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On Some General Inequalities Related to Operator AM-GM Inequality

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1. Introduction

Let $\mathbb{B}(\mathcal{H})$ be the C^*–algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. As usual, we reserve m, M for scalars and I for the identity operator on $\mathcal{H}$. A self adjoint operator A is said to be positive (written $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, while it is said to be strictly positive (written $A > 0$) if A is positive and invertible. If A and B are self adjoint, we write $B \geq A$ in case $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$–isomorphism between the C^*–algebra $C(\sigma(A))$ of continuous functions on the spectrum $\sigma(A)$ of a self adjoint operator A and the C^*–algebra generated by A and the identity operator I. This is called the functional calculus of A. If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)$ $(t \in \sigma(A))$ implies $f(A) \geq g(A)$ (see [11, p. 3]). A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It's said to be unital if $\Phi(I) = I$.

For any strictly positive operator $A, B \in \mathbb{B}(\mathscr{H})$ and $0 \leq v \leq 1$, we write

$$A!_v B = \left(A^{-1}\nabla_v B^{-1}\right)^{-1},\ A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{v}A^{\frac{1}{2}},\ A\nabla B = (1-v)A + vB.$$

For the case $v = \frac{1}{2}$, we write $!$, $\sharp$, and ∇, respectively. We use the same notions for scalars.

It is well-known that the arithmetic–geometric mean inequality (in short, AM-GM inequality), with respect to operator order, says that

$$A\sharp_v B \leq A\nabla_v B. \tag{1}$$

The Löwner–Heinz theorem [11, Theorem 1.8] says that if $A, B \in \mathbb{B}(\mathcal{H})$ are positive, then for $0 \leq p \leq 1$,

$$A \leq B \Rightarrow A^p \leq B^p. \tag{2}$$

In general (2) is not true for $p > 1$.

Lin [7] nicely reduced the study of squared operator inequalities to that of some norm inequalities. Actually, he found that a reverse of operator AM-GM inequality can be squared

$$(A\nabla B)^2 \leq K(h)^2(A\sharp B)^2, \tag{3}$$

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whenever $A, B \in \mathbb{B}(\mathcal{H})$ are two positive operators satisfying $mI \le A, B \le MI$ for some scalars $0 < m < M$, and $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$. It follows from (2) and (3) that

$$(A\nabla B)^p \le K(h)^p (A\sharp B)^p \quad (0 < p \le 2). \tag{4}$$

It is natural to ask whether inequality (4) is true for $p \ge 2$? Recently, an affirmative answer to this question has been given by Fu and He [2], where it has been proved that

$$(A\nabla B)^p \le \left(\frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right)^p (A\sharp B)^p.$$

The problem of squaring operator inequalities has been studied extensively in the literature. We refer the reader to [4, 8–10, 12] as sample of this work.
As mentioned above, Lin's method was based on some observations about the operator norm and an arithmetic-geometric mean inequality of Bhatia and Kittaneh (see [7, Lemma 2.2]). This paper intends to square a reverse of operator AM-GM inequality in a different way. Moreover, we square the operator Pólya–Szegö inequality [6, 10].

2. Main Results

To prove our generalized operator AM-GM inequalities, we need several well known lemmas. The first lemma is a simple consequence of the Jensen inequality concerning the convexity of certain power function [11, Theorem 1.4].

Lemma 2.1. *(Hölder–McCarthy inequality) Let $A \in \mathbb{B}(\mathcal{H})$ be a positive operator. Then for any unit vector $x \in \mathcal{H}$,*

$$\langle Ax, x\rangle^p \le \langle A^p x, x\rangle \quad (p \ge 1).$$

The second lemma is the converses of Hölder–McCarthy inequality [11, Theorem 1.29].

Lemma 2.2. *Let $A \in \mathbb{B}(\mathcal{H})$ such that $mI \le A \le MI$ for some scalars $0 < m < M$. Then for any unit vector $x \in \mathcal{H}$,*

$$\left\langle A^2 x, x\right\rangle \le K(h)\langle Ax, x\rangle^2,$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$.

The third lemma is a reverse of operator AM–GM inequality, which has been proved in [5, Theorem 1].

Lemma 2.3. *Let $A, B \in \mathbb{B}(\mathcal{H})$ such that $mI \le A, B \le MI$ for some scalars $0 < m < M$. Then for any $0 \le v \le 1$,*

$$A\nabla_v B \le \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A\sharp_v B,$$

where $\lambda = \min\{v, 1-v\}$. In particular,

$$A\nabla B \le \sqrt{K(h)} A\sharp B,$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$.

Our first result is a generalization of the inequality (3).

Theorem 2.4. *Let $A, B \in \mathbb{B}(\mathcal{H})$ such that $mI \le A, B \le MI$ for some scalars $0 < m < M$. Then for any $0 \le v \le 1$,*

$$(A\nabla_v B)^2 \le K(h)\left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^2 (A\sharp_v B)^2, \tag{5}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$, and $\lambda = \min\{v, 1-v\}$. In particular,

$$(A\nabla B)^2 \le K(h)^2 (A\sharp B)^2.$$

Proof. One can see that Lemma 2.3 implies

$$\langle A\nabla_v Bx, x\rangle \leq \left\langle \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A\sharp_v Bx, x \right\rangle \tag{6}$$

for any unit vector $x \in \mathcal{H}$. Taking the square in (6), we have

$$\begin{aligned}\langle A\nabla_v Bx, x\rangle^2 &\leq \left\langle \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A\sharp_v Bx, x \right\rangle^2 \\ &= \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^2 \langle A\sharp_v Bx, x\rangle^2 \\ &\leq \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^2 \left\langle (A\sharp_v B)^2 x, x\right\rangle \quad \text{(by Lemma 2.1).}\end{aligned} \tag{7}$$

On the other hand, $mI \leq A, B \leq MI$ implies

$$(1-v)\, mI \leq (1-v)\, A \leq (1-v)\, MI, \tag{8}$$

and

$$vmI \leq vB \leq vMI. \tag{9}$$

It follows from (8) and (9) that

$$mI \leq A\nabla_v B \leq MI.$$

By applying Lemma 2.2, we get

$$\frac{1}{K(h)} \left\langle (A\nabla_v B)^2 x, x\right\rangle \leq \langle A\nabla_v Bx, x\rangle^2. \tag{10}$$

Combining (7) and (10) we infer

$$\left\langle (A\nabla_v B)^2 x, x\right\rangle \leq K(h) \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^2 \left\langle (A\sharp_v B)^2 x, x\right\rangle$$

for any unit vector $x \in \mathcal{H}$. This completes the proof. □

Another result of this type is the following one:

Corollary 2.5. *Let $A, B \in \mathbb{B}(\mathcal{H})$ such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then for any $0 \leq v \leq 1$,*

$$(A\nabla_v B)^2 \leq \left(K(h) \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^2 - 1\right) M^2 I + (A\sharp_v B)^2,$$

where $\lambda = \min\{v, 1-v\}$.

Proof. It follows from (5) that

$$\begin{aligned}(A\nabla_v B)^2 - (A\sharp_v B)^2 &\leq \left(K(h) \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^2 - 1\right)(A\sharp_v B)^2 \\ &\leq \left(K(h) \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^2 - 1\right) M^2 I,\end{aligned}$$

where the second inequality follows from the fact that

$$A, B \leq MI \quad \Rightarrow \quad A\sharp_v B \leq MI\sharp_v MI = MI.$$

□

As pointed out by Fujii and Nakamura in their paper [3, Theorem 2], if $A \in \mathbb{B}(\mathcal{H})$ is a positive operator such that $mI \leq A \leq MI$ for some scalars $0 < m < M$, and $x \in \mathcal{H}$ is a unit vector, then for any $p \geq 1$,

$$\langle A^p x, x\rangle \leq K(m, M, p) \langle Ax, x\rangle^p, \tag{11}$$

where

$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p} \cdot \frac{M^p - m^p}{mM^p - Mm^p}\right)^p. \tag{12}$$

We note that $K(m, M, -1) = K(m, M, 2) = \frac{(M+m)^2}{4Mm}$ is the original Kantorovich constant.

Now, by employing (11) and applying a same arguments as in the proof of Theorem 2.4 we reach the following result.

Corollary 2.6. *Let $A, B \in \mathbb{B}(\mathcal{H})$ such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then for any $p \geq 1$ and $0 \leq v \leq 1$,*

$$(A\nabla_v B)^p \leq K(m, M, p) \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^p (A\sharp_v B)^p,$$

where $K(m, M, p)$ is defined as in (12), and $\lambda = \min\{v, 1-v\}$.

Since $A^{-1}\nabla_v B^{-1} \leq \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A^{-1}\sharp_v B^{-1}$ and $\left(A^{-1}\sharp_v B^{-1}\right)^{-1} = A\sharp_v B$, it follows that

$$A\sharp_v B \leq \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A!_v B. \tag{13}$$

The following result concerning (13) may be stated:

Corollary 2.7. *Let $A, B \in \mathbb{B}(\mathcal{H})$ such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then for any $p \geq 1$ and $0 \leq v \leq 1$,*

$$(A\sharp_v B)^p \leq K(m, M, p) \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^p (A!_v B)^p,$$

where $K(m, M, p)$ is defined as in (12), and $\lambda = \min\{v, 1-v\}$.

The following result is interesting in itself as well.

Proposition 2.8. *Let $A \in \mathbb{B}(\mathcal{H})$ be a positive operator with the spectra contained in the interval J. If $f : J \to \mathbb{R}$ is a convex function, then for any unit vector $x \in \mathcal{H}$*

$$\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p + p\alpha,$$

where

$$\alpha = \sup_{\substack{x\in\mathcal{H} \\ \|x\|=1}} \left\{\langle A^p x, x\rangle - \langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle\right\}.$$

Proof. It is well known that if f is a convex function on an interval J, then for each point $(s, f(s))$, there exists a real number C_s such that

$$f(s) + C_s(t-s) \leq f(t)$$

for all $t \in J$. If f is a differentiable at s, then

$$f(s) + f'(s)(t-s) \leq f(t). \tag{14}$$

Since $f(t) = t^p$ $(p \geq 1)$ is a convex and differentiable function, then from (14) we obtain

$$s^p + ps^{p-1}(t-s) \leq t^p.$$

Applying functional calculus we get

$$A^p + \left(ptA^{p-1} - pA^p\right) \leq t^p I.$$

Hence for any unit vector $x \in \mathcal{H}$,

$$\langle A^p x, x\rangle + \left(pt\left\langle A^{p-1}x, x\right\rangle - p\langle A^p x, x\rangle\right) \leq t^p. \tag{15}$$

Now, since the spectra of $\langle Ax, x\rangle$, contained in the interval J, by replacing t by $\langle Ax, x\rangle$ in (15), we infer (see also [1, Theorem 2.1])

$$\langle A^p x, x\rangle + \left(p\langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle - p\langle A^p x, x\rangle\right) \leq \langle Ax, x\rangle^p.$$

On the other hand, by Lemma 2.1, we know that

$$\langle A^p x, x\rangle - \langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle \geq 0.$$

Therefore,

$$\langle A^p x, x\rangle - \langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle \leq \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \left\{\langle A^p x, x\rangle - \langle Ax, x\rangle\left\langle A^{p-1}x, x\right\rangle\right\}$$

Consequently,

$$\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p + p\alpha$$

which completes the proof of this proposition. □

We now present our next main result.

Theorem 2.9. *Let $A, B \in \mathbb{B}(\mathcal{H})$ such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then for any $p \geq 1$ and $0 \leq v \leq 1$,*

$$(A\nabla_v B)^p \leq \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^p (A\sharp_v B)^p + p\beta I,$$

where $\lambda = \min\{v, 1-v\}$ and

$$\beta = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \left\{\langle (A\nabla_v B)^p x, x\rangle - \langle A\nabla_v Bx, x\rangle\left\langle (A\nabla_v B)^{p-1}x, x\right\rangle\right\}.$$

Furthermore,

$$(A\sharp_v B)^p \leq \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^p (A!_v B)^p + \gamma pI,$$

where

$$\gamma = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \left\{\left\langle (A\sharp_v B)^p x, x\right\rangle - \langle A\sharp_v Bx, x\rangle\left\langle (A\sharp_v B)^{p-1}x, x\right\rangle\right\}.$$

Proof. Employing Proposition 2.8 for two inequalities

$$\langle A\nabla_v Bx, x\rangle^p \leq \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^p \left\langle (A\sharp_v B)^p x, x\right\rangle,$$

and

$$\langle A\sharp_v Bx, x\rangle^p \leq \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^p \langle (A!_v B)^p x, x\rangle.$$

□

It has been shown in [6], that if $A, B \in \mathbb{B}(\mathcal{H})$ are two positive operators satisfying $mI \leq A, B \leq MI$ for some scalars $0 < m < M$, and $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ is a unital positive linear map, then

$$\Phi(A)\sharp\Phi(B) \leq \sqrt{K(h)}\Phi(A\sharp B). \tag{16}$$

Remark 2.10. *We give a simple proof of* (16)*. One can write*

$$\begin{aligned}\Phi(A)\sharp\Phi(B) &\leq \Phi(A)\nabla\Phi(B) \quad \textit{(by (1))}\\ &= \Phi(A\nabla B)\\ &\leq \Phi\left(\sqrt{K(h)}A\sharp B\right) \quad \textit{(by Lemma 2.3)}\\ &= \sqrt{K(h)}\Phi(A\sharp B)\end{aligned}$$

i.e.,

$$\Phi(A)\sharp\Phi(B) \leq \Phi(A)\nabla\Phi(B) \leq \sqrt{K(h)}\Phi(A\sharp B)$$

which actually refines the inequality (16)*.*

We next present the generalizations of (16).

Theorem 2.11. *Let $A, B \in \mathbb{B}(\mathcal{H})$ such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then for any $p \geq 1$,*

$$(\Phi(A)\sharp\Phi(B))^p \leq K(h)^{\frac{p}{2}} K(m, M, p)\, \Phi(A\sharp B)^p, \tag{17}$$

and

$$(\Phi(A)\sharp\Phi(B))^p - \Phi(A\sharp B)^p \leq \left(K(h)^{\frac{p}{2}} K(m, M, p) - 1\right) M^p I, \tag{18}$$

where $K(m, M, p)$ is defined as in (12)*.*

Proof. It follows from (16) that

$$\begin{aligned}\langle\Phi(A)\sharp\Phi(B)x, x\rangle^p &\leq K(h)^{\frac{p}{2}}\langle\Phi(A\sharp B)x, x\rangle^p\\ &\leq K(h)^{\frac{p}{2}}\left\langle\Phi(A\sharp B)^p x, x\right\rangle\end{aligned} \tag{19}$$

for any unit vector $x \in \mathcal{H}$.
Since $mI \leq A, B \leq MI$ and Φ is a unital positive linear mapping, then $mI \leq \Phi(A), \Phi(B) \leq MI$. Thus, $mI \leq \Phi(A)\sharp\Phi(B) \leq MI$. Hence from (11),

$$\left\langle(\Phi(A)\sharp\Phi(B))^p x, x\right\rangle \leq K(m, M, p)\langle\Phi(A)\sharp\Phi(B)x, x\rangle^p \tag{20}$$

for any unit vector $x \in \mathcal{H}$. Combining (19) and (20), we get the desired inequality (17).
For (18),

$$\begin{aligned}(\Phi(A)\sharp\Phi(B))^p - \Phi(A\sharp B)^p &\le \left(K(h)^{\frac{p}{2}}K(m,M,p)-1\right)\Phi(A\sharp B)^p \\ &\le \left(K(h)^{\frac{p}{2}}K(m,M,p)-1\right)M^pI.\end{aligned}$$

where we have used the fact that

$$mI \le \Phi(A\sharp B) \le MI.$$

□

It is immediate to see from (17) and (18) that

$$(\Phi(A)\sharp\Phi(B))^2 \le K(h)^2\Phi(A\sharp B)^2,$$

and

$$(\Phi(A)\sharp\Phi(B))^2 - \Phi(A\sharp B)^2 \le \left(K(h)^2-1\right)M^2I.$$

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