Filomat 34:11 (2020), 3609–3626
https://doi.org/10.2298/FIL2011609T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A New Compact Alternating Direction Implicit Method for Solving Two Dimensional Time Fractional Diffusion Equation With Caputo-Fabrizio Derivative

Mehran Taghipour[a], Hossein Aminikhah[b]

[a]*Department of Applied Mathematics and Computer Science, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*
[b]*Center of Excellence for Mathematical Modelling, Optimization and Combinational Computing (MMOCC), University of Guilan, Rasht, Iran*

Abstract. In this paper, a new compact alternating direction implicit (ADI) difference scheme is proposed for the solution of two dimensional time fractional diffusion equation. Theoretical considerations are discussed. We show that the proposed method is fourth order accurate in space and two order accurate in time. The stability and convergence of the compact ADI method are presented by the Fourier analysis method. Numerical examples confirm the theoretical results and high accuracy of the proposed scheme.

1. Introduction

Fractional differential equations (FDEs), which deal with derivatives and integrals of any arbitrary real or complex order, have been highly regarded by researchers. Fractional calculus has a history of more than 300 years, yet its applicability in different domains has been realized recently. In the last three decades, the subject witnessed a significant growth of research [1, 2]. Recently, fractional differential equations have been widely used in various fields such as muscular blood vessel modeling [3], non-linear oscillation of earthquakes [4], control theory [5], financial economics [6], biotechnology [7] and etc [8–11]. Fractional derivatives model the various dynamical processes and they carry information regarding their present as well as past states. In order to characterize memory property of complex systems, one needs to employ the non-integer order derivatives because these operators give a complete description of different physical processes with dissipation and long-range interaction [12]. There are many numerical methods proposed for solving the FDEs up to now, e.g., finite difference method [13], spectral methods [14–16], finite element method [17, 18], RBF [19] and etc. Low-order finite difference schemes are not accurate enough for solving many problems in science. Recently the focus has shifted to high order compact finite difference methods [20–23]. The advantage of the high order compact finite difference is that they give high accuracy on small stencils with greater computational efficiency [24]. The original split type method was introduced by Peaceman and Rachford in 1955. Their method is called the alternating direction implicit method. Such schemes split multi-dimensional problems to a series of one-dimensional ones which are much easier to solve [25].

2010 *Mathematics Subject Classification*. Primary 65L05; Secondary 26A33, 34A08
Keywords. Fractional differential equation; Compact scheme; Finite difference; Alternating direction implicit method; Caputo-Fabrizio derivative
Received: 26 January 2020; Accepted: 16 February 2020
Communicated by Maria Alessandra Ragusa
Email addresses: mtp20222@yahoo.com (Mehran Taghipour), hossein.aminikhah@gmail.com (Hossein Aminikhah)

So far, various types of fractional derivatives have been studied. In 1835 Liouville considered the fractional derivative of function $f(x)$, denoted by $D^{\alpha} f(x)$, of order α [26]. After him, Riemann, Grunwald, Sonine, Hadamard, Marchaud, Hille, Riez, Chen, Burlak, Kalisch, Caputo, Agarwal and others proposed definitions of the fractional derivatives. A review of definitions for fractional derivatives can be found in [27]. A compact ADI method for two-dimensional nonlinear reaction-diffusion equations was given recently in [28], and a compact ADI method for solving two-dimensional Riesz space fractional diffusion equation proposed in [29].

Caputo and Fabrizio have suggested a new definition of a fractional derivative which we will use in this paper [30]. In this paper, we consider the following two-dimensional time fractional diffusion equation describing sub-diffusive phenomena with a non-homogeneous term

$$ {}_{0}^{CF}D_t^{\gamma} u(x,y,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t),\ (x,y) \in \Omega,\ 0 \leq t \leq T, \tag{1} $$

with the initial condition

$$ u(x,y,0) = w(x,y),\ (x,y) \in \Omega, \tag{2} $$

and the Dirichlet boundary conditions

$$ \begin{aligned} &u(L_1,y,t) = \varphi_1(y,t), u(L_2,y,t) = \varphi_2(y,t), \\ &u(x,L_3,t) = \psi_1(x,t), u(x,L_4,t) = \psi_2(x,t), t \geq 0, \end{aligned} \tag{3} $$

where $\Omega = (L_1, L_2) \times (L_3, L_4)$. Here $0 < \gamma < 1$ and ${}_{0}^{CF}D_t^{\gamma} u$ denotes the Caputo-Fabrizio fractional derivative of the function $u(x,y,t)$ defined as:

$$ {}_{0}^{CF}D_t^{\gamma} u(x,y,t) = \frac{M(\gamma)}{1-\gamma} \int_0^t \frac{\partial u(x,y,s)}{\partial s} e^{-\sigma(t-s)} ds, \tag{4} $$

where $M(\gamma)$ is a normalization function such that $M(0) = M(1) = 1$ and $\sigma = \frac{\gamma}{1-\gamma}$.
Our goal is to present a compact ADI scheme to solve (1)-(3) based on the new fractional derivative. The unconditionally stable result is derived.

The paper is organized as follows. In Section 2, we give a compact ADI difference scheme for two-dimensional time fractional diffusion equation. Then in Section 3, we present the analysis of stability and convergence for the presented scheme. In Section 4, some numerical results using the fourth order compact finite difference scheme are carried out. Finally, this paper ends with conclusions in Section 5.

2. Compact ADI scheme for 2D time-fractional diffusion equation with Caputo-Fabrizio derivative

The domain $\Omega \times [0,T]$ of (1)–(3) is divided into a uniform grid of mesh points (x_j, y_k, t_n) with $x_j = L_1 + jh_x$, $j = 0,1,\ldots,N_x+1$, $y_k = kh_y$, $k = 0,1,\ldots,N_y+1$, and $t_n = n\tau$, $n = 0,1,\ldots,N$. Here N_x, N_y and N are positive integers, $h_x = (L_2 - L_1)/(N_x+1)$ and $h_y = (L_4 - L_3)/(N_y+1)$ are the mesh-widths in x, y, respectively, with h_x/h_y bounded from below and above, and $\tau = T/N$ is the time step.

A discrete approximation to the ${}_{0}^{CF}D_t^{\gamma} u(x,y,t)$ at (x_j, y_k, t_n) can be obtained by the following approximation

$$\begin{aligned}
{}_0^{CF}D_t^{\gamma}u(x_j,y_k,t_n) &= \frac{M(\gamma)}{1-\gamma}\int_0^{t_n}\frac{\partial u(x_j,y_k,s)}{\partial s}e^{-\sigma(t_n-s)}ds\\
&= \frac{M(\gamma)}{1-\gamma}\sum_{l=1}^{n}\int_{t_{n-1}}^{t_n}\frac{\partial u(x_j,y_k,s)}{\partial s}e^{-\sigma(t_n-s)}ds\\
&= \frac{M(\gamma)}{1-\gamma}\sum_{l=1}^{n}\int_{t_{n-1}}^{t_n}\frac{u_{j,k}^{l}-u_{j,k}^{l-1}}{\tau}e^{-\sigma(t_n-s)}ds+R\\
&= \frac{M(\gamma)}{\gamma\tau}\sum_{l=1}^{n}\left(u_{j,k}^{l}-u_{j,k}^{l-1}\right)d_{l,n}+R\\
&= \frac{M(\gamma)}{\gamma\tau}\left(d_{n,n}u_{j,k}^{n}-\sum_{l=1}^{n-1}\left(d_{l+1,n}-d_{l,n}\right)u_{j,k}^{l}-d_{1,n}u_{j,k}^{0}\right)+R,
\end{aligned} \tag{5}$$

where $u_{j,k}^{n}=u(x_j,y_k,t_n)$, R is the error term and $d_{l,n}$ is obtained as follows:

$$d_{l,n}=e^{-\sigma\tau(n-l)}-e^{-\sigma\tau(n-l+1)}.$$

We introduce the following notation

$$\frac{\delta_x^2u_{j,k}^n}{h_x^2}=\frac{u_{j-1,k}^n-2u_{j,k}^n+u_{j+1,k}^n}{h_x^2}.$$

We need the following lemmas in derivation of compact finite difference scheme.

Lemma 2.1. *Consider the partial differential equation*

$$\frac{\partial^2u(x,y,t)}{\partial x^2}+\frac{\partial^2u(x,y,t)}{\partial y^2}=f(x,y,t). \tag{6}$$

A fourth-order compact finite difference scheme for this equation is given as

$$\left(\frac{\delta_x^2}{h_x^2}+\frac{1}{12h_x^2}\delta_x^2\delta_y^2+\frac{\delta_y^2}{h_y^2}+\frac{1}{12h_y^2}\delta_x^2\delta_y^2\right)u_{j,k}^n=\left(\frac{\delta_x^2}{12}+\frac{\delta_x^2}{12}+1\right)f_{j,k}^n+O(h_x^4+h_y^4). \tag{7}$$

Proof. Based on the definition of operator δ_x^2, we have the following relation for (6) at point (x_j,y_k,t_n)

$$\frac{\delta_x^2}{h_x^2}u_{j,k}^n+\frac{\delta_y^2}{h_y^2}u_{j,k}^n-\tau_{j,k}^n=f_{j,k}^n, \tag{8}$$

where

$$\tau_{j,k}^n=\frac{h_x^2}{12}\Big(\frac{\partial^4u}{\partial x^4}\Big)_{j,k}^n+\frac{h_y^2}{12}\Big(\frac{\partial^4u}{\partial y^4}\Big)_{j,k}^n+O(h_x^4+h_y^4). \tag{9}$$

In order to obtain a fourth-order scheme, the fourth derivatives of u in Eq. (9) should be approximated. Eq. (6) gives

$$\frac{\partial^4u}{\partial x^4}\Big|_{j,k}^n=\Big(\frac{\partial^2f}{\partial x^2}-\frac{\partial^4u}{\partial x^2\partial y^2}\Big)_{j,k}^n=\frac{\delta_x^2}{h_x^2}f_{j,k}^n-\frac{\delta_x^2\delta_y^2}{h_x^2h_y^2}u_{j,k}^n, \tag{10}$$

$$\frac{\partial^4u}{\partial y^4}\Big|_{j,k}^n=\Big(\frac{\partial^2f}{\partial y^2}-\frac{\partial^4u}{\partial y^2\partial x^2}\Big)_{j,k}^n=\frac{\delta_y^2}{h_y^2}f_{j,k}^n-\frac{\delta_x^2\delta_y^2}{h_x^2h_y^2}u_{j,k}^n. \tag{11}$$

By substituting Eqs. (10) and (11) in Eq. (9) we get

$$\tau_{j,k}^n = \frac{1}{12}(\delta_x^2 f_{j,k}^n + \delta_y^2 f_{j,k}^n) - \frac{1}{12}(\frac{\delta_x^2\delta_y^2}{h_y^2}u_{j,k}^n + \frac{\delta_x^2\delta_y^2}{h_x^2}u_{j,k}^n) + O(h_x^4 + h_y^4), \tag{12}$$

hence by substituting Eq. (12) in Eq. (8) the lemma is proved. □

Lemma 2.2. *Suppose $v(t) \in C^2[0, t_n]$. Let*

$$A = \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \left[v'(\tau) - \frac{v(t_l) - v(t_{l-1})}{\Delta t} \right] e^{-\sigma(t_n-\tau)} d\tau,$$

then

$$|A| \leq \frac{(\Delta t)^2}{\sigma} \max_{0\leq t\leq t_n} |v''(t)| \left| \frac{\sigma^2}{12} - \frac{\sigma^3 \Delta t}{24} + \cdots \right| t_n.$$

Proof. Using the Taylor series expansion with integral remainder, we have

$$\begin{aligned} A &= \frac{1}{\Delta t} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \left[\int_{t_{l-1}}^{\tau} v''(s)(s - t_{l-1}) ds - \int_{\tau}^{t_l} v''(s)(t_l - s) ds \right] e^{-\sigma(t_n-\tau)} d\tau \\ &= \frac{1}{\sigma \Delta t} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \left[(s - t_{l-1}) \left(e^{-\sigma(t_n-t_l)} - e^{-\sigma(t_n-s)} \right) - (t_l - s) \left(e^{-\sigma(t_n-s)} - e^{-\sigma(t_n-t_{l-1})} \right) \right] v''(s) ds. \end{aligned}$$

Since,

$$\begin{aligned} &\int_{t_{l-1}}^{t_l} \left[(s - t_{l-1}) \left(e^{-\sigma(t_n-t_l)} - e^{-\sigma(t_n-s)} \right) - (t_l - s) \left(e^{-\sigma(t_n-s)} - e^{-\sigma(t_n-t_{l-1})} \right) \right] ds \\ &\quad = (\Delta t)^4 e^{-\sigma\Delta t(n-l)} \left(\frac{\sigma^2}{12} - \frac{\sigma^3 \Delta t}{24} + \cdots \right) \\ &\quad \leq (\Delta t)^4 \left(\frac{\sigma^2}{12} - \frac{\sigma^3 \Delta t}{24} + \cdots \right), \end{aligned}$$

and $e^{-\sigma\Delta t(n-l)} < 1$ for $l = 1, 2, \ldots, n$ and $t_n = n\Delta t$, hence the result will be achieved. □

Now, using the Lemma (2.2) we get

$$\begin{aligned} &\frac{M(\gamma)}{1-\gamma} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \left(\frac{\partial u}{\partial s}\Big|_{(x_j,y_k,s)} - \frac{u_{j,k}^l - u_{j,k}^{l-1}}{\tau} \right) e^{-\sigma(t_n-s)} ds \\ &\quad \leq \frac{M(\gamma)}{1-\gamma} \frac{\tau^2}{\sigma} \max_{0\leq t\leq t_n} |u''(t)| \left| \left(\frac{\sigma^2}{12} - \frac{\sigma^3 \tau}{24} + \cdots \right) \right| \sum_{l=1}^{n} e^{-\sigma\tau(n-l)} = O(\tau^2). \end{aligned}$$

Let

$${}_0^{CF}D_t^{\gamma} u(x, y, t) = \frac{M(\gamma)}{1-\gamma} \int_0^t \frac{\partial u(x, y, s)}{\partial s} e^{-\sigma(t-s)} ds = g(x, y, t), \tag{13}$$

so Eq. (1) can be written as follows

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y, t) - f(x, y, t). \tag{14}$$

Using lemma (2.1), we can write Eq. (14) at the point (x_j, y_k, t_n) as follows

$$\left(\frac{\delta_x^2}{h_x^2} + \frac{1}{12h_x^2}\delta_x^2\delta_y^2 + \frac{\delta_y^2}{h_y^2} + \frac{1}{12h_y^2}\delta_x^2\delta_y^2\right)u_{j,k}^n = \left(\frac{\delta_x^2}{12} + \frac{\delta_x^2}{12} + 1\right)\left(g_{j,k}^n - f_{j,k}^n\right) + O(h^4). \tag{15}$$

Therefore, from Eq. (5) and lemma (2.2) we have

$$\left(\frac{\delta_x^2}{h_x^2} + \frac{1}{12h_x^2}\delta_x^2\delta_y^2 + \frac{\delta_y^2}{h_y^2} + \frac{1}{12h_y^2}\delta_x^2\delta_y^2\right)u_{j,k}^n = \left(\frac{\delta_x^2}{12} + \frac{\delta_x^2}{12} + 1\right)\times \tag{16}$$
$$\left[\frac{1}{\gamma\tau}\left(d_{n,n}u_{j,k}^n - \sum_{l=1}^{n-1}\left(d_{l+1,n} - d_{l,n}\right)u_{j,k}^l - d_{1,n}u_{j,k}^0\right)\right] - \left(\frac{\delta_x^2}{12} + \frac{\delta_x^2}{12} + 1\right)f_{j,k}^n + O(h^4 + \tau^2),$$

where $M(\gamma) = 1$. Let $v = \frac{d_{n,n}}{\gamma\tau}$ then

$$\left(\frac{\delta_x^2}{12} + \frac{\delta_x^2}{12} + 1\right)u_{j,k}^n - \left(\frac{\delta_x^2}{vh_x^2} + \frac{1}{12vh_x^2}\delta_x^2\delta_y^2 + \frac{\delta_y^2}{vh_y^2} + \frac{1}{12vh_y^2}\delta_x^2\delta_y^2\right)u_{j,k}^n = \frac{1}{d_{n,n}}\left(\frac{\delta_x^2}{12} + \frac{\delta_y^2}{12} + 1\right)$$
$$\left[\sum_{l=1}^{n-1}\left(d_{l+1,n} - d_{l,n}\right)u_{j,k}^l + d_{1,n}u_{j,k}^0\right] + \frac{1}{v}\left(\frac{\delta_x^2}{12} + \frac{\delta_x^2}{12} + 1\right)f_{j,k}^n + O(h^4 + \tau^2). \tag{17}$$

Using factorization, we can rewrite Eq. (17) as

$$\left(1+\left(\frac{1}{12} - \frac{1}{vh_x^2}\right)\delta_x^2\right)\left(1 + \left(\frac{1}{12} - \frac{1}{vh_y^2}\right)\delta_y^2\right)u_{j,k}^n$$
$$=\left(\frac{1}{144} + \frac{1}{v^2h_x^2h_y^2}\right)u_{j,k}^n + \frac{1}{d_{n,n}}\left(\frac{\delta_x^2}{12} + \frac{\delta_y^2}{12} + 1\right)\left[\sum_{l=1}^{n-1}\left(d_{l+1,n} - d_{l,n}\right)u_{j,k}^l + d_{1,n}u_{j,k}^0\right]$$
$$\frac{1}{v}\left(\frac{\delta_x^2}{12} + \frac{\delta_x^2}{12} + 1\right)f_{j,k}^n + O(h^4 + \tau^2). \tag{18}$$

To give a numerical scheme we need to approximate the term $\left(\frac{1}{144} + \frac{1}{v^2h_x^2h_y^2}\right)u_{j,k}^n$ on the right hand side Eq. (18) and omit the last term. As in paper [31] we use the following approximation

$$u_{j,k}^n = \begin{cases} 2u_{j,k}^{n-1} - u_{j,k}^{n-2}, & n > 1, \\ u_{j,k}^0, & n = 1. \end{cases}$$

Using the Taylor series expansion we have

$$u(x_j, y_k, t_{n-1}) = u(x_j, y_k, t_n) + \tau u'(x_j, y_k, t_n) + \frac{\tau^2}{2}u''(x_j, y_k, t_n) + O(\tau^3),$$
$$u(x_j, y_k, t_{n-2}) = u(x_j, y_k, t_n) + 2\tau u'(x_j, y_k, t_n) + 2\tau^2 u''(x_j, y_k, t_n) + O(\tau^3).$$

So $u_{j,k}^n = 2u_{j,k}^{n-1} - u_{j,k}^{n-2} + O(\tau^2)$. Define $\mu_x = \frac{1}{vh_x^2}$ and $\mu_y = \frac{1}{vh_y^2}$, for $n = 1$, we have

$$\left(1+\left(\frac{1}{12} - \mu_x\right)\delta_x^2\right)\left(1 + \left(\frac{1}{12} - \mu_y\right)\delta_y^2\right)u_{j,k}^n = \left(\frac{1}{144} + \mu_x\mu_y\right)u_{j,k}^0$$
$$+\left(\frac{\delta_x^2}{12} + \frac{\delta_y^2}{12} + 1\right)u_{j,k}^0 + \frac{1}{v}\left(\frac{\delta_x^2}{12} + \frac{\delta_x^2}{12} + 1\right)f_{j,k}^1, \tag{19}$$

and for $n \geq 2$, we obtain

$$\begin{aligned}
&\Big(1+\big(\frac{1}{12}-\mu_x\big)\delta_x^2\Big)\Big(1+\big(\frac{1}{12}-\mu_y\big)\delta_y^2\Big)u_{j,k}^n \\
&=\big(\frac{1}{144}+\mu_x\mu_y\big)\big(2u_{j,k}^{n-1}-u_{j,k}^{n-2}\big)+\frac{1}{d_{n,n}}\Big(\frac{\delta_x^2}{12}+\frac{\delta_y^2}{12}+1\Big)\Big[\sum_{l=1}^{n-1}\left(d_{l+1,n}-d_{l,n}\right)u_{j,k}^l+d_{1,n}u_{j,k}^0\Big] \\
&\quad+\frac{1}{v}\Big(\frac{\delta_x^2}{12}+\frac{\delta_x^2}{12}+1\Big)f_{j,k}^n.
\end{aligned} \tag{20}$$

Therefore, for $n = 1$ and $n \geq 2$ we obtain the following compact ADI scheme that is implemented in two steps

$$\begin{cases}
\Big(1+\big(\frac{1}{12}-\mu_x\big)\delta_x^2\Big)U_{j,k}^{1*} \\
\quad=\big(\frac{1}{144}+\mu_x\mu_y\big)u_{j,k}^0+\Big(\frac{\delta_x^2}{12}+\frac{\delta_y^2}{12}+1\Big)u_{j,k}^0+\frac{1}{v}\Big(\frac{\delta_x^2}{12}+\frac{\delta_x^2}{12}+1\Big)f_{j,k}^1, \\
\Big(1+\big(\frac{1}{12}-\mu_y\big)\delta_y^2\Big)U_{j,k}^1=U_{j,k}^{1*}, \\
\\
\Big(1+\big(\frac{1}{12}-\mu_x\big)\delta_x^2\Big)U_{j,k}^{n*} \\
=\big(\frac{1}{144}+\mu_x\mu_y\big)\big(2u_{j,k}^{n-1}-u_{j,k}^{n-2}\big) \\
\quad+\frac{1}{d_{n,n}}\Big(\frac{\delta_x^2}{12}+\frac{\delta_y^2}{12}+1\Big)\Big[\sum\limits_{l=1}^{n-1}\left(d_{l+1,n}-d_{l,n}\right)u_{j,k}^l+d_{1,n}u_{j,k}^0\Big]+\frac{1}{v}\Big(\frac{\delta_x^2}{12}+\frac{\delta_x^2}{12}+1\Big)f_{j,k}^n, \\
\Big(1+\big(\frac{1}{12}-\mu_y\big)\delta_y^2\Big)U_{j,k}^n=U_{j,k}^{n*}.
\end{cases} \tag{21}$$

Let the (known) right-hand side of Eq. (21) be denoted by $\mathbf{f}$, then

$$U_{j,k}^{n*}+\big(\frac{1}{12}-\mu_x\big)\big(U_{j+1,k}^{n*}-2U_{j,k}^{n*}+U_{j-1,k}^{n*}\big)=\mathbf{f}_{j,k}^n, \tag{22}$$

Thus, for each $1 \leq k \leq N_y, n = 1, 2, \ldots, N$, a simple tridiagonal system is solved for $U_{j,k}^{n*}$.

After calculating $U_{j,k}^{n*}$, we obtain $U_{j,k}^n$ from

$$U_{j,k}^n+\big(\frac{1}{12}-\mu_x\big)\big(U_{j,k+1}^n-2U_{j,k}^n+U_{j,k-1}^n\big)=U_{j,k}^{n*}.$$

For each $1 \leq j \leq N_x, n = 1, 2, \ldots, N$, a tridiagonal system of equations is solved for $U_{j,k}^n$.

There is an important point that needs to be addressed with regard to the solution of the system (22). When $j = 1$ or $j = N_x$ boundary values for $U_{j,k}^{n*}$ are required in the form of $U_{0,k}^{n*}$ or $U_{N_x+1,k}^{n*}$. We can obtain the required boundary conditions for $U_{j,k}^{n*}$ from the second and fourth equations in (21). For example, at the $j = 1$ we get

$$U_{0,k}^{n*}=\Big(1+\big(\frac{1}{12}-\mu_y\big)\delta_y^2\Big)U_{j,k}^n=\Big(1+\big(\frac{1}{12}-\mu_y\big)\delta_y^2\Big)\varphi_1(y_k,t_n).$$

Theorem 2.3. *The coefficient matrices in the scheme* (21) *is invertible.*

Proof. It is clear that the coefficient matrices for (21) are strictly diagonally dominant. Thus these matrices are invertible. □

3. Error analysis

3.1. Stability analysis

To study the stability analysis of the proposed scheme, we use the Fourier method. This method is a very useful technique for analyzing the stability of a finite difference method. It uses mutually orthogonal vectors that form a basis for n-dimensional space. Thus the error term can be expanded as a linear combination of these basis vectors. Let $\tilde{U}^l_{mnk}$ be the approximate solution of the scheme, and define

$$\zeta^n_{jk} = U^n_{jk} - \tilde{U}^n_{jk}, \qquad 1 \le j \le N_x, \quad 1 \le k \le N_y, \quad 0 \le n \le N,$$

with corresponding vector

$$\zeta^n = (\zeta^n_{1,1}, \zeta^n_{1,2}, \dots, \zeta^n_{N_x,N_y})^T.$$

Thus we have

$$\begin{aligned}&\left(1 + \left(\frac{1}{12} - \mu_x\right)\delta_x^2\right)\left(1 + \left(\frac{1}{12} - \mu_y\right)\delta_y^2\right)\zeta^1_{j,k}\\ &= \left(\frac{1}{144} + \mu_x\mu_y\right)\delta_x^2\delta_y^2\zeta^0_{j,k} + \left(\frac{\delta_x^2}{12} + \frac{\delta_y^2}{12} + 1\right)\zeta^0_{j,k},\end{aligned} \tag{23}$$

and for $n \ge 2$,

$$\begin{aligned}&\left(1+\left(\frac{1}{12} - \mu_x\right)\delta_x^2\right)\left(1 + \left(\frac{1}{12} - \mu_y\right)\delta_y^2\right)\zeta^n_{j,k}\\ &=\left(\frac{1}{144} + \mu_x\mu_y\right)\delta_x^2\delta_y^2\left(2\zeta^{n-1}_{j,k} - \zeta^{n-2}_{j,k}\right) + \frac{1}{d_{n,n}}\left(\frac{\delta_x^2}{12} + \frac{\delta_y^2}{12} + 1\right)\left[\sum_{l=1}^{n-1} (d_{l+1,n} - d_{l,n})\,\zeta^l_{j,k} + d_{1,n}\zeta^0_{j,k}\right].\end{aligned} \tag{24}$$

The Fourier series for $\zeta^n(x, y)$ is

$$\zeta^n(x, y) = \sum_{m_2=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} \rho_n(m_1, m_2)e^{i2\pi\{m_1 \frac{x-L_1}{L_2-L_1} + m_2 \frac{y-L_3}{L_4-L_3}\}},$$

and the discrete Fourier coefficients are

$$\begin{aligned}\rho_n(m_1, m_2) = &\frac{1}{\sqrt{(L_2 - L_1)(L_4 - L_3)}}\\ &\times \int_{L_3}^{L_4} \int_{L_2}^{L_1} e^{-i2\pi\{m_1 \frac{\xi-L_1}{L_2-L_1} + m_2 \frac{\eta-L_3}{L_4-L_3}\}}\zeta^n(\xi, \eta)d\xi d\eta.\end{aligned}$$

For any vector $Z = (z_{1,1}, z_{1,2}, \dots, z_{N_x,N_y})^T$, we define the discrete l^2 norm as follows

$$\|Z\|_{l^2} = \left(h_x h_y \sum_{l=1}^{N_y} \sum_{j=1}^{N_x} z^2_{j,k}\right)^{\frac{1}{2}}.$$

So we have Parseval's equality for the discrete Fourier transform, that is,

$$\begin{aligned}\|\zeta^n\|^2_{l^2} &= h_x h_y \sum_{l=1}^{N_y} \sum_{j=1}^{N_x} |\zeta^n_{j,k}|^2\\ &= \int_{L_3}^{L_4} \int_{L_2}^{L_1} |\zeta^n(\xi, \eta)|^2 d\xi d\eta = \sum_{m_2=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} |\rho_n(m_1, m_2)|^2.\end{aligned}$$

We can expand $\zeta_{j,k}^n$ into Fourier series, and Because the difference equations are linear, we can analyze the behavior of the total error by tracking the behavior of an arbitrary nth component. So we can suppose that the solution of Eq. (23) and Eq. (24) has the following form

$$\zeta_{j,k}^n = \frac{1}{\sqrt{L_2 - L_1}\sqrt{L_4 - L_3}}\rho_n e^{i(\sigma_x j h_x + \sigma_y k h_y)},$$

where $\sigma_x = 2m_1\pi/(L_2 - L_1)$, $\sigma_y = 2m_2\pi/(L_4 - L_3)$.

Substituting the above expression into Eq. (23) and Eq. (24) and for simplicity we let $s_1 = \sin^2 \frac{\sigma_x h_x}{2}$, $s_2 = \sin^2 \frac{\sigma_y h_y}{2}$, ($thus$ $0 \le s_i \le 1$, $i = 1, 2$), then for $n = 1$ we have

$$\Big(1 - 4\Big(\frac{1}{12} - \mu_x\Big)s_1\Big)\Big(1 - 4\Big(\frac{1}{12} - \mu_y\Big)s_2\Big)\rho_1 = \Big[16\mu_x\mu_y s_1 s_2 + \big(1 - \frac{1}{3}s_1\big)\big(1 - \frac{1}{3}s_2\big)\Big]\rho_0,$$

and for $2 \le n \le N$, we get

$$\begin{aligned}&\Big(1 - 4\Big(\frac{1}{12} - \mu_x\Big)s_1\Big)\Big(1 - 4\Big(\frac{1}{12} - \mu_y\Big)s_2\Big)\rho_n = \\ &\Big[32\big(\frac{1}{144} + \mu_x\mu_y\big)s_1 s_2 + \big(\frac{d_{n,n} - d_{n-1,n}}{d_{n,n}}\big)\big(- \frac{1}{3}(s_1 + s_2) + 1\big)\Big]\rho_{n-1} \\ &\Big[-16\big(\frac{1}{144} + \mu_x\mu_y\big)s_1 s_2 + \big(\frac{d_{n-1,n} - d_{n-2,n}}{d_{n,n}}\big)\big(- \frac{1}{3}(s_1 + s_2) + 1\big)\Big]\rho_{n-2} \\ &+ \big(- \frac{1}{3}(s_1 + s_2) + 1\big)\left(\sum_{l=1}^{n-3}\left(\frac{d_{l+1,n} - d_{l,n}}{d_{n,n}}\right)\rho_l + \frac{d_{1,n}}{d_{n,n}}\rho_0,\right)\Big).\end{aligned}$$

Consequently, we obtain

$$\rho_1 = \frac{16\mu_x\mu_y s_1 s_2 + \big(1 - \frac{1}{3}s_1\big)\big(1 - \frac{1}{3}s_2\big)}{\big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\big)\big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\big)}\rho_0 \equiv \frac{\eta_0}{\mu}\rho_0, \tag{25}$$

$$\begin{aligned}\rho_n = &\frac{32\big(\frac{1}{144} + \mu_x\mu_y\big)s_1 s_2 + \big(\frac{d_{n,n} - d_{n-1,n}}{d_{n,n}}\big)\big(- \frac{1}{3}(s_1 + s_2) + 1\big)}{\big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\big)\big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\big)}\rho_{n-1} \\ &\frac{-16\big(\frac{1}{144} + \mu_x\mu_y\big)s_1 s_2 + \big(\frac{d_{n-1,n} - d_{n-2,n}}{d_{n,n}}\big)\big(- \frac{1}{3}(s_1 + s_2) + 1\big)}{\big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\big)\big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\big)}\rho_{n-2} \\ &+ \frac{1 - \frac{1}{3}(s_1 + s_2)}{\big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\big)\big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\big)}\left(\sum_{l=1}^{n-3}\left(\frac{d_{l+1,n} - d_{l,n}}{d_{n,n}}\right)\rho_l + \frac{d_{1,n}}{d_{n,n}}\rho_0\right) \\ &\equiv \sum_{l=1}^{n}\frac{\eta_{n-l}}{\mu}\rho_{n-l}, \quad 2 \le n \le N,\end{aligned} \tag{26}$$

where $\mu = \big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\big)\big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\big)$.

Remark 3.1. *The coefficient μ satisfies in the following relation* $0 < \frac{1}{\mu} \leq \frac{9}{4}$.

Now, we can give the following estimate.

Theorem 3.2. *Suppose that ρ_n $(1 \leq n \leq N)$ are defined by* (25) *and* (26), *then for $\gamma \in (0,1)$*

$$|\rho_n| \leq \Big(1 + \tfrac{C\tau}{\mu}\Big)^n |\rho_0|, \quad n = 1, 2, \cdots, N.$$

Proof. We will prove this claim by mathematical induction. For $n = 1$ we have

$$|\rho_1| = |\frac{\eta_0}{\mu}||\rho_0| \leq \Big(1 + \frac{C\tau}{\mu}\Big)|\rho_0|.$$

Assume that

$$|\rho_m| \leq \Big(1 + \frac{C\tau}{\mu}\Big)^m |\rho_0|, \qquad m = 1, 2, \ldots, n-1,$$

then we get

$$\begin{aligned}
|\rho_n| &= \left| \sum_{l=1}^{n} \frac{\eta_{n-l}}{\mu} \rho_{n-l} \right| \\
&\leq \left| \frac{32\Big(\frac{1}{144} + \mu_x\mu_y\Big)s_1 s_2 + \Big(\frac{d_{n,n}-d_{n-1,n}}{d_{n,n}}\Big)\Big(-\frac{1}{3}(s_1 + s_2) + 1\Big)}{\Big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\Big)\Big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\Big)} \right| \Big(1 + \frac{C\tau}{\mu}\Big)^{n-1} |\rho_0| \\
&\left| \frac{-16\Big(\frac{1}{144} + \mu_x\mu_y\Big)s_1 s_2 + \Big(\frac{d_{n-1,n}-d_{n-2,n}}{d_{n,n}}\Big)\Big(-\frac{1}{3}(s_1 + s_2) + 1\Big)}{\Big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\Big)\Big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\Big)} \right| \Big(1 + \frac{C\tau}{\mu}\Big)^{n-1} |\rho_0| \\
&+ \frac{1 - \frac{1}{3}(s_1 + s_2)}{\Big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\Big)\Big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\Big)} \Big(1 + \frac{C\tau}{\mu}\Big)^{n-1} \left(\sum_{l=1}^{n-3} \Big(\frac{d_{l+1,n} - d_{l,n}}{d_{n,n}}\Big)\rho_0| + \frac{d_{1,n}}{d_{n,n}}|\rho_0| \right) \\
&\leq \frac{48\mu_x\mu_y s_1 s_2 + \Big(1 - \frac{1}{3}s_1\Big)\Big(1 - \frac{1}{3}s_2\Big)}{\Big(1 - \frac{1}{3}s_1 + 4\mu_x s_1\Big)\Big(1 - \frac{1}{3}s_2 + 4\mu_y s_2\Big)} \Big(1 + \frac{C\tau}{\mu}\Big)^{n-1} |\rho_0| \leq \Big(1 + \frac{C\tau}{\mu}\Big)^n |\rho_0|.
\end{aligned}$$

and the proof is completed. □

Theorem 3.3. *The compact finite difference scheme* (21) *is unconditionally stable for $\gamma \in (0,1)$.*

Proof. Suppose that $\tilde{U}^n$ is the approximate solution of (21). Applying Theorem 3.2, Parseval's equality and $n\tau \leq T$ we obtain

$$
\begin{aligned}
\|U^n - \tilde{U}^n\|_{l^2}^2 &= \|\zeta^n\|_{l^2}^2 \\
&= h_x h_y \sum_{k=1}^{N_y} \sum_{j=1}^{N_x} |\zeta_{j,k}^n|^2 \\
&= \frac{h_x}{L_2 - L_1} \frac{h_y}{L_4 - L_3} \sum_{k=1}^{N_y} \sum_{j=1}^{N_x} |\rho_n e^{i(\sigma_x j h_x + \sigma_y k h_y)}|^2 \\
&= \frac{h_x}{L_2 - L_1} \frac{h_y}{L_4 - L_3} \sum_{k=1}^{N_y} \sum_{j=1}^{N_x} |\rho_n|^2 \\
&\le \frac{h_x}{L_2 - L_1} \frac{h_y}{L_4 - L_3} \sum_{k=1}^{N_y} \sum_{j=1}^{N_x} \Big(1 + \frac{C\tau}{\mu}\Big)^{2n} |\rho_0|^2 \\
&= \frac{h_x}{L_2 - L_1} \frac{h_y}{L_4 - L_3} \Big(1 + \frac{C\tau}{\mu}\Big)^{2n} \sum_{k=1}^{N_y} \sum_{j=1}^{N_x} |\rho_0 e^{i(\sigma_x j h_x + \sigma_y k h_y)}|^2 \\
&= \Big(1 + \frac{C\tau}{\mu}\Big)^{2n} \|\zeta^0\|_{l^2}^2 \le e^{\frac{2\times 9Cn\tau}{4}} \|U^0 - \tilde{U}^0\|_{l^2}^2, \quad n = 1, 2, \cdots, N.
\end{aligned}
$$

So that

$$\|U^n - \tilde{U}^n\|_{l^2} \le e^{\frac{9}{4}TC} \|U^0 - \tilde{U}^0\|_{l^2},$$

which means that the scheme is unconditionally stable. □

3.2. Convergence

We will use the Fourier analysis for the scheme (21).

For $1 \le j \le N_x, 1 \le k \le N_y$, Let $e_{j,k}^n = u_{j,k}^n - U_{j,k}^n$ and $e^n = (e_{1,1}^n, e_{1,2}^n, \dots, e_{N_x,N_y}^n)^T$, $\mathbf{R}^n = (R_{1,1}^n, R_{1,2}^n, \dots, R_{N_x,N_y}^n)^T$, $1 \le n \le N$. Then

$$\Big(1 + \Big(\frac{1}{12} - \mu_x\Big)\delta_x^2\Big)\Big(1 + \Big(\frac{1}{12} - \mu_y\Big)\delta_y^2\Big) e_{j,k}^1 = R_{j,k}^1, \tag{27}$$

and for $n \ge 2$

$$
\begin{aligned}
\Big(1 + \Big(\frac{1}{12} - \mu_x\Big)\delta_x^2\Big)\Big(1 + \Big(\frac{1}{12} - \mu_y\Big)\delta_y^2\Big) e_{j,k}^n &= \Big(\frac{1}{144} + \mu_x \mu_y\Big)\Big(2e_{j,k}^{n-1} - e_{j,k}^{n-2}\Big) \\
&\quad + \frac{1}{d_{n,n}}\Big(\frac{\delta_x^2}{12} + \frac{\delta_y^2}{12} + 1\Big)\Big[\sum_{l=1}^{n-1} (d_{l+1,n} - d_{l,n})\, e_{j,k}^l\Big] + R_{j,k}^n.
\end{aligned} \tag{28}
$$

Similar to the stability we have

$$\|e^n\|_{l^2}^2 = h_x h_y \sum_{l=1}^{N_y} \sum_{j=1}^{N_x} |e_{j,k}^n|^2 = \sum_{m_2=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} |c_n(m_1, m_2)|^2, \tag{29}$$

and

$$\|R^n\|_{l^2}^2 = h_x h_y \sum_{l=1}^{N_y} \sum_{j=1}^{N_x} |R_{j,k}^n|^2 = \sum_{m_2=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} |d_n(m_1, m_2)|^2. \tag{30}$$

Now we assume that $e^n_{j,k}$ and $R^n_{j,k}$ are

$$e^n_{j,k} = \frac{1}{\sqrt{L_2 - L_1}\sqrt{L_4 - L_3}} c_n e^{i(\sigma_x jh_x + \sigma_y kh_y)},$$

$$R^n_{j,k} = \frac{1}{\sqrt{L_2 - L_1}\sqrt{L_4 - L_3}} d_n e^{i(\sigma_x jh_x + \sigma_y kh_y)},$$

where $\sigma_x = 2m_1\pi/(L_2 - L_1)$ and $\sigma_y = 2m_2\pi/(L_4 - L_3)$.
Substituting the above expression into Eqs. (27) and (28) and for simplicity we let $s_1 = \sin^2 \dfrac{\sigma_x h_x}{2}$, $s_2 = \sin^2 \dfrac{\sigma_y h_y}{2}$, (*thus* $0 \le s_i \le 1$, $i = 1, 2$), then for $n = 1$ we have

$$c_1 = \frac{1}{\mu} d_1, \tag{31}$$

and for $n \ge 2$

$$\begin{aligned} c_n = {} & \frac{32\left(\frac{1}{144} + \mu_x\mu_y\right)s_1s_2 + \left(\frac{d_{n,n} - d_{n-1,n}}{d_{n,n}}\right)\left(-\frac{1}{3}(s_1 + s_2) + 1\right)}{\mu} c_{n-1} \\ & \frac{-16\left(\frac{1}{144} + \mu_x\mu_y\right)s_1s_2 + \left(\frac{d_{n-1,n} - d_{n-2,n}}{d_{n,n}}\right)\left(-\frac{1}{3}(s_1 + s_2) + 1\right)}{\mu} c_{n-2} \\ & + \frac{1 - \frac{1}{3}(s_1 + s_2)}{\mu}\left(\sum_{l=1}^{n-3}\left(\frac{d_{l+1,n} - d_{l,n}}{d_{n,n}}\right)c_l + \frac{d_{1,n}}{d_{n,n}}c_0\right) + \frac{d_n}{\mu}, \end{aligned} \tag{32}$$

where $\mu = \left(1 - \frac{1}{3}s_1 + 4\mu_x s_1\right)\left(1 - \frac{1}{3}s_2 + 4\mu_y s_2\right)$.

Also from convergence of the series in the right hand side of (30), there is a positive constant C_2 such that

$$|d_n| \le C_2 L\tau |d_1|, \quad n = 1, 2, \ldots, N. \tag{33}$$

Theorem 3.4. *If c_n be the solutions of equations* (31) *and* (32)*, then there is positive constants C_2 and B_n such that*

$$|c_n| \le C_2 B_n \left(1 + \frac{\tau L}{\mu}\right)^n |d_1|, \quad n = 1, 2, \ldots, N. \tag{34}$$

Proof. We use the mathematical induction for proof. For $n = 1$, from Eq. (31) we have

$$|c_1| \le \frac{1}{\mu}|d_1| \le \frac{\tau L}{\mu} C_2 |d_1| \le C_2\left(1 + \frac{\tau L}{\mu}\right)|d_1|.$$

Now, suppose that

$$|c_m| \le C_2 B_m \left(1 + \frac{\tau L}{\mu}\right)^m |d_1|, \quad m = 1, 2, \ldots, n - 1.$$

Using Eq. (32), we have

$$
\begin{aligned}
|c_n| \le & \left| \frac{32\left(\frac{1}{144} + \mu_x\mu_y\right)s_1s_2 + \left(\frac{d_{n,n}-d_{n-1,n}}{d_{n,n}}\right)\left(-\frac{1}{3}(s_1+s_2)+1\right)}{\mu} \right| C_2 B_{n-1}\left(1+\frac{\tau L}{\mu}\right)^{n-1} |d_1| \\
& \left| \frac{-16\left(\frac{1}{144} + \mu_x\mu_y\right)s_1s_2 + \left(\frac{d_{n-1,n}-d_{n-2,n}}{d_{n,n}}\right)\left(-\frac{1}{3}(s_1+s_2)+1\right)}{\mu} \right| C_2 B_{n-2}\left(1+\frac{\tau L}{\mu}\right)^{n-1} |d_1| \\
& + \frac{1-\frac{1}{3}(s_1+s_2)}{\mu}\left(\sum_{l=1}^{n-3}\left(\frac{d_{l+1,n}-d_{l,n}}{d_{n,n}}\right) C_2 B_l \left(1+\frac{\tau L}{\mu}\right)^{n-1}|d_1|\right)\Bigg| + \left|\frac{d_n}{\mu}\right|.
\end{aligned}
$$

Now assume that

$$C' = \max\{B_1, B_2, \ldots, B_{n-1}\},$$

so

$$
\begin{aligned}
|c_n| \le & \left| \frac{32\left(\frac{1}{144} + \mu_x\mu_y\right)s_1s_2 + \left(\frac{d_{n,n}-d_{n-1,n}}{d_{n,n}}\right)\left(-\frac{1}{3}(s_1+s_2)+1\right)}{\mu} \right| C_2 C'\left(1+\frac{\tau L}{\mu}\right)^{n-1} |d_1| \\
& \left| \frac{-16\left(\frac{1}{144} + \mu_x\mu_y\right)s_1s_2 + \left(\frac{d_{n-1,n}-d_{n-2,n}}{d_{n,n}}\right)\left(-\frac{1}{3}(s_1+s_2)+1\right)}{\mu} \right| C_2 C'\left(1+\frac{\tau L}{\mu}\right)^{n-1} |d_1| \\
& + \frac{1-\frac{1}{3}(s_1+s_2)}{\mu}\left(\sum_{l=1}^{n-3}\left(\frac{d_{l+1,n}-d_{l,n}}{d_{n,n}}\right) C_2 C' \left(1+\frac{\tau L}{\mu}\right)^{n-1}|d_1|\right) + C_2\left(1+\frac{\tau L}{\mu}\right)^{n}|d_1| \\
\le & \frac{48\left(\frac{1}{144} + \mu_x\mu_y\right)s_1s_2 + \left(-\frac{1}{3}(s_1+s_2)+1\right)}{\mu} C_2 C'\left(1+\frac{\tau L}{\mu}\right)^{n-1}|d_1| + C_2\left(1+\frac{\tau L}{\mu}\right)^{n}|d_1| \\
\le & C_2 C'\left(1+\frac{\tau L}{\mu}\right)^{n}|d_1| + C_2\left(1+\frac{\tau L}{\mu}\right)^{n}|d_1| = C_2(1+C')\left(1+\frac{\tau L}{\mu}\right)^{n}|d_1| \\
= & C_2 B_n\left(1+\frac{\tau L}{\mu}\right)^{n}|d_1|.
\end{aligned}
$$

This completes the proof. □

Theorem 3.5. *The compact ADI finite difference scheme* (21) *is convergent, and the order of convergence is* $O(\tau + h^4)$.

Proof. By Theorem (3.4) and using (29) and (30), we can obtain

$$\|e^n\|_{l^2} \le C_2 B_n \left(1+\frac{\tau L}{\mu}\right)^{n} \|R^1\|_{l^2} \le C_2 B_n e^{\frac{n\tau L}{\mu}} C_3(\tau + h^4).$$

By remark (3.1) $0 \le \frac{1}{\mu} \le \frac{9}{4}$ and since $n\tau \le T$, we obtain

$$\|e^n\|_{l^2} \le C(\tau + h^4), \quad C = C_2 B_n C_3 e^{\frac{9LT}{4}}.$$

□

4. Numerical Results

In this section, we perform some numerical calculations to test the convergence of our scheme. All the computations are performed using Matlab. In all examples we assume that $h_x = h_y = h$ and use the error norm

$$\|e^n\|_{l_\infty} = \max \left| u(x_j, y_k, t_n) - U^n_{j,k} \right| : 0 \le j \le N_x, 0 \le k \le N_y,$$

where $e^n_{j,k} = u^n_{j,k} - U^n_{j,k}$. Also we calculated the computational orders of the method presented in this paper with the following formula

$$r(\tau, h) = \log_2(\|e(4\tau, 2h)\|_{l_\infty} / \|e(\tau, h)\|_{l_\infty}).$$

Example 4.1. *We first consider the following equation with the initial and boundary conditions*

$$\begin{cases} {}^{CF}_{0}D^{\gamma}_{t} u(x, y, t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \ 0 < x, y < 1, \ 0 \le t \le 1, \\ u(0, y, t) = 0, u(1, y, t) = 0, \\ u(x, 0, t) = 0, u(x, 1, t) = 0, \\ u(x, y, 0) = 0. \end{cases}$$

We give exact solution $u(x, y, t) = xy(1 - x)(1 - y)\sin(t)$, and for different γ, we have different $f(x, y, t)$. Numerical solution and pointwise errors have been demonstrated in Fig. (1). *Table* (1) *gives the approximation errors and convergence rates for the fourth order compact difference scheme. We choose different space and time step sizes to obtain the numerical convergence rate in space.*

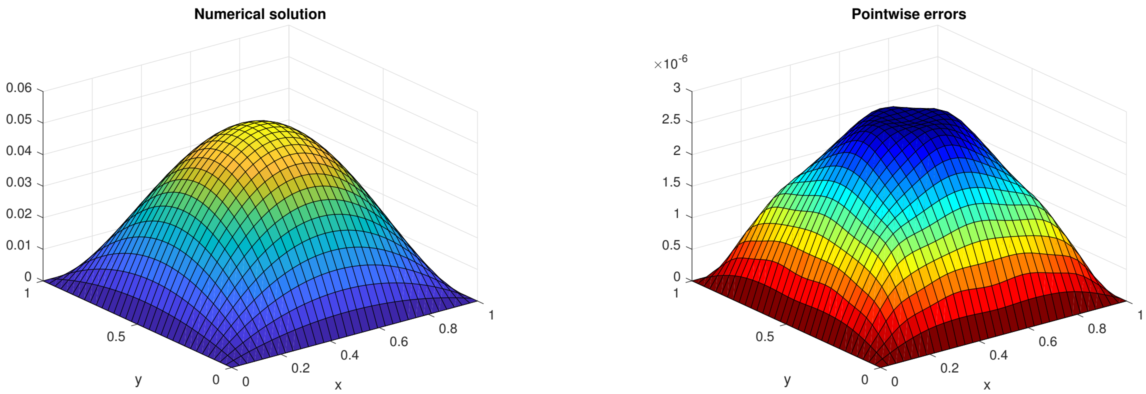


Figure 1: The plot of numerical solution and pointwise errors at $\tau = 1/256$ and $h_x = h_y = 1/32$ with $\gamma = 0.2$ for example 4.1.

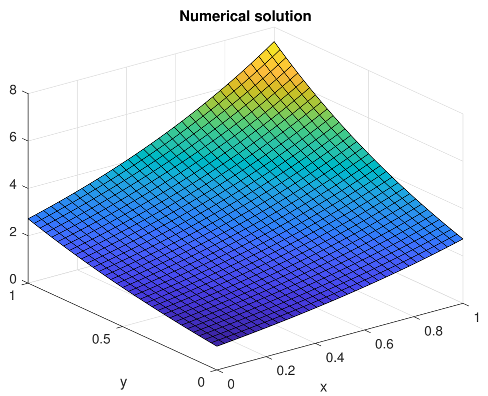
Example 4.2. *Consider the two dimensional time fractional diffusion equation as follows*

$$\begin{cases} {}^{CF}_{0}D^{\gamma}_{t} u(x, y, t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \ 0 < x, y < 1, \ 0 \le t \le 1, \\ u(0, y, t) = t^2 e^y, u(1, y, t) = t^2 e^{1+y}, \\ u(x, 0, t) = t^2 e^x, u(x, 1, t) = t^2 e^{x+1}, \\ u(x, y, 0) = 0. \end{cases}$$

The exact solution for this problem is $u(x, y, t) = t^2 e^{x+y}$. Table (2) *gives the approximation errors and rates of u for the compact difference scheme of Example* (4.2). *Results show that the space rates are almost $O(h^4)$ and are consistent with our theoretical analysis. Also Numerical solution and pointwise errors have been demonstrated in Fig.* (2).

Table 1: Error and experiment order of convergence for different values of γ, for example 4.1

γ	h	τ	$\|e^n\|_{l_\infty}$	Order
0.1	$\frac{1}{4}$	$\frac{1}{4}$	$1.3683e-02$	–
0.1	$\frac{1}{8}$	$\frac{1}{16}$	$7.0627e-04$	4.2760
0.1	$\frac{1}{16}$	$\frac{1}{64}$	$5.6257e-05$	3.6501
0.1	$\frac{1}{32}$	$\frac{1}{256}$	$3.3538e-06$	4.0682
0.3	$\frac{1}{4}$	$\frac{1}{4}$	$9.5564e-03$	–
0.3	$\frac{1}{8}$	$\frac{1}{16}$	$2.8612e-04$	5.0618
0.3	$\frac{1}{16}$	$\frac{1}{64}$	$3.9921e-05$	2.4989
0.3	$\frac{1}{32}$	$\frac{1}{256}$	$2.3441e-06$	4.0900
0.5	$\frac{1}{4}$	$\frac{1}{4}$	$5.0387e-03$	–
0.5	$\frac{1}{8}$	$\frac{1}{16}$	$4.6054e-04$	3.4516
0.5	$\frac{1}{16}$	$\frac{1}{64}$	$2.4932e-05$	4.2072
0.5	$\frac{1}{32}$	$\frac{1}{256}$	$1.6940e-06$	3.8795
0.7	$\frac{1}{4}$	$\frac{1}{4}$	$1.0617e-03$	–
0.7	$\frac{1}{8}$	$\frac{1}{16}$	$3.4366e-04$	1.6273
0.7	$\frac{1}{16}$	$\frac{1}{64}$	$1.7136e-05$	4.3258
0.7	$\frac{1}{32}$	$\frac{1}{256}$	$1.0063e-06$	4.0899
0.9	$\frac{1}{4}$	$\frac{1}{4}$	$1.2361e-03$	–
0.9	$\frac{1}{8}$	$\frac{1}{16}$	$1.0447e-04$	3.5646
0.9	$\frac{1}{16}$	$\frac{1}{64}$	$1.6022e-05$	4.2301
0.9	$\frac{1}{32}$	$\frac{1}{256}$	$3.3166e-07$	5.5942



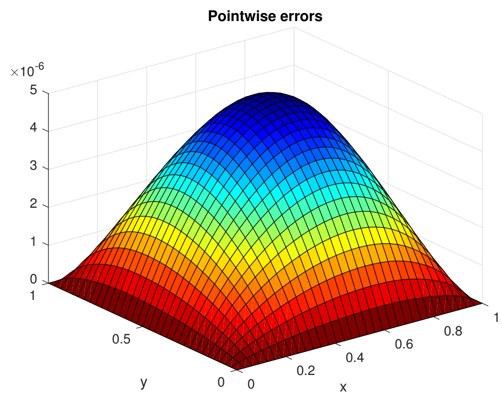


Figure 2: The plot of numerical solution and pointwise errors at $\tau = 1/256$ and $h_x = h_y = 1/32$ with $\gamma = 0.2$ for example 4.2.

Example 4.3. *Consider the following equation*

$$\begin{cases} {}_0^{CF}D_t^{\gamma} u(x,y,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t),\ 0 < x,y < 1,\ 0 \le t \le 1, \\ u(0,y,t) = 0, u(1,y,t) = 0, \\ u(x,0,t) = 0, u(x,1,t) = 0, \\ u(x,y,0) = \sin(\pi x)\sin(\pi y). \end{cases}$$

We give exact solution $u(x,y,t) = (t^2+1)\sin(\pi x)\sin(\pi y)$, and for different γ, we have different $f(x,y,t)$. The maximum absolute errors and their estimated convergence rates approximated by the compact ADI method are shown in Table (3). *The numerical results of Example* (4.3) *are provided to show that the proposed approximation method is computationally efficient.*

Table 2: Error and experiment order of convergence for different values of γ, for example 4.2

γ	h	τ	$\|e^n\|_{l_\infty}$	Order
0.1	$\frac{1}{4}$	$\frac{1}{4}$	$2.2886e-02$	–
0.1	$\frac{1}{8}$	$\frac{1}{16}$	$1.2102e-03$	4.2411
0.1	$\frac{1}{16}$	$\frac{1}{64}$	$8.9767e-05$	3.7529
0.1	$\frac{1}{32}$	$\frac{1}{256}$	$5.5718e-06$	4.0010
0.3	$\frac{1}{4}$	$\frac{1}{4}$	$1.9323e-02$	–
0.3	$\frac{1}{8}$	$\frac{1}{16}$	$1.0779e-03$	4.1640
0.3	$\frac{1}{16}$	$\frac{1}{64}$	$6.6135e-05$	4.0267
0.3	$\frac{1}{32}$	$\frac{1}{256}$	$4.1376e-06$	3.9985
0.5	$\frac{1}{4}$	$\frac{1}{4}$	$1.4057e-02$	–
0.5	$\frac{1}{8}$	$\frac{1}{16}$	$7.0760e-04$	4.3122
0.5	$\frac{1}{16}$	$\frac{1}{64}$	$4.0324e-05$	4.1332
0.5	$\frac{1}{32}$	$\frac{1}{256}$	$2.5167e-06$	4.0020
0.7	$\frac{1}{4}$	$\frac{1}{4}$	$5.2053e-03$	–
0.7	$\frac{1}{8}$	$\frac{1}{16}$	$1.2603e-04$	5.3681
0.7	$\frac{1}{16}$	$\frac{1}{64}$	$6.2757e-06$	4.3278
0.7	$\frac{1}{32}$	$\frac{1}{256}$	$3.6523e-07$	4.1029
0.9	$\frac{1}{4}$	$\frac{1}{4}$	$1.6284e-02$	–
0.9	$\frac{1}{8}$	$\frac{1}{16}$	$1.1551e-03$	3.8174
0.9	$\frac{1}{16}$	$\frac{1}{64}$	$7.6274e-05$	3.9207
0.9	$\frac{1}{32}$	$\frac{1}{256}$	$4.8040e-06$	3.9889

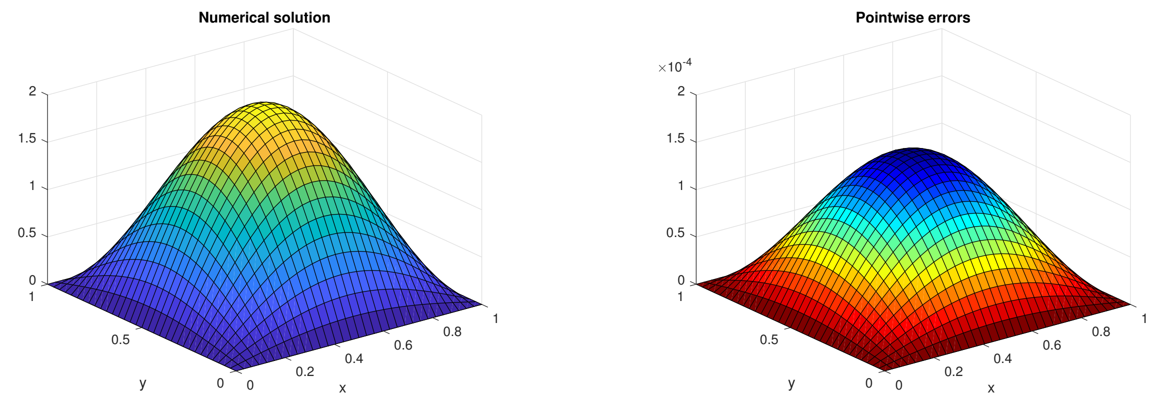


Figure 3: The plot of numerical solution and pointwise errors at $\tau = 1/256$ and $h_x = h_y = 1/32$ with $\gamma = 0.2$ for example 4.3.

Example 4.4. *In this example the new compact ADI scheme is used to solve the following equation*

$$\begin{cases} {}_0^{CF}D_t^{\gamma} u(x,y,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t),\ 0 < x,y < 1,\ 0 \le t \le 1, \\ u(0,y,t) = 0, u(1,y,t) = 0, \\ u(x,0,t) = 0, u(x,1,t) = 0, \\ u(x,y,0) = x^2(1-x)^3 \sin(\pi y) e^{x+2y}. \end{cases}$$

The exact solution for this problem is $u(x,y,t) = (t^2+1)x^2(1-x)^3 \sin(\pi y)e^{x+2y}$. Numerical results for time and space with $\gamma = 0.1, 0.3, 0.5, 0.7, 0.9$ are presented in Table (4), respectively. Figure (4) shows the numerical solutions and pointwise errors.

Table 3: Error and experiment order of convergence for different values of γ, for example 4.3

γ	h	τ	$\|e^n\|_{l_\infty}$	Order
0.1	$\frac{1}{4}$	$\frac{1}{4}$	$5.3409e-01$	–
0.1	$\frac{1}{8}$	$\frac{1}{16}$	$2.8900e-02$	4.2079
0.1	$\frac{1}{16}$	$\frac{1}{64}$	$2.1592e-03$	3.7425
0.1	$\frac{1}{32}$	$\frac{1}{256}$	$1.5036e-04$	3.8440
0.3	$\frac{1}{4}$	$\frac{1}{4}$	$4.7311e-01$	–
0.3	$\frac{1}{8}$	$\frac{1}{16}$	$2.7209e-02$	4.1200
0.3	$\frac{1}{16}$	$\frac{1}{64}$	$1.9246e-03$	3.8214
0.3	$\frac{1}{32}$	$\frac{1}{256}$	$1.7690e-04$	3.4435
0.5	$\frac{1}{4}$	$\frac{1}{4}$	$3.9814e-01$	–
0.5	$\frac{1}{8}$	$\frac{1}{16}$	$2.2690e-02$	4.1331
0.5	$\frac{1}{16}$	$\frac{1}{64}$	$1.7522e-03$	3.6948
0.5	$\frac{1}{32}$	$\frac{1}{256}$	$2.1784e-04$	3.0078
0.7	$\frac{1}{4}$	$\frac{1}{4}$	$3.0489e-01$	–
0.7	$\frac{1}{8}$	$\frac{1}{16}$	$1.5001e-02$	4.3451
0.7	$\frac{1}{16}$	$\frac{1}{64}$	$1.4322e-03$	3.3888
0.7	$\frac{1}{32}$	$\frac{1}{256}$	$2.2312e-04$	2.6823
0.9	$\frac{1}{4}$	$\frac{1}{4}$	$1.8515e-01$	–
0.9	$\frac{1}{8}$	$\frac{1}{16}$	$5.1344e-03$	5.1724
0.9	$\frac{1}{16}$	$\frac{1}{64}$	$3.5669e-04$	3.8474
0.9	$\frac{1}{32}$	$\frac{1}{256}$	$4.4063e-05$	3.0170

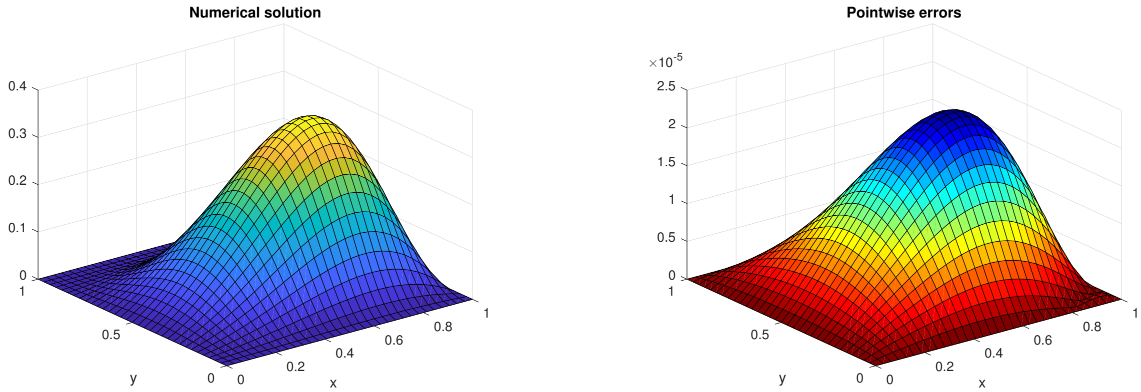


Figure 4: The plot of numerical solution and pointwise errors at $\tau = 1/256$ and $h_x = h_y = 1/32$ with $\gamma = 0.2$ for example 4.4.

5. Conclusions

In this paper, we constructed a high-order compact alternating direction implicit method for the solution of two dimensional time-fractional diffusion equation with Caputo-Fabrizio derivative. The time fractional derivative of the mentioned equation approximated by a scheme of order $O(\tau^2)$ and spatial derivatives replaced with a fourth order compact difference scheme. We prove that the scheme is unconditionally stable for $\gamma \in (0, 1)$. Numerical results confirmed the theoretical results of the proposed scheme, i.e the scheme has fourth order of accuracy in space.

Table 4: Error and experiment order of convergence for different values of γ, for example 4.4

γ	h	τ	$\|e^n\|_{l_\infty}$	Order
0.1	$\frac{1}{4}$	$\frac{1}{4}$	$9.7182e-02$	–
0.1	$\frac{1}{8}$	$\frac{1}{16}$	$7.9542e-03$	3.6109
0.1	$\frac{1}{16}$	$\frac{1}{64}$	$5.3281e-04$	3.9000
0.1	$\frac{1}{32}$	$\frac{1}{256}$	$3.6008e-05$	3.8872
0.3	$\frac{1}{4}$	$\frac{1}{4}$	$8.7864e-02$	–
0.3	$\frac{1}{8}$	$\frac{1}{16}$	$6.4657e-03$	3.7644
0.3	$\frac{1}{16}$	$\frac{1}{64}$	$4.4755e-04$	3.8526
0.3	$\frac{1}{32}$	$\frac{1}{256}$	$3.4778e-05$	3.6858
0.5	$\frac{1}{4}$	$\frac{1}{4}$	$75959e-02$	–
0.5	$\frac{1}{8}$	$\frac{1}{16}$	$5.01520e-03$	3.9208
0.5	$\frac{1}{16}$	$\frac{1}{64}$	$3.6417e-04$	3.7836
0.5	$\frac{1}{32}$	$\frac{1}{256}$	$3.6070e-05$	3.3357
0.7	$\frac{1}{4}$	$\frac{1}{4}$	$6.0359e-02$	–
0.7	$\frac{1}{8}$	$\frac{1}{16}$	$3.1687e-03$	4.2516
0.7	$\frac{1}{16}$	$\frac{1}{64}$	$2.6388e-04$	3.5859
0.7	$\frac{1}{32}$	$\frac{1}{256}$	$3.3105e-05$	2.9948
0.9	$\frac{1}{4}$	$\frac{1}{4}$	$4.0009e-02$	–
0.9	$\frac{1}{8}$	$\frac{1}{16}$	$1.1959e-03$	5.0642
0.9	$\frac{1}{16}$	$\frac{1}{64}$	$7.1557e-05$	4.0629
0.9	$\frac{1}{32}$	$\frac{1}{256}$	$6.4832e-06$	4.0629

References

[1] Santanu Saha Ray, Sahoo Subhadarshan, *Generalized Fractional Order Differential Equations Arising in Physical Models*, Chapman and Hall/CRC, 2018.

[2] Varsha Daftardar-Gejji, *Fractional Calculus and Fractional Differential Equations*, Springer Nature Singapore Pte Ltd, 2018.

[3] M.P. Aghababa, M. Borjkhani, *Chaotic fractional order model for muscular blood vessel and its control via fractional control scheme*, Complexity, 20 (2014) 37-46.

[4] A.E. El-Misiery, E. Ahmed, *On a fractional model for earthquakes*, Applied Mathematics and Computation, 178 (2006) 207-211.

[5] D. Baleanu, J.A. Machado, A.C. Luo, editors, *Fractional dynamics and control*, Springer Science and Business Media, 2011.

[6] H. Fallahgoul, S. Focardi, F. Fabozzi, *Fractional calculus and fractional processes with applications to financial economics: theory and application*, Academic Press, 2016.

[7] R.L. Magin, *Fractional calculus in bioengineering*, Redding: Begell House, 2006.

[8] D. Baleanu, A.M Lopes, *Handbook of Fractional Calculus with Applications Volume 7, Part A*, Walter de Gruyter, 2019.

[9] R. Ameen, F. Jarad,T. Abdeljawad, *Ulam Stability for Delay Fractional Differential Equations with a Generalized Caputo Derivative*, Filomat 32 (15), 5265-5274, (2018).

[10] M. Kratou, *Ground State Solutions of p-Laplacian singular Kirchhoff Problem involving a Riemann-Liouville fractional derivative*, Filomat 33 (7), 2073-2088, (2019).

[11] M.A. Ragusa, *Necessary and sufficient condition for a VMO function*, Applied Mathematics and Computation 218 (24), 11952-11958, (2012).

[12] K.A. Abro, J.F. Gómez-Aguilar, *Dual Fractional Analysis of Blood Alcohol Model Via Non-integer Order Derivatives*, In Fractional Derivatives with Mittag-Leffler Kernel (pp. 69-79), Springer Nature Switzerland AG, 2019.

[13] L. Changpin, F. Zeng, *Finite difference methods for fractional differential equations*, International Journal of Bifurcation and Chaos, 22(4) (2012) 1230014-28.

[14] E. Doha, A. Bhrawy, S. Ezz-Eldien, *Numerical approximations for fractional diffusion equations via a Chebyshev spectral-tau method*, Open Physics, 11(10) (2013) 1494-1503.

[15] M. Zheng, F. Liu, V. Anh, I. Turner, *A high-order spectral method for the multi-term time-fractional diffusion equations*, Applied Mathematical Modelling, 40(7-8) (2016) 4970-4985.

[16] J. Shen, C. Sheng, *Spectral methods for fractional differential equations using generalized Jacobi functions*, Numerical Methods, Walter de Gruyter, 2019.

[17] W. Fan, F. Liu, X. Jiang, I. Turner, *A novel unstructured mesh finite element method for solving the time-space fractional wave equation on a two-dimensional irregular convex domain*, Fractional Calculus and Applied Analysis, 20(2) (2017) 352-383.

[18] Z. Zhao, C. Li, *Fractional difference/finite element approximations for the time–space fractional telegraph equation*, Applied Mathematics and Computation, 219(6) (2012) 2975-2988.

[19] Q. Liu, Y. Gu, P. Zhuang, F. Liu, Y.F. Nie, *An implicit RBF meshless approach for time fractional diffusion equations*, Computational Mechanics, 48(1) (2011) 1-12.
[20] J. Zhao, M. Davison, R.M. Corless, *Compact finite difference method for American option pricing*, Journal of Computational and Applied Mathematics, 206(1) (2007) 306-321.
[21] F. Wu, D. Li, J. Wen, J. Duan, *Stability and convergence of compact finite difference method for parabolic problems with delay*, Applied Mathematics and Computation, 322 (2018) 129-139.
[22] H. Wang, X. Ma, J. Lu, W. Gao, *An efficient time-splitting compact finite difference method for Gross–Pitaevskii equation*, Applied Mathematics and Computation, 297 (2017) 131-144.
[23] J.C. Strikwerda, *Finite difference schemes and partial differential equations*, Siam, 2004.
[24] W.F. Spotz, G.F. Carey, *High-order compact finite difference methods*, Houston, TX, (1996) 397-408.
[25] Z. Li, Z. Qiao, T. Tang, *Numerical solution of differential equations: introduction to finite difference and finite element methods*, Cambridge University Press, 2017.
[26] J.L. Liouville, *Memoire sur le changement de la variable dans le calcul des differentielle a indices quelconques*, Journal de L'École Polytechnique, 15(24), 1835, 17-54.
[27] X.J. Yang, *General Fractional Derivatives: Theory, Methods and Applications*, Chapman and Hall/CRC, 2019.
[28] F. Wu, X. Cheng, D. Li, J. Duan, *A two-level linearized compact ADI scheme for two-dimensional nonlinear reaction–diffusion equations*, Computers Mathematics with Applications, 75(8) (2018) 2835-2850.
[29] S. Valizadeh, A. Borhanifar, A. Malek, *Compact ADI method for solving two-dimensional Riesz space fractional diffusion equation*, arXiv preprint arXiv, (2018) 1802.02015.
[30] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl, 1(2) (2015) 1-13.
[31] H. Wang, K. Wang, T. Sircar, *A direct O (N log2 N) finite difference method for fractional diffusion equations*, Journal of Computational Physics, 229(21) (2010) 8095-8104.
[32] M. Cui, *Convergence analysis of high-order compact alternating direction implicit schemes for the two-dimensional time fractional diffusion equation*, Numerical Algorithms, 62(3) (2013) 383-409.