aA + bB - (a + b)AA^\sharp B\Big) = r\Big(cA + dB - (c + d)ABB^\sharp\Big) = r(A - B)$.

Proof. The following identities will hold by direct calculations:

$$(I_n - \frac{a + b}{a}AA^\sharp)\Big(aA + bB - (a + b)AA^\sharp B\Big) = b(B - A). \tag{2.6}$$

$$\Big(cA + dB - (c + d)ABB^\sharp\Big)(I_n - \frac{c + d}{d}BB^\sharp) = c(A - B). \tag{2.7}$$

Since $AA^\sharp$ and $BB^\sharp$ are both idempotent matrices and $\frac{a+b}{a} \neq 1$, $\frac{c+d}{d} \neq 1$, then $I_n - \frac{a+b}{a}AA^\sharp$ and $I_n - \frac{c+d}{d}BB^\sharp$ are both nonsingular matrices. Hence the proof follows by (2.6) and (2.7). □

Theorem 2.8. *Let $A, B \in \mathbb{C}_n^{GI}$, $a, b, c, d \in \mathbb{C} \setminus \{0\}$. The following are equivalent:*
(a) $A - B$ is nonsingular.
(b)$aA + bB - (a + b)AA^\sharp B$ is nonsingular, and

$$[aA + bB - (a + b)AA^\sharp B]^{-1} = \frac{a+b}{ab}(A - B)^{-1}AA^\sharp - \frac{1}{b}(A - B)^{-1}.$$

(c)$cA + dB - (c + d)ABB^\sharp$ is nonsingular, and

$$[cA + dB - (c + d)ABB^\sharp]^{-1} = \frac{1}{c}(A - B)^{-1} - \frac{c+d}{cd}BB^\sharp(A - B)^{-1}.$$

Proof. The proof follows using (2.6) and (2.7) by direct calculations. □

In Theorems 2.7,2.8, the coefficients in $c_1A + c_2B - c_3AA^\sharp B$ are demanded to satisfy $c_1 + c_2 = c_3$. If $c_1 + c_2 \neq c_3$, we have the following theorem:

Theorem 2.9. *Let $A, B \in \mathbb{C}_n^{GI}$, $a, b, c \in \mathbb{C}$, $ab \neq 0$, $a + b \neq c$, $AA^\sharp B = ABB^\sharp$. The following hold:*
(a) $\mathcal{N}(aA + bB - cAA^\sharp B) \cong \mathcal{N}(A + B)$.
(b) $\mathcal{R}(aA + bB - cAA^\sharp B) \cong \mathcal{R}(A + B)$.
(c) $r(A + B) = r(aA + bB - cAA^\sharp B)$.
(d) $A + B$ is nonsingular $\Leftrightarrow$ $aA + bB - cAA^\sharp B$ is nonsingular.
(e) When $A + B$ is nonsingular, we have

$$(aA + bB - cAA^\sharp B)^{-1} = \frac{a+b-c}{2ab}(A + B)^{-1} + \frac{a+c-b}{2ab}BB^\sharp(A + B)^{-1}$$

$$+\frac{b+c-a}{2ab}(A + B)^{-1}AA^\sharp + \frac{(a+c-b)(b+c-a)}{2ab(a+b-c)}BB^\sharp(A + B)^{-1}AA^\sharp.$$

Proof. Since $AA^\sharp$ and $BB^\sharp$ are both idempotent matrices and $\frac{b+c-a}{a+b-c} \neq -1$, $\frac{a+c-b}{a+b-c} \neq -1$, then $I_n + \frac{b+c-a}{a+b-c}AA^\sharp$ and $I_n + \frac{a+c-b}{a+b-c}BB^\sharp$ are both nonsingular matrices. Let

$$T = I_n + \frac{b+c-a}{a+b-c}AA^\sharp, \quad K = I_n + \frac{a+c-b}{a+b-c}BB^\sharp.$$

Direct calculation shows that

$$T(aA + bB - cAA^\sharp B)K = \frac{2ab}{a+b-c}(A + B). \tag{2.8}$$

(a) From the identity (2.8), the nonsingularity of T, K and $\frac{2ab}{a+b-c} \neq 0$, we have

$$\mathcal{N}(A + B) = \mathcal{N}((aA + bB - cAA^\sharp B)K)$$

$$= \{K^{-1}\alpha | \alpha \in \mathcal{N}(aA + bB - cAA^\sharp B)\} \cong \mathcal{N}(aA + bB - cAA^\sharp B).$$

(b) From the identity (2.8), the nonsingularity of T, K and $\frac{2ab}{a+b-c} \neq 0$, we have

$$\mathcal{R}(A + B) = \mathcal{R}(T(aA + bB - cAA^\sharp B)) = \{T\alpha | \alpha \in \mathcal{R}(aA + bB - cAA^\sharp B)\} \cong \mathcal{R}(aA + bB - cAA^\sharp B).$$

(c) and (d) can be derived directly from (a).
(e) can be obtained from the identity (2.8). □

In Theorems 2.7,2.9 if we take $A, B \in \mathbb{C}_n^P$ then we have that $A, B \in \mathbb{C}_n^{GI}$ and $A^\sharp = A$, $B^\sharp = B$, so we get:

$$r(aA + bB - cAB) = \begin{cases} r(A - B), & \text{when } c = a + b \\ r(A + B), & \text{when } c \neq a + b \end{cases} \quad (ab \neq 0). \tag{2.9}$$

The rank identity (2.9) is an important result of reference [25].

Theorem 2.10. *Let $A, B \in \mathbb{C}_n^{GI}$, $a_i, b_j \in \mathbb{C}(i = 0, 1, \cdots, 2k, j = 0, 1, \cdots, 2l)$, satisfying the following identity:*

$$\begin{gathered} a_0A + a_1(BB^\sharp AA^\sharp)A + a_2AA^\sharp(BB^\sharp AA^\sharp)A + a_3(BB^\sharp AA^\sharp)^2A+ \\ a_4AA^\sharp(BB^\sharp AA^\sharp)^2A + \cdots + a_{2k}AA^\sharp(BB^\sharp AA^\sharp)^kA+ \\ b_0B + b_1(AA^\sharp BB^\sharp)B + b_2BB^\sharp(AA^\sharp BB^\sharp)B + b_3(AA^\sharp BB^\sharp)^2B \\ +b_4BB^\sharp(AA^\sharp BB^\sharp)^2B + \cdots + b_{2l}BB^\sharp(AA^\sharp BB^\sharp)^lB = 0. \end{gathered} \tag{2.10}$$

Then if $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_2(a_0 + a_1 + \cdots + a_{2k}) \neq c_1(b_0 + b_1 + \cdots + b_{2l})$, we have

$$\mathcal{N}(c_1A + c_2B) = \mathcal{N}(A) \cap \mathcal{N}(B).$$

Proof. It is obvious that $\mathcal{N}(A) \cap \mathcal{N}(B) \subseteq \mathcal{N}(c_1A + c_2B)$. On the other hand, for any $\alpha \in \mathcal{N}(c_1A + c_2B)$, we have

$$(c_1A + c_2B)\alpha = 0. \tag{2.11}$$

Multiplying the last equality from the left side by $AA^\sharp$ and $BB^\sharp$, respectively, we get:

$$(c_1A + c_2AA^\sharp B)\alpha = 0 = (c_1BB^\sharp A + c_2B)\alpha. \tag{2.12}$$

From equalities (2.11) and (2.12), we have $AA^\sharp B\alpha = B\alpha$, $BB^\sharp A\alpha = A\alpha$. Therefore

$$\begin{gathered} A\alpha = (BB^\sharp AA^\sharp)A\alpha = AA^\sharp(BB^\sharp AA^\sharp)A\alpha = \\ (BB^\sharp AA^\sharp)^2A\alpha = \cdots = AA^\sharp(BB^\sharp AA^\sharp)^kA\alpha, \\ B\alpha = (AA^\sharp BB^\sharp)B\alpha = BB^\sharp(AA^\sharp BB^\sharp)B\alpha = \\ (AA^\sharp BB^\sharp)^2B\alpha = \cdots = BB^\sharp(AA^\sharp BB^\sharp)^lB\alpha. \end{gathered}$$

Combining with (2.10), we have

$$(a_0 + a_1 + \cdots + a_{2k})A\alpha + (b_0 + b_1 + \cdots + b_{2l})B\alpha = 0. \tag{2.13}$$

From (2.11) and (2.13), observing that $c_2(a_0 + a_1 + \cdots + a_{2k}) \neq c_1(b_0 + b_1 + \cdots + b_{2l})$, we have $A\alpha = B\alpha = 0$. Therefore $\alpha \in \mathcal{N}(A) \cap \mathcal{N}(B)$. □

Theorem 2.11. *Let $A, B \in \mathbb{C}_n^{GI}$, $a_i, b_j \in \mathbb{C}(i = 0, 1, \cdots, 2k, j = 0, 1, \cdots, 2l)$, satisfying the identity (2.10). Then*

$$r(c_1A + c_2B) = r(d_1A + d_2B)$$

for all $c_1, c_2, d_1, d_2 \in \mathbb{C} \setminus \{0\}$ satisfying $c_2(a_0 + a_1 + \cdots + a_{2k}) \neq c_1(b_0 + b_1 + \cdots + b_{2l})$ and $d_2(a_0 + a_1 + \cdots + a_{2k}) \neq d_1(b_0 + b_1 + \cdots + b_{2l})$. In particular, if $d_1A + d_2B$ is nonsingular for some $d_1, d_2 \in \mathbb{C} \setminus \{0\}$ with $d_2(a_0 + a_1 + \cdots + a_{2k}) \neq d_1(b_0 + b_1 + \cdots + b_{2l})$, then $c_1A + c_2B$ is nonsingular for all $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ with $c_2(a_0 + a_1 + \cdots + a_{2k}) \neq c_1(b_0 + b_1 + \cdots + b_{2l})$.

Proof. By Theorem 2.10 we have that $\mathcal{N}(c_1A + c_2B) = \mathcal{N}(A) \cap \mathcal{N}(B) = \mathcal{N}(d_1A + d_2B)$, so $r(c_1A + c_2B) = n - \dim\mathcal{N}(c_1A + c_2B) = n - \dim\mathcal{N}(d_1A + d_2B) = r(d_1A + d_2B)$. □

In Theorems 2.10,2.11, let $a_2 = a_3 = \cdots = a_{2k} = 0 = b_2 = b_3 = \cdots = b_{2l}$, $A, B \in \mathbb{C}_n^{k-P} (k \geq 3)$, then $A, B \in \mathbb{C}_n^{GI}$ and $A^\sharp = A^{k-2}, B^\sharp = B^{k-2}$. Hence the Theorems 2.10,2.11 are the Theorems 2.1,2.2 in reference [4], respectively. So the Theorems 2.10,2.11 are the generalizations of the two Theorems of [4].

Theorem 2.12. *Let $A, B \in \mathbb{C}_n^{GI}$, $a_i, b_j \in \mathbb{C}(i = 0, 1, \cdots, 2k, j = 0, 1, \cdots, 2l)$, satisfying the following identity:*

$$a_0A + a_1A(AA^\sharp BB^\sharp) + a_2A(AA^\sharp BB^\sharp)AA^\sharp + a_3A(AA^\sharp BB^\sharp)^2 + a_4A(AA^\sharp BB^\sharp)^2AA^\sharp +$$

$$\cdots + a_{2k}A(AA^\sharp BB^\sharp)^kAA^\sharp + b_0B + b_1B(BB^\sharp AA^\sharp) + b_2B(BB^\sharp AA^\sharp)BB^\sharp +$$

$$b_3B(BB^\sharp AA^\sharp)^2 + b_4B(BB^\sharp AA^\sharp)^2BB^\sharp + \cdots + b_{2l}B(BB^\sharp AA^\sharp)^lBB^\sharp = 0. \tag{2.14}$$

Then if $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_2(a_0 + a_1 + \cdots + a_{2k}) \neq c_1(b_0 + b_1 + \cdots + b_{2l})$, we have

$$\mathcal{R}(c_1A + c_2B) = \mathcal{R}(A) + \mathcal{R}(B).$$

Proof. Taking conjugate transpose to both sides of (2.14) and observing that $A^*, B^* \in \mathbb{C}_n^{GI}$ and $\bar{c_2}(\bar{a_0} + \bar{a_1} + \cdots + \bar{a_{2k}}) \neq \bar{c_1}(\bar{b_0} + \bar{b_1} + \cdots + \bar{b_{2l}})$, then we have from the Theorem 2.10 that

$$\mathcal{N}(\bar{c_1}A^* + \bar{c_2}B^*) = \mathcal{N}(A^*) \cap \mathcal{N}(B^*). \tag{2.15}$$

The result is obtained by taking orthogonal complement to both sides of the equality (2.15). □

The following Theorem 2.13 can be easily obtained from the Theorem 2.12.

Theorem 2.13. *Let $A, B \in \mathbb{C}_n^{GI}$, $a_i, b_j \in \mathbb{C}(i = 0, 1, \cdots, 2k, j = 0, 1, \cdots, 2l)$, satisfying the identity (2.14). Then*

$$r(c_1A + c_2B) = r(d_1A + d_2B)$$

for all $c_1, c_2, d_1, d_2 \in \mathbb{C} \setminus \{0\}$ satisfying $c_2(a_0 + a_1 + \cdots + a_{2k}) \neq c_1(b_0 + b_1 + \cdots + b_{2l})$ and $d_2(a_0 + a_1 + \cdots + a_{2k}) \neq d_1(b_0 + b_1 + \cdots + b_{2l})$. In particular, if $d_1A + d_2B$ is nonsingular for some $d_1, d_2 \in \mathbb{C} \setminus \{0\}$ with $d_2(a_0 + a_1 + \cdots + a_{2k}) \neq d_1(b_0 + b_1 + \cdots + b_{2l})$, then $c_1A + c_2B$ is nonsingular for all $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ with $c_2(a_0 + a_1 + \cdots + a_{2k}) \neq c_1(b_0 + b_1 + \cdots + b_{2l})$.

In Theorems 2.12,2.13, let $a_2 = a_3 = \cdots = a_{2k} = 0 = b_2 = b_3 = \cdots = b_{2l}$, $A, B \in \mathbb{C}_n^{k-P} (k \geq 3)$, then $A, B \in \mathbb{C}_n^{GI}$ and $A^\sharp = A^{k-2}, B^\sharp = B^{k-2}$. Hence Theorems 2.12,2.13 are Theorems 2.3,2.4 in reference [4], respectively. So Theorems 2.12,2.13 are the generalizations of the two Theorems of [4].

Theorem 2.14. *Let $A, B \in \mathbb{C}_n^{GI}$, $c_1, c_2, c_3 \in \mathbb{C}$. The following are equivalent:*
(a) $c_1ABB^\sharp + c_2BAA^\sharp - c_3ABAA^\sharp$ is nonsingular.
(b) $c_1BB^\sharp A + c_2AA^\sharp B - c_3BB^\sharp AB$ is nonsingular.
(c) $c_1A + c_2B - c_3AB$ and $I_n - AA^\sharp - BB^\sharp$ are nonsingular.

Proof. The result is an immediate consequence of the equalities:

$$(c_1A + c_2B - c_3AB)(I_n - AA^\sharp - BB^\sharp) = -(c_1ABB^\sharp + c_2BAA^\sharp - c_3ABAA^\sharp)$$

and

$$(I_n - AA^\sharp - BB^\sharp)(c_1A + c_2B - c_3AB) = -(c_1BB^\sharp A + c_2AA^\sharp B - c_3BB^\sharp AB).$$

□

Theorem 2.14 is the generalization of Theorem 2.6 of reference [19].

3. The nonsingularity and group invertibility of combinations of two group invertible matrices under some special conditions

In this section we discuss the nonsingularity and group invertibility of combinations of two group invertible matrices under the following special five conditions:
(a) $AB = BA = 0$.
(b) $A \in \mathbb{C}_n^P$, $AB = BA$.
(c) $AA^\sharp B = BB^\sharp A$ or $ABB^\sharp = BAA^\sharp$.
(d) $AA^\sharp B = BB^\sharp A$, A or B is nonsingular.
(e) $ABB^\sharp = BAA^\sharp$, A or B is nonsingular.

Lemma 3.1. *Let $A, B \in \mathbb{C}_n^{GI}$, $AB = BA = 0$, $c_1, c_2 \in \mathbb{C}$, $c_1c_2 \neq 0$. Then*
(a) $\mathcal{N}(c_1A + c_2B) = \mathcal{N}(A) \cap \mathcal{N}(B)$.
(b)$\mathcal{R}(c_1A + c_2B) = \mathcal{R}(A) + \mathcal{R}(B)$.

Proof. Since by $AB = BA = 0$ we have that $AA^\sharp BB^\sharp = BB^\sharp AA^\sharp = 0$, so we can easily get using Theorem 2.10 that $\mathcal{N}(c_1A + c_2B) = \mathcal{N}(A) \cap \mathcal{N}(B)$ or by direct verification. Similarly, by using the Theorem 2.12, we have that $\mathcal{R}(c_1A + c_2B) = \mathcal{R}(A) + \mathcal{R}(B)$. □

Theorem 3.2. *Let $A, B \in \mathbb{C}_n^{GI}$, $AB = BA = 0$, $c_1, c_2 \in \mathbb{C}$, $c_1c_2 \neq 0$. The following are equivalent:*
(a) $A + B$ is nonsingular.
(b)$A - B$ is nonsingular.
(c) $c_1A + c_2B$ is nonsingular, then $(c_1A + c_2B)^{-1} = (\frac{1}{2c_1} + \frac{1}{2c_2})(A + B)^{-1} + (\frac{1}{2c_1} - \frac{1}{2c_2})(A - B)^{-1}$.

Proof. Since $A \in \mathbb{C}_n^{GI}$, Lemma 1.2 assured that there exists a nonsingular matrix X such that

$$A = X^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}X,$$

where $D \in \mathbb{C}^{r\times r}$, D is a nonsingular matrix of order r. Let

$$B = X^{-1}\begin{pmatrix} M & N \\ K & L \end{pmatrix}X,$$

where $M \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{(n-r)\times(n-r)}$, $N \in \mathbb{C}^{r\times(n-r)}$, $K \in \mathbb{C}^{(n-r)\times r}$. The conditions $AB = BA = 0$ and D is nonsingular yield that $M = 0, K = 0, N = 0$. Therefore

$$B = X^{-1}\begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}X.$$

Then

$$A + B = X^{-1}\begin{pmatrix} D & 0 \\ 0 & L \end{pmatrix}X,$$

$$A - B = X^{-1}\begin{pmatrix} D & 0 \\ 0 & -L \end{pmatrix}X,$$

$$c_1A + c_2B = X^{-1}\begin{pmatrix} c_1D & 0 \\ 0 & c_2L \end{pmatrix}X.$$

Hence $c_1A + c_2B$ is nonsingular if and only if L is nonsingular if and only if $A + B$ is nonsingular if and only if $A - B$ is nonsingular . When $c_1A + c_2B$ is nonsingular , we have

$$(c_1A + c_2B)^{-1} = X^{-1}\begin{pmatrix} \frac{1}{c_1}D^{-1} & 0 \\ 0 & \frac{1}{c_2}L^{-1} \end{pmatrix}X.$$

$$= (\frac{1}{2c_1} + \frac{1}{2c_2})X^{-1}\begin{pmatrix} D^{-1} & 0 \\ 0 & L^{-1} \end{pmatrix}X + (\frac{1}{2c_1} - \frac{1}{2c_2})X^{-1}\begin{pmatrix} D^{-1} & 0 \\ 0 & (-L)^{-1} \end{pmatrix}X$$

$$= (\frac{1}{2c_1} + \frac{1}{2c_2})(A+B)^{-1} + (\frac{1}{2c_1} - \frac{1}{2c_2})(A-B)^{-1}.$$

□

Theorem 3.3. *Let $A, B \in \mathbb{C}_n^{GI}$, $AB = BA = 0$, $c_1, c_2 \in \mathbb{C}$, $c_1c_2 \neq 0$. Then*
(a) $c_1A + c_2B \in \mathbb{C}_n^{GI}$.
(b)$(c_1A + c_2B)^\sharp = (\frac{1}{2c_1} + \frac{1}{2c_2})(A+B)^\sharp + (\frac{1}{2c_1} - \frac{1}{2c_2})(A-B)^\sharp$.

Proof. From the proof of Theorem 3.2, we can suppose that

$$A = X^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}X,$$

where D is an invertible matrix of order r and

$$B = X^{-1}\begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}X.$$

Since $B \in \mathbb{C}_n^{GI}$, then $L \in \mathbb{C}_{n-r}^{GI}$. Therefore

$$c_1A + c_2B = X^{-1}\begin{pmatrix} c_1D & 0 \\ 0 & c_2L \end{pmatrix}X \in \mathbb{C}_n^{GI}.$$

Direct calculations will show that

$$(c_1A + c_2B)^\sharp = (\frac{1}{2c_1} + \frac{1}{2c_2})(A+B)^\sharp + (\frac{1}{2c_1} - \frac{1}{2c_2})(A-B)^\sharp.$$

□

Theorem 3.4. *Let $A \in \mathbb{C}_n^P$, $B \in \mathbb{C}_n^{GI}$, $AB = BA$, $c_1, c_2, c_3 \in \mathbb{C}$, $c_1c_2 \neq 0$ and $\mathcal{B}$ is the linear transformation determined by B. Then*
(a) if $c_2 + c_3 = 0$, then $c_1A + c_2B + c_3AB$ is nonsingular $\Leftrightarrow$ $\mathcal{B}|_{\mathcal{N}(A)}$ is nonsingular.
(b) if $c_2 + c_3 \neq 0$, then $c_1A + c_2B + c_3AB$ is nonsingular $\Leftrightarrow$ $-\frac{c_1}{c_2+c_3} \notin \sigma(\mathcal{B}|_{\mathcal{R}(A)})$ and $\mathcal{B}|_{\mathcal{N}(A)}$ is nonsingular.
(c) if $c_2 + c_3 = 0$, then $c_1A + c_2B + c_3AB \in \mathbb{C}_n^{GI}$.
(d) if $c_2 + c_3 \neq 0$, then $c_1A + c_2B + c_3AB \in \mathbb{C}_n^{GI}$ $\Leftrightarrow$ the orders of blocks with diagonal element $-\frac{c_1}{c_2+c_3}$ in the Jordan canonical form of $\mathcal{B}|_{\mathcal{R}(A)}$ are all ones or $-\frac{c_1}{c_2+c_3} \notin \sigma(\mathcal{B}|_{\mathcal{R}(A)})$.

Proof. Since $A^2 = A$, then there exists a nonsingular matrix X such that

$$A = X^{-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}X, \quad r = r(A).$$

Let

$$B = X^{-1}\begin{pmatrix} M & N \\ K & L \end{pmatrix}X.$$

Then $AB = BA$ implies that

$$B = X^{-1}\begin{pmatrix} M & 0 \\ 0 & L \end{pmatrix}X,$$

where $\sigma(\mathcal{B}|_{\mathcal{R}(A)}) = \sigma(M)$, $\sigma(\mathcal{B}|_{\mathcal{N}(A)}) = \sigma(L)$. Since $B \in \mathbb{C}_n^{GI}$, we have $M \in \mathbb{C}_r^{GI}$, $L \in \mathbb{C}_{n-r}^{GI}$. Direct calculations show that

$$c_1A + c_2B + c_3AB = X^{-1}\begin{pmatrix} c_1I_r + (c_2 + c_3)M & 0 \\ 0 & c_2L \end{pmatrix}X. \tag{3.1}$$

From (3.1), we know that $c_1A + c_2B + c_3AB$ is nonsingular if and only if $c_1I_r + (c_2 + c_3)M$ and c_2L are both nonsingular. Hence (a) and (b) can be obtained by $\sigma(\mathcal{B}|_{\mathcal{R}(A)}) = \sigma(M)$ and $\sigma(\mathcal{B}|_{\mathcal{N}(A)}) = \sigma(L)$.

(c) If $c_2 + c_3 = 0$, since $L \in \mathbb{C}_{n-r}^{GI}$, we have $c_2L \in \mathbb{C}_{n-r}^{GI}$. Therefore $c_1A + c_2B + c_3AB \in \mathbb{C}_n^{GI}$.

(d) If $c_2 + c_3 \neq 0$, the identity (3.1) yields that $c_1A + c_2B + c_3AB \in \mathbb{C}_n^{GI}$ if and only if $c_1I_r + (c_2 + c_3)M \in \mathbb{C}_r^{GI}$ and $c_2L \in \mathbb{C}_{n-r}^{GI}$ if and only if $c_1I_r + (c_2 + c_3)M \in \mathbb{C}_r^{GI}$. From the Lemma 1.2, we have $c_1I_r + (c_2 + c_3)M \in \mathbb{C}_r^{GI}$ if and only if the orders of blocks with diagonal elements 0 of Jordan canonical form of $c_1I_r + (c_2 + c_3)M$ are all ones or $0 \notin \sigma\big(c_1I_r + (c_2 + c_3)M\big)$ if and only if the orders of blocks with diagonal elements $-\frac{c_1}{c_2+c_3}$ of Jordan canonical form of M are all ones or $-\frac{c_1}{c_2+c_3} \notin \sigma(M)$. Now observe that $\sigma(\mathcal{B}|_{\mathcal{R}(A)}) = \sigma(M)$, then (d) is derived. □

Theorem 3.5. *Let $A, B \in \mathbb{C}_n^{GI}$ be such that $AA^\sharp B = BB^\sharp A$ or $ABB^\sharp = BAA^\sharp$. Then the nonsingularity of $c_1A + c_2B + c_3AA^\sharp B$, where $c_1, c_2, c_3 \in \mathbb{C}$ and $c_1, c_2, c_1 + c_2 + c_3 \in \mathbb{C} \setminus \{0\}$ is independent of choice of scalars c_1, c_2, c_3.*

Proof. Let $\alpha \in \mathcal{N}(c_1A + c_2B + c_3AA^\sharp B)$. Since $AA^\sharp B = BB^\sharp A$, then we have

$$(c_1A + c_2B + c_3AA^\sharp B)\alpha = 0. \tag{3.2}$$

Multiplying the last equality from the left sid by $AA^\sharp$ and $BB^\sharp$, respectively, we get

$$(c_1A + c_2AA^\sharp B + c_3AA^\sharp B)\alpha = 0, \tag{3.3}$$

$$(c_1BB^\sharp A + c_2B + c_3BB^\sharp AA^\sharp B)\alpha = 0. \tag{3.4}$$

Adding last two equalities and using (3.2) we get that $AA^\sharp B\alpha = 0$ which further gives

$$(c_1A + c_2B)\alpha = 0. \tag{3.5}$$

Multiplying the last equality from the left side by $AA^\sharp$, we get $A\alpha = 0$. Similarly, we get $B\alpha = 0$. Hence if $\alpha \in \mathcal{N}(c_1A + c_2B + c_3AA^\sharp B)$ for some choice of $c_1, c_2, c_3 \in \mathbb{C}$ such that all $c_1, c_2, c_1 + c_2 + c_3$ are non zero then $\alpha \in \mathcal{N}(d_1A + d_2B + d_3AA^\sharp B)$ for any such choice of $d_1, d_2, d_3 \in \mathbb{C}$. Hence nonsingularity of $c_1A + c_2B + c_3AA^\sharp B$ is independent of choice of scalars.

If $ABB^\sharp = BAA^\sharp$, then the result can be obtained by similar discussions like that above. □

Theorem 3.6. *Let $A, B \in \mathbb{C}_n^{GI}$ be such that $AA^\sharp B = BB^\sharp A$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_3, c_4 \in \mathbb{C}$, A or B is nonsingular. The following hold:*

(a) If A is nonsingular and $c_3 + c_4 = 0$, then $c_1A + c_2B + c_3AB + c_4BA$ is nonsingular $\Leftrightarrow c_1 + c_2 \neq 0$.

(b) If A is nonsingular and $c_3 + c_4 \neq 0$, then $c_1A + c_2B + c_3AB + c_4BA$ is nonsingular $\Leftrightarrow -\frac{c_1+c_2}{c_3+c_4} \notin \sigma(B) \setminus \{0\}$.

(c) If A is nonsingular, then $c_1A + c_2B + c_3AB + c_4BA \in \mathbb{C}_n^{GI} \Leftrightarrow (c_1 + c_2)I_n + (c_3 + c_4)B \in \mathbb{C}_n^{GI}$.

(d) If B is nonsingular and $c_3 + c_4 \neq 0$, then $c_1A + c_2B + c_3AB + c_4BA$ is nonsingular $\Leftrightarrow c_1 + c_2 \neq 0$.

(e) If B is nonsingular and $c_3 + c_4 \neq 0$, then $c_1A + c_2B + c_3AB + c_4BA$ is nonsingular $\Leftrightarrow -\frac{c_1+c_2}{c_3+c_4} \notin \sigma(A) \setminus \{0\}$.

(f) If B is nonsingular, then $c_1A + c_2B + c_3AB + c_4BA \in \mathbb{C}_n^{GI} \Leftrightarrow (c_1 + c_2)I_n + (c_3 + c_4)A \in \mathbb{C}_n^{GI}$.

Proof. Since $AA^\sharp B = BB^\sharp A$, then it is only to consider the proof under the case A is nonsingular from the symmetric position of A and B. The case B is nonsingular can be obtained similarly.

Suppose that A is nonsingular and since $B \in \mathbb{C}_n^{GI}$, then there exists a nonsingular matrix X such that

$$B = X^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}X,$$

where D is a nonsingular matrix of order r, $r = r(B)$. Let

$$A = X^{-1}\begin{pmatrix} M & N \\ K & L \end{pmatrix}X,$$

where $M \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{(n-r)\times(n-r)}$, $N \in \mathbb{C}^{r\times(n-r)}$, $K \in \mathbb{C}^{(n-r)\times r}$. Since A is nonsingular, then $A^\sharp = A^{-1}$. Then from the condition $AA^\sharp B = BB^\sharp A$, we have $B^2 = BA$. Substituting the above block matric expressions of A and B into $B^2 = BA$ yields that $M = D, N = 0$. Therefore

$$A = X^{-1}\begin{pmatrix} D & 0 \\ K & L \end{pmatrix}X.$$

Since A is nonsingular, we have L is also a nonsingular matrix of order $n-r$. Let $T = c_1A + c_2B + c_3AB + c_4BA$. Direct calculations show that

$$T = X^{-1}\begin{pmatrix} (c_1+c_2)D + (c_3+c_4)D^2 & 0 \\ c_1K + c_3KD & c_1L \end{pmatrix}X. \tag{3.6}$$

(a) If $c_3 + c_4 = 0$ and since D, L are nonsingular and $c_1 \neq 0$, we have

$$T \text{ is nonsingular} \Leftrightarrow c_1 + c_2 \neq 0.$$

(b) If $c_3 + c_4 \neq 0$ and observe that $\sigma(B) \setminus \{0\} = \sigma(D)$, we have T is nonsingular if and only if $(c_1 + c_2)D + (c_3 + c_4)D^2$ is nonsingular if and only if $\det\Big((c_1 + c_2)I_r + (c_3 + c_4)D\Big) \neq 0$ if and only if $-\frac{c_1+c_2}{c_3+c_4} \notin \sigma(D)$ if and only if $-\frac{c_1+c_2}{c_3+c_4} \notin \sigma(B) \setminus \{0\}$.

(c) From the identity (3.6), nonsingularity of D and c_1L, we have $T \in \mathbb{C}_n^{GI}$ if and only if $r(T) = r(T^2)$ if and only if $r\Big((c_1 + c_2)D + (c_3 + c_4)D^2\Big) = r\Big(\big((c_1 + c_2)D + (c_3 + c_4)D^2\big)^2\Big)$ if and only if $r\Big((c_1 + c_2)I_r + (c_3 + c_4)D\Big) = r\Big(\big((c_1+c_2)I_r+(c_3+c_4)D\big)^2\Big)$ if and only if $(c_1+c_2)I_r+(c_3+c_4)D \in \mathbb{C}_r^{GI}$ if and only if $(c_1+c_2)I_n+(c_3+c_4)B \in \mathbb{C}_n^{GI}$. $\square$

If $ABB^\sharp = BAA^\sharp$, A or B is nonsingular, then some results can be obtained by similar discussions like that above.

References

[1] O.M. Baksalary and G. Trenkler. Column space equlities for orthogonal projectors. *Appl. Math. and Computation*, 212:519–529, 2009.

[2] J.K. Baksalary and O.M. Baksalary. Nonsingularity of linear combinations of idempotent matrices. *Linear Algebra Appl.*, 388:25-29, 2004.

[3] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and applications, second ed.*. Springer-Verlag, New-York, 2003.

[4] J. Benítez, X. Liu and T. Zhu. Nonsingularity and group invertibility of linear combinations of two k-potent matrices. *Linear and Multilinear Algebra*, 58:1023-1035, 2010.

[5] D.S. Cvetković Ilić and C. Deng. The Drazin invertibility of the difference and the sum of two idempotent operators. *Journal of Computational and Applied Mathematics*, 233(8):1717-1722, 2010.

[6] D.S. Cvetković Ilić and R. Harte. On a Jacobson's lemma and Drazin invertibility. *Appl. Math. Letters*, 23:417-420, 2010.

[7] D.S. Cvetković Ilić and M. Kostadinov. Invertibility of linear combination in B(H). *Linear Multilinear Algebra*, 66(11):2139-2150, 2018.

[8] D.S. Cvetković Ilić and Y. Wei. Algebraic Properties of Generalized Inverses. *Series: Development in Mathematics*, Vol. 52, Springer, 2017.

[9] C.Y. Deng. The Drazin inverses of sum and difference of idempotents. *Linear Algebra Appl.*, 439:1282-1291, 2009.

[10] C.Y. Deng. The Drazin inverses of products and differences of orthogonal projectors. *J. Math. Anal. Appl.*, 335:64-71, 2007.

[11] C. Deng, D.S. Cvetković Ilić and Y. Wei. Properties of the combinations of commutative idempotents. *Linear Algebra Appl.*, 436:202-221, 2012.

[12] C. Deng, D.S. Cvetković Ilić and Y. Wei. On invertibility of combinations of k-potent operators. *Linear Algebra Appl.*, 437(1):376-387, 2012.

[13] H.K. Du, X.Y. Yao and C.Y. Deng. Invertibility of linear combinations of two idempotents. *Proc. Amer. math. soc.*, 134:1451-1457, 2005.

[14] J. Groß and G. Trenkler. Nonsingularity of the difference of two oblique projectors. *SIAM J. Matrix Anal. Appl.*, 21:390-395, 2009.

[15] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University press, Cambridge, 1985.

[16] J.J. Koliha, D.S. Cvetković Ilić and C. Deng. Generalized Drazin invertibility of combinations of idempotents. *Linear Algebra Appl.*, 437(9):2317-2324, 2012.

[17] J.J. Koliha, V. Rakočević and I. Straškraba. The difference and sum of projectors. *Linear Algebra Appl.*, 388:279-288, 2004.

[18] J.J. Koliha and V. Rakočević. The nullity and rank of linear combinations of idempotent matrices. *Linear Algebra Appl.*, 418:11-14, 2006.

[19] X. Liu, S. Wu and J. Benítez. On nonsingularity of combinations of two group invertible matrices and two tripotent matrices. *Linear and Multilinear Algebra*, to appear.

[20] X. Liu, S. Wu and Y. Yu. The group inverse of the combinations of two idempotent matrices. *Linear and Multilinear Algebra*, 59:101-115, 2011.

[21] M. Tošić and D.S. Cvetković Ilić. The invertibility of the difference and the sum of commuting generalized and hypergeneralized projectors. *Linear Algebra.*, 61(4):482-493, 2013.

[22] M. Tošić and D.S. Cvetković Ilić. Invertibility of a linear combination of two matrices and partial orderings. *Appl. Math. Comp.*, 218:4651-4657, 2012.

[23] Y. Tian and G.P.H. Styan. Rank equalities for idempotent and involutary matrices. *Linear Algebra Appl.*, 355:101-107, 2011.

[24] Y. Tian and G.P.H. Styan. Rank equalities for idempotent matrices with applications. *J. Comput. and Appl. Math.*, 191:77-97, 2006.

[25] K. Zuo. Nonsingularity of the difference and the sum of two idempotent matrices. *Linear Algebra Appl.*, 433:476-482, 2010.

[26] K. Zuo. The nullity and rank of combinations of idempotent matrices. *Journal of Math.(PRC)*, 28:619-623, 2008.